1 Examples of guarded signatures and guarded algebras

This is a first attempt to take the examples from wga04.tex and recast them in the new formulation.

1.1 Examples of guarded signatures

We describe d and c in a signature $\Gamma = (S, UF, PF, d, c, \delta, \gamma)$ by giving the inverse of the function

$$\begin{aligned} (d,c) : (UF \cup PF) &\to S^* \times S \\ f &\mapsto \langle d(f), c(f) \rangle \end{aligned}$$

The examples to be attempted include: Naturals with Bool, Chess-boards and Arrays.

Example 1.1 The signature **Naturals**

$$S = \{nat,\ bool\}$$

$$UF = \{0,\ 1,\ 2, \ldots,\ true,\ false,\ suc,\ add,\ mult,\ eq,\ leq,\ dnat,\ dbool,\ not,\ and\}$$

$$PF = \{prd,\ sbt,\ div\}$$

$$\begin{aligned}
&(d,c)^{-1}(\lambda,\ nat) = \{0,\ 1,\ 2, \ldots\} \\
&(d,c)^{-1}(\lambda,\ bool) = \{true,\ false\} \\
&(d,c)^{-1}(nat,\ nat) = \{suc,\ prd\} \\
&(d,c)^{-1}(nat.nat,\ nat) = \{add,\ mult,\ sbt,\ div\} \\
&(d,c)^{-1}(nat,\ bool) = \{eq,\ leq,\ dnat,\ dbool\} \\
&(d,c)^{-1}(bool,\ bool) = \{not\} \\
&(d,c)^{-1}(bool.bool,\ bool) = \{and\}
\end{aligned}$$

$$\begin{aligned}
&\delta(nat) = (bool,\ true,\ dnat) \\
&\delta(bool) = (bool,\ true,\ dbool)
\end{aligned}$$

$$\begin{aligned}
&\gamma(prd) = (bool,\ false,\ eq(x_1, 0)) \\
&\gamma(sbt) = (bool,\ true,\ leq) \\
&\gamma(div) = (bool,\ false,\ eq(x_1, 0))
\end{aligned}$$

□

Example 1.2 The signature **Chess-board**

$$S = \{pos,\ bool\}$$

$$UF = \{dpos,\ dbool,\ true,\ false,\ not,\ dup,\ ddown,\ dleft,\ dright\}$$

$PF = \{up,\ down,\ right,\ left\}$

$(d,c)^{-1}(pos,\ bool) = (dpos,\ dup,\ ddown,\ dleft,\ dright\}$
$(d,c)^{-1}(\lambda,\ bool)$
$(d,c)^{-1}(bool,\ bool) = \{dbool,\ not\}$
$(d,c)^{-1}(bool.bool,\ bool) = \{and\}$
$(d,c)^{-1}(pos,\ pos) = \{up,\ down,\ left,\ right\}$

$\delta(pos) = (bool,\ true,\ dpos)$
$\delta(bool) = (bool,\ true,\ dbool)$

$\gamma(up) = (bool,\ true,\ dup)$
$\gamma(down) = (bool,\ true,\ ddown)$
$\gamma(left) = (bool,\ true,\ dleft)$
$\gamma(right) = (bool,\ true,\ dright)$

□

Example 1.3 The signature **Array**
The idea here is to specify array of elements of some set, *Elt*, subject to two guards

- legal(a,i), which checks to see if the index i is appropriate for the array a – for an array of length n the le gal indices are 0 through $n-1$.
- present(a,i) which checks if the ith index has b een given a value in the array a.

We assume that all the operations from the signature **Natural** are included and only present the new operations.

$S = \{nat,\ elt, array,\ bool\}$

$UF = \{create,\ length,\ legal,\ present\}$

$PF = \{legal,\ present\}$

$(d,c)^{-1}(nat,\ array) = \{create\}$
$(d,c)^{-1}(array,\ nat) = \{length\}$
$(d,c)^{-1}(array.nat,\ bool) = \{legal,\ present\}$
$(d,c)^{-1}(array.nat.elt,\ array) = \{insert\}$
$(d,c)^{-1}(array.nat,\ elt) = \{access\}$

$\delta(array) = (bool,\ true,\ legal)$

$\gamma(insert) = (bool,\ true,\ legal(a,i))$
$\gamma(access) = (bool,\ true,\ present(a,i))$

□

1.2 Examples of Guarded Algebras

We give two examples for each of the above three signatures.

Example 1.4 The algebra **NAT1**

carriers $\texttt{NAT1}(nat) = \{0, 1, 2, 3, \ldots\}$, $\texttt{NAT1}(bool) = \{\texttt{true}, \ \texttt{false}\}$.

operations: Let the operations other than `dnat, dbool, pr d, sub,` and `div`, have the expected interpretation. Take

$\texttt{NAT1}(dnat)(n) = true$

$\texttt{NAT1}(dbool)(b) = true$

$$\texttt{NAT1}(prd)(n) = \begin{cases} n-1 & \text{if } n > 0 \\ n & \text{if } n = 0 \end{cases}$$

$$\texttt{NAT1}(sbt)(n, m) = \begin{cases} n-m & \text{if } n \geq m \\ n & \text{otherwise} \end{cases}$$

$$\texttt{NAT1}(div)(n, m) = \begin{cases} n/m & \text{if } m \neq 0 \\ n & \text{if } n = 0 \end{cases}$$

where for $n \neq 0$ we define n/m to be the greatest natural number q such that $q * m \leq n$.
□

Example 1.5 The algebra **NAT2**.

carriers:

$\texttt{NAT2}(nat) = \{0, 1, 2, 3, \ldots\} \cup \{errnat\}$
$\texttt{NAT2}(bool) = \{\texttt{true}, \ \texttt{false}, \ errbool\}$.

operations: Let the operations of sort *nat* other than `dnat, dbool, prd, sub,` and `div` have the expected interpretation on $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ or $\{true, \ false\}$, but preserve `errnat`, e.g., `add(3,4)=7` but $add(errnat, 4) = add(3, errnat) = add(errnat, errnat) = errnat$. Then take

$$\texttt{NAT2}(dnat)(n) = \begin{cases} true & \text{if } n \in \mathbf{N} \\ errbool & \text{otherwise} \end{cases}$$

$$\texttt{NAT2}(dbool)(b) = \begin{cases} true & \text{if } b \in \{true, \ false\} \\ errbool & \text{otherwise} \end{cases}$$

$$\texttt{NAT2}(prd)(n) = \begin{cases} n-1 & \text{if } n > 0, \ n \neq \texttt{errnat} \\ \texttt{errnat} & \text{if } n = 0 \text{ or } n = \texttt{errnat} \end{cases}$$

$$\texttt{NAT2}(sbt)(n, m) = \begin{cases} n-m & \text{if } n \geq m, \ n, m \neq \texttt{errnat} \\ \texttt{errnat} & \text{otherwise} \end{cases}$$

$$\texttt{NAT2}(div)(n,m) = \begin{cases} n/m & \text{if } m \neq 0,\ n,m \neq \texttt{errnat} \\ \texttt{errnat} & \text{otherwise} \end{cases}$$

where for $n \neq 0$ we define n/m to be the greatest natural number q such that $q*m \leq n$.

□

Example 1.6 The algebra **CB1**.

carriers:

$\mathbf{CB1}(pos) = \mathbf{Z} \times \mathbf{Z}$.

$\mathbf{CB1}(bool) = \{true,\ false\}$.

operations:

$\mathbf{CB1}(up)(\langle x,y\rangle) = \langle x, y+1\rangle$

$\mathbf{CB1}(down)(\langle x,y\rangle) = \langle x, y-1\rangle$

$\mathbf{CB1}(left)(\langle x,y\rangle) = \langle x-1, y\rangle$

$\mathbf{CB1}(right)(\langle x,y\rangle) = \langle x+1, y\rangle$

$$\mathbf{CB1}(dpos)(\langle x,y\rangle) = \begin{cases} true & \langle x,y\rangle \in [8]\times[8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB1}(dup)(\langle x,y\rangle) = \begin{cases} true & \langle x,y\rangle, up(\langle x,y\rangle) \in [8]\times[8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB1}(ddown)(\langle x,y\rangle) = \begin{cases} true & \langle x,y\rangle, down(\langle x,y\rangle) \in [8]\times[8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB1}(dleft)(\langle x,y\rangle) = \begin{cases} true & \langle x,y\rangle, left(\langle x,y\rangle) \in [8]\times[8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB1}(dright)(\langle x,y\rangle) = \begin{cases} true & \langle x,y\rangle, right(\langle x,y\rangle) \in [8]\times[8] \\ false & \text{otherwise} \end{cases}$$

$\mathbf{CB1}(dbool)(b) = true.$

□

Example 1.7 The algebra **CB2**

carriers:

$\mathbf{CB2}(pos) = [8] \times [8]$

$\mathbf{CB2}(bool) = \{true,\ false\}$.

operations:

$$\mathbf{CB2}(up)(\langle x,y\rangle) = \begin{cases} \langle x, y+1\rangle & \text{if } x \in [8],\ y \in [7] \\ \langle x,y\rangle & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(down)(\langle x,y\rangle) = \begin{cases} \langle x, y-1\rangle & \text{if } x \in [8],\ y \in \{2,3,4,5,6,7,8\} \\ \langle x,y\rangle & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(left)(\langle x,y\rangle) = \begin{cases} \langle x-1, y+1\rangle & \text{if } x \in \{2,3,4,5,6,7,8\},\ y \in [8] \\ \langle x,y\rangle & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(right)(\langle x,y\rangle) = \begin{cases} \langle x+1, y\rangle & \text{if } x \in [7],\ y \in [8] \\ \langle x,y\rangle & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(dpos)(\langle x,y\rangle) = \begin{cases} true & \text{if } \langle x,y\rangle \in [8] \times [8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(dup)(\langle x,y\rangle) = \begin{cases} true & \text{if } \langle x,y\rangle \in [8] \times [7] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(ddown)(\langle x,y\rangle) = \begin{cases} true & \text{if } x \in [8],\ y \in \{2,3,4,5,6,7,8\} \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(dleft)(\langle x,y\rangle) = \begin{cases} true & \text{if } x \in \{2,3,4,5,6,7,8\},\ y \in [8] \\ false & \text{otherwise} \end{cases}$$

$$\mathbf{CB2}(dright)(\langle x,y\rangle) = \begin{cases} true & \text{if } x \in [7],\ y \in [8] \\ false & \text{otherwise} \end{cases}$$

$\mathbf{CB2}(dbool)(b) = true.$

□

Example 1.8 The algebra **ARR1**:

carriers:

$\mathbf{ARR1}(arr) = (Elt + \{B\})^*$, where B is a new symbol which we may think of informally as "blank".

operations:

$\mathbf{ARR1}(create)(n) = \langle B, \ldots, B\rangle \in (Elt + \{B\})^n.$

$\mathbf{ARR1}(length)(a) = |a|.$

$$\mathbf{ARR1}(present)(a,i) = \begin{cases} true & \text{if } n \le |a| \text{ and } a_{i+1} \ne B \\ false & \text{otherwise} \end{cases}$$

$\mathbf{ARR1}(legal)(a,i) = i < |a|$

$$\mathbf{ARR1}(insert)(a,i,e) = \begin{cases} \langle a_1, \ldots, a_i, e, a_{i+2}, \ldots, a_n\rangle & \text{if } legal(a,i) = true \\ ? & \text{otherwise} \end{cases}$$

$$\mathbf{ARR1}(access)(a,i) = \begin{cases} a_{i+1} & \text{if } legal(a,i) = true \text{ and } present(a,i) = true \\ ? & \text{otherwise} \end{cases}$$

$\mathbf{ARR1}(darr)(a) = true$

Where "?" can be any element of the appropriate carrier. □

Example 1.9 The algebra **ARR2**:

carriers:

ARR2(arr) = The set of all pairs $\langle n,\ f\rangle$ where f is a single-valued subset of $\{0, \ldots, n-1\} \times Elt$ (here "single-valued" means that $\langle i, e\rangle, \langle i, e'\rangle \in f$ implies $e = e'$). For any such pair, $\langle n, f\rangle$ and $i < n$ let

$$val(f, i) = \begin{cases} \{\langle i-1, e\rangle\} & \text{if } \langle i-1, e\rangle \in f \\ \emptyset & \text{if no such } e \text{ exists} \end{cases}$$

operations:

$\mathbf{ARR2}(create)(n) = \langle n, \emptyset\rangle$

$\mathbf{ARR2}(length)(\langle n, f\rangle) = n$

$\mathbf{ARR2}(present)(\langle n, f\rangle, i) = \exists e,\ \langle i-1, e\rangle \in f.$

$\mathbf{ARR2}(legal)(\langle n, f\rangle, i) = i < n$

$$\mathbf{ARR2}(insert)(\langle n, f\rangle, i, e) = \begin{cases} (f \backslash val(f, i)) \cup \{\langle i-1, e\rangle\} & \text{if } i < n \\ ? & \text{otherwise} \end{cases}$$

$$\mathbf{ARR2}(access)(\langle n, f\rangle, i) = \begin{cases} e & \text{if } \exists e,\ val(f, i) = \{\langle i-1, e\rangle\} \\ ? & \text{otherwise} \end{cases}$$

$\mathbf{ARR2}(darr)(a) = true$

Where "?" can be any element of the appropriate carrier. □