Algebras, Categories and Programming Languages: Part II

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13 Cartesian Closed Categories

The recent rise in interest in category theory by computer scientists is largely due to applications of category theory to the understanding of various version of the $\lambda - calculus$. The key categorical concept underlying much of that work is the concept of a Cartesian closed category. Cartesian closed categories are defined in terms of adjoints. By way of motivation before the definition, we look at a special adjunction in **Set**.

Example 13.149 Let S be a set, then we can define a functor

$$\begin{aligned} \_ \times S : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto A \times S \\ f &\mapsto f \times 1_S \end{aligned}$$

where "$\times$" denotes the Cartesian product. Then $\_ \times S$ has a right-adjoint

$$\begin{aligned} \_^S : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto A^S \\ g &\mapsto g^S \end{aligned}$$

where $A^S = \mathbf{Set}(S, A)$, the set of all total mappings $f : S \to A$ and

$$\begin{aligned} g^S : A^S &\to B^S \\ h &\mapsto g \bullet h. \end{aligned}$$

□

Proof For $A \in |\mathbf{Set}|$, define

$$\begin{aligned} \varepsilon_A : A^S \times S &\to A \\ \langle \alpha, s \rangle &\mapsto \alpha(s) \end{aligned}$$

that is, ε_A *evaluates* the function $\alpha \in A^S$ at argument $s \in S$. Let $C \in |\mathbf{Set}|$, then $(\_ \times S)(C) = C \times S$. Now let $f : C \times S \to A$. We must show that there exists a unique morphism $\overline{f} : C \to A^S$ such that $\varepsilon_A \bullet (\_ \times S)(\overline{f}) = f$. But since $(\_ \times S)(\overline{f}) = \overline{f} \times 1_S$ we see that the desired $\overline{f} : C \to A^S$ is the one that takes each $c \in C$ to the mapping

$$\begin{aligned} f_c : S &\to A \\ s &\mapsto f(c, s). \end{aligned}$$

□

Readers familiar with λ-calculus should recognize an intuitive tie between the counit ε and the evaluation operation in the λ-calculus, and between $\psi : \mathbf{Set}(\_ \times S, \_) \to \mathbf{Set}(\_, \_^S)$ (from Proposition 12.144 (of the notes handed out earlier)and the abstraction operation $(e(x) \mapsto \lambda x.e(x))$ of the λ-calculus. In particular, we draw the readers attention to the similarity between the λ-calculus identity

$$(\lambda x.f(x)a) = f(a)$$

and, the identity

$$\varepsilon \bullet (\psi(f(x)) \times a) = f(a)$$

arising from the above adjunction.

Definition 13.150 A category $\mathbf{C}$ is *Cartesian closed* when each of the functors $U_{\mathbf{1}} : \mathbf{C} \to \mathbf{1}$, $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$, and $\_ \times b : \mathbf{C} \to \mathbf{C}$ has a specified right adjoint. That is, $\mathbf{C}$ comes with a specified terminal object, a specified product object $X \times Y$ for all $X, Y \in |\mathbf{C}|$ with specified projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, and, finally, for each $S \in |\mathbf{C}|$ we have a specific right adjoint of $\_ \times S$, denoted $\_^b$ and taking each $A \in |\mathbf{C}|$ to an object A^S, and a corresponding isomorphism

$$\Lambda_{A,B} : \mathbf{C}(A \times S, B) \cong \mathbf{C}(A, B^S)$$

natural in A and B. The corresponding component of the counit of this adjunction is then a morphism $\varepsilon_B : B^S \times S \to B$ which is natural in B and universal from $\_ \times S$ to B. □

For a detailed treatment of the relationship between cartesian closed categories and the λ-calculus see Lambek and Scott, [2].

Instead of defining a cartesian closed category as above we can define it equationally (as if it was an equationally defined algebra), but it is important to realize that many examples, such as that of **Set**, are not algebras as they involve proper classes (non-sets).

The equational presentation goes as follows: For all $A, B \in |\mathbf{C}|$ we have operators $\varepsilon_{A,B} : B^A \times A \to B$ and for all $X \in |\mathbf{C}|$ we have an operator $\Lambda_{A,B,X} : \mathbf{C}(X \times A, B) \to `bfC(X, B^A)$ satisfying the axioms

$$\varepsilon_{A,B} \bullet \langle \Lambda_{A,B,C}(f) \times 1_A \rangle = f$$

and

$$\Lambda_{A,B,X}(\varepsilon_{A,B} \bullet (h \times 1_A)) = h$$

Note that the first axiom above guarantees that we have the commuting diagram needed to define the adjoint while the second equation establishes the necessary uniqueness.

The whole definition can be made equational by adding axioms for a category (i.e., equational presentations of the properties of composition and the identity morphisms) and equations for binary product and the terminal object, as follows:

$$p_{A,B} \bullet \langle f, g \rangle = f$$

$$q_{A,B} \bullet \langle f, g \rangle = g$$

and, for any $h : C \to A \times B$,

$$\langle p_{A,B} \bullet h, q_{A,B} \bullet h \rangle = h$$

and, for each $A \in |\mathbf{C}|$ we have $t_A : A \to \mathbf{1}$ and the axiom that for all $f : A \to \mathbf{1}$,

$$f = t_A.$$

Proposition 13.151 *If* $\mathbf{C}$ *is a cartesian closed category and* $\mathbf{C}$ *has designated finite coproducts then* $\mathbf{C}$ *is distributive.*

Proof For every $S \in |\mathbf{C}|$ the functor $\_ \times S$ is a left adjoint and so preserves colimits and thus coproducts in particular. Given a coproduct $\langle A+B, i : A \to A+B, j : B \to A+B \rangle$, its image under $\_ \times S$ is $\langle (A+B) \times S,\ i \times 1_S : A \times S \to (A+B) \times S,\ j \times 1_S : B \times S \to (A+B) \times S \rangle$ and so this cocone must be a coproduct in $\mathbf{C}$ for $A \times S$ and $B \times S$. Now let $\langle (A \times S) + (B \times S), i', j' \rangle$ be the designated coproduct for $(A \times S)$ and $(B \times S)$ then $[i \times 1_S,\ j \times 1_S] : (A \times S) + (B \times S) \to (A + B) \times S$ is the unique morphism, h, such that $h \bullet i' = i \times 1_S$ and $h \bullet j' = j \times 1_S$ and, by the earlier argument that all coproducts for $(A \times S)$ and $(B \times S)$ are isomorphic, it follows that it is an isomorphism. □

Example 13.152 The category **Set** is a cartesian closed category that is also a distributive category. □

Definition 13.153 A category **C** has *fixpoints* if for every $f : A \times B \to B$ there exists a morphism $Y(f) : A \to B$ such that $f \bullet \langle 1_A,\ Y(f)\rangle = Y(f)$. □

There exist cartesian closed categories with fixpoints. A good example is the category **CPO** of continuous partial orders which is defined as follows:

> Let $P = \langle P, \leq\rangle$ be a poset with a *bottom* $\perp$.
>
> $D \subseteq P$ is said to be *directed* if for all $d, d' \in D$ there exists $d'' \in D$ such that $d, d' \leq d''$.
>
> P is *(directed-) complete* (is a CPO = complete partial order) if every directed subset of D of P has a least upper bound $\bigsqcup D$ in P. There are other options here such a ω- and bounded- complete. They have many properties in common which have been studied under the title of Z -completeness with Z a variable.
>
> Let P and Q be CPOs then $f : P \to Q$ is
>
> > *monotonic* if it preserves order, i.e., $p \leq q$ implies $f(p) \leq f(q)$.
> > *continuous* if for every directed subset $D \subseteq P$, $f(\bigsqcup D) = \bigsqcup\langle f(d) \mid d \in D\rangle$. Note that f continuous implies f monotonic.

Proposition 13.154 *Let* **CPO** *be the category with all CPOs as objects and with continuous maps as morphisms. Then* **CPO** *is a cartesian closed category with fixpoints.*

Proof Let the products be as in **Set**. Define $Q^P = \mathbf{CPO}(P, Q)$, note that this set is itself a CPO under the ordering that $f \leq g$ iff $f(p) \leq g(p)$ for all $p \in P$. Let

$$\begin{gathered} \varepsilon_{P,Q} : Q^P \times P \to Q \\ \langle f, p\rangle \mapsto f(p) \end{gathered}$$

It follows then that **CPO** is cartesian closed by essentially the same argument as for **Set**.

Now, given $f : P \times Q \to Q$ we want $Y(f) : P \to Q$ such that $f \bullet \langle 1_P, Y(f)\rangle = Y(f)$. Given $p \in P$ let

$$a^{(0)} = \perp \text{ and } a^{(n+1)} = f(p, a^{(n)}).$$

then

$$\begin{gathered} \perp = a^{(0)} \leq a^{(1)} \\ \text{and} \\ a^{(n)} \leq a^{(n+1)} \text{ implies } a^{(n+1)} = f(p, a^{(n)}) \leq f(p, a^{(n+1)}) = a^{(n+2)} \end{gathered}$$

by monotonicity. So $\langle a^{(n)} \mid n \in \omega\rangle$ is an ω-chain and thus a directed set so $\bigsqcup\langle a^{(n)} \mid n \in \omega\rangle$ exists. Now take $Y(f) = \bigsqcup\langle a^{(n)} \mid n \in \omega\rangle$ then

$$f(p, \bigsqcup\langle a^{(n)} \mid n \in \omega\rangle) = \bigsqcup\langle f(p, a^{(n)}) \mid n \in \omega\rangle = \bigsqcup\langle a^{(n+1)} \mid n \in \omega\rangle = \bigsqcup\langle a^{(n)} \mid n \in \omega\rangle$$

□

Proposition 13.155 *In any cartesian category $A \times \mathbf{1} \cong A$.*

Proof Let $t_A : A \to \mathbf{1}$ be the unique morphism given by the terminality of $\mathbf{1}$, then let $\langle 1_A, t_A \rangle : A \to A \times \mathbf{1}$ be the unique product mediating morphism for 1_A and t_A. It is immediate that $p \bullet \langle 1_A, t_A \rangle = 1_A$. Now consider $\langle 1_A, t_A \rangle \bullet p$, we have:

$$p \bullet \langle 1_A, t_A \rangle \bullet p = 1_A \bullet p = p$$

and

$$q \bullet \langle 1_A, t_A \rangle \bullet p = t_A \bullet p = t_{A \times \mathbf{1}} = q$$

The last two equalities holding because there is one, and only one, morphism $A \times \mathbf{1} \to \mathbf{1}$ since $\mathbf{1}$ is terminal. But then $\langle 1_A, t_A \rangle \bullet p = \langle p, q \rangle = 1_{A \times \mathbf{1}}$ and thus $\langle 1_A, t_A \rangle$ is an isomorphism. □

Proposition 13.156 *If $\mathbf{C}$ is a cartesian closed category with an initial object $\mathbf{0}$ and fixpoint operator Y then $\mathbf{C}$ is trivial (i.e., all objects are isomorphic).*

Proof Consider $p : \mathbf{1} \times \mathbf{0} \to \mathbf{0}$ then $Y(p) : \mathbf{1} \to \mathbf{0}$ but there exists a unique morphism $0_{\mathbf{1}} : \mathbf{0} \to \mathbf{1}$ since $\mathbf{0}$ is initial and

$$0_{\mathbf{0}} \bullet Y(p) : \mathbf{1} \to \mathbf{1} \text{ so } = 1_{\mathbf{1}}$$

and

$$Y(p) \bullet O_{\mathbf{O}} : \mathbf{0} \to \mathbf{0} \text{ so } = 1_{\mathbf{0}}$$

so $\mathbf{0} \cong \mathbf{1}$, but then for any $A \in |\mathbf{C}|$

$$A \cong A \times \mathbf{1} \cong A \times \mathbf{0} \cong \mathbf{0}$$

The last equality holding since $\_ \times A$, being a left adjoint, preserves colimits and thus initial objects in particular, thus $\mathbf{0} \times A$ is an initial object and thus isomorphic to $\mathbf{0}$. And, of course, because they are both products for A and $\mathbf{0}$, $A \times \mathbf{0} \cong \mathbf{0} \times A$. □

Example 13.157 Let X be a set and let P be the poset $P = \langle \mathcal{P}, \subseteq \rangle$, where $\mathcal{P}(X)$ is the set of all subsets of X ordered by inclusion. This poset P, viewed as a category with an arrow $A \to B$ iff $A \subseteq B$ is a cartesian closed category. The product operation is intersection of sets, i.e., for sets A and B $A \times B = A \cap B$ with the projections being the evident inclusions. Now given sets A and B define $A^B = ((X - B) \cup A)$, the morphism $\varepsilon_{A,B} : A^B \times B \to A$ is then just the inclusion $((X - B) \cup A) \cap B \subseteq A$.

Then we see that for for any subsets A, B and C of X, that if $f : C \times B \to A$ is the inclusion if $C \cup B \subseteq A$ then $\Lambda(f) : C \to A^B$ is the inclusion $C \subseteq ((X - B) \cup A)$, and $\Lambda(f) \times 1_B$ is the inclusion $C \cap B \subseteq ((X - B) \cup A) \cap B$. Note that the choice of A^B is such that $C \cap B \subseteq A$ iff $C \subseteq ((X - B) \cup A)$ and this is exactly the property captured by saying that P is a cartesian closed category. □

14 Typed λ-Calculus

In this section we will illustrate the use of Cartesian closed categories to provide models for the λ-calculus. This treatment is a modification of that given by Asperti and Longo [1].

14.1 A Review of λ-Calculus

Let `Atom` be a given set (the set of *ground types*).

Define the set `Typ` of *type labels over* `Atom` as follows:

1. `Atom`$\subseteq$ `Typ`.
2. If $A, B \in$ `Typ` then $A \to B \in$ `Typ`.
3. If $A, B \in$ `Typ` then $A \times B \in$ `Typ`.

For each $A \in$ `Typ` let there be given a denumerable set of *variables* of type A. If we view `Typ` as a discrete category then we can think of the variables as being (given by) a functor var : `Typ` $\to$ **Set**.

The set of well typed λ-terms (w.t.t.) and their types is defined as follows:

1. Every variable $x : A$ is a w.t.t.
2. If $x : A$ is a variable and $M : B$ is a w.t.t. then $\lambda x : A.M : B$ is a w.t.t.
3. If $M : A \to B$ is a w.t.t. and $N : A$ is a w.t.t. then $MN : B$ is a w.t.t.
4. If $M : A$ is a w.t.t. and $N : B$ is a w.t.t. then so is $\langle M, N\rangle : A \times B$
5. if $M : A \times B$ is a w.t.t. then $fst(M) : A$ is a w.t.t.
6. if $M : A \times B$ is a w.t.t. then $snd(M) : B$ is a w.t.t.

If $M : B$ is a w.t.t. then the set $FV(M)$ of *free variables in* M is defined as follows:

1. If $N = x$ then $FV(M) = \{x\}$.
2. If $M = \lambda x : A.N$ then $FV(M) = FV(N) - \{x\}$.
3. If $M = NP$ then $FV(M) = FV(N) \cup FV(P)$.
4. If $M = \langle N, p\rangle$ then $FV(M) = FV(N) \cup FV(P)$.
5. if $M = fst(N)$ then $FV(M) = FV(N)$.
6. if $M = snd(N)$ then $FV(M) = FV(N)$.

Definition 14.158 The *substitution* $[M/x]N : B$, "substituting $M : A$ for $x : A$ in $N : B$", is defined as follows:

1. If $N : B = x : A$ then $[M/x]N : B = M : A$.
2. If $N : B = y : A$, $y \neq x$, then $[M/x]N : B = N : B$.

3. If $N : B = \lambda x : C.P : B$ then $[M/x]N : B = \lambda x : C.P : B$.
4. If $N : B = \lambda y : C.P : B$, $y \neq x$, $y \notin FV(M)$ then $[M/x]N : B = \lambda y : C.[M/x]P : B$.
5. If $N : B = PQ : B$ then $[M/x]N : B = [M/x]P[M/x]Q : B$.
6. If $N : B = \langle P,\ Q\rangle : B$ then $[M/x]N : B = \langle [M/x]P,\ [M/x]Q\rangle$.
7. If $N : B = fst(P) : B$ then $[M/x]N : B = fst([M/x]P) : B$.
8. If $N : B = snd(P) : B$ then $[M/x]N : B = snd([M/x]P) : B$.

□

We take the quotient of the set of λ-terms given by the following equation schemata:

(α) $\lambda y : A.M = \lambda y : A.[y/x]M$ if $y \notin FV(M)$.

$(\to \beta)$ $(\lambda x : A.M)N = [N/x]M$.

$(\times\beta_1)$ $fst(\langle M, N\rangle) = M$.

$(\times\beta_2)$ $snd(\langle M, N\rangle) = N$.

$(\to \eta)$ $\lambda x : A.(Mx) = M$ if $x \notin FV(M)$.

$(\times\eta)$ $\langle fst(P), snd(P)\rangle = P$.

14.2 Modeling the λ-calculus within a CCC

Given a CCC $\mathbf{C}$ with designated terminal object $\mathbf{1}$ and projections fst and snd, let $I : \mathtt{Atom} \to |\mathbf{C}|$ be an interpretation of the atoms and extend I to a mapping $[\_] : \mathtt{Typ} \to |\mathbf{C}|$ as follows:

$[\sigma] = I(\sigma)$ for $\sigma \in \mathtt{Atom}$.

$[\sigma \to \tau] = [\tau]^{[\sigma]}$.

$[\sigma \times \tau] = [\sigma] \times [\tau]$.

Given $u \in \mathtt{Typ}^*$ let $A_i = [u_i]$ and $n = |u|$. Define a projection

$$p_i^u : (\cdots(\mathbf{1} \times A_1)\cdots \times A_n) \to A_i$$

for each $i = 1, \ldots, n$. Note that $(\cdots(\mathbf{1} \times A_1)\cdots \times A_n) \cong A_1 \times \cdots \times A_n$, however the use of the terminal object simplifies the following presentation. We will write A^u for $(\cdots(\mathbf{1} \times A_1)\cdots \times A_n \to A)i$.

Given $u \in \mathtt{Typ}^*$, $\Delta \in \mathtt{Var}^u$ and $M : \sigma$ with $FV(M) \subseteq \{\Delta_1, \ldots, \Delta_n\}$ $(n = |u|)$ we *interpret M with respect to* Δ as a morphism

$$[M^\sigma]_\Delta : A^u \to A_0 \ (= [\sigma])$$

such that

1. $[x^{u_i}]_\Delta = p_i^u$.

2. $[M^{\sigma\to\tau}N_\sigma]_\Delta = eval_\sigma \circ \langle [M^{\sigma\to\tau}]_\Delta,\ [N^\tau]_\Delta\rangle$.

3. $[\lambda x^\tau . M^\sigma]_\Delta = \Lambda_{\tau,\sigma}([M]_{\Delta\cdot x^\tau})$, note $\Delta \cdot x^\tau \in \mathtt{Var}^{u\cdot\tau}$.

4. $[\langle M, N\rangle]_\Delta = \langle [M]_\Delta,\ [N]_\Delta\rangle$.

5. $[fst(M^{\sigma\times\tau})]_\Delta = fst^{\sigma,\tau} \circ [M]_\Delta$.

6. $[snd(M^{\sigma\times\tau})]_\Delta = snd^{\sigma,\tau} \circ [M]_\Delta$.

Lemma 14.159 *(The substitution lemma)*

1. If $y^\sigma \notin FV(N)$ then $[N]_{\Delta\cdot y^\sigma} = [N]_\Delta \circ fst^{u,\sigma}$.

2. $[[N/x^\sigma]M]_\Delta = [M]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle$.

Proof

1. We proceed by induction on the structure of M:

   (a) If $M = x_i^{u_i}$ then the result follows from the commuting of the diagram

$$\begin{tikzcd} & A^{u\cdot\sigma} \arrow[dl, "fst^{u,\sigma}"'] \arrow[dr, "{[x_i^{u_i}]_{\Delta\cdot y^\sigma} = p_i^{u\cdot\sigma}}"] & \\ A^u \arrow[rr, "{[x_i^{u_i}]_\Delta = p_i^u}"'] & & A_i \end{tikzcd}$$

   (b) If $M = PQ$ then we have

$$\begin{aligned} [PQ]_{\Delta\cdot y^\sigma} &= eval \circ \langle [P]_{\Delta\cdot y^\sigma}, [Q]_{\Delta\cdot y^\sigma}\rangle \\ &= eval \circ \langle [P]_\Delta \circ fst^{u,\sigma}, [Q]_\Delta \circ fst^{u,\sigma}\rangle \quad \text{by induction} \\ &= eval \circ \langle [P]_\Delta, [Q]_\Delta\rangle \circ fst^{u,\sigma} \\ &= [PQ]_\Delta \circ fst^{u,\sigma} \end{aligned}$$

   (c) If $M = [\lambda x^\tau . P]$ then

$$[\lambda x^\tau . P^\rho]_{\Delta\cdot y^\sigma} = \Lambda_{\tau,\rho}([P]_{\Delta\cdot y^\sigma\cdot x^\tau})$$

   which, taking $\xi : (A^u \times A_\sigma) \times A_\tau \cong (A^u \times T_\tau) \times A_\sigma$ to be the evident isomorphism

$$\begin{aligned} &= \Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau\cdot y^\sigma} \circ \xi) \\ &= \Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau} \circ fst^{u\cdot\tau,\sigma} \circ \xi) \end{aligned}$$

   but $fst^{u\cdot\tau,\sigma} \circ \xi = fst^{u,\sigma} \times 1_\tau$, so the above

$$= \Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau} \circ fst^{u,\sigma\times 1_\tau}) \quad (**)$$

   Now

$$\begin{aligned} & eval_{\tau,\rho} \circ (\Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau} \circ fst^{u,\sigma} \times 1_\tau \\ = \;& [P]_{\Delta\cdot x^\tau} \circ (fst^{u,\sigma} \times 1_\tau & \text{def } \Lambda \text{ and } eval \\ = \;& eval_{\tau,\rho} \circ (\Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau}) \circ (fst^{u,\sigma} \times 1_\tau) \\ = \;& eval_{\tau,\rho} \circ (\Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau}) \circ fst^{u,\sigma} \times 1_\tau) \end{aligned}$$

   so $\Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau} \circ fst^{u,\sigma} \times 1_\tau) = \Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau}) \circ fst^{u,\sigma}$ by universality (the uniqueness condition). Whence

$$(**) = \Lambda_{\tau,\rho}([P]_{\Delta\cdot x^\tau}) \circ fst^{u,\sigma}$$
$$= [\lambda x^\tau.P]_\Delta \circ fst^{u\sigma}.$$

We leave the remaining cases of the proof of the first statement as an exercise for the reader.

2. (Proof of second statement in the substitution lemma.) We again proceed by induction on the structure of M.

   (a) If $M = x^\sigma$ then
$$\begin{array}{ll} & [x^\sigma]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle \\ = & snd^{u,\sigma} \circ \langle 1,\ [N]_\Delta\rangle \\ = & [N]_\Delta \\ = & [[/x^\sigma]x^\sigma]_\Delta \end{array}$$

   (b) If $M = x^{u_i}$, $u_i \neq \sigma$, then
$$\begin{array}{ll} & [x^{u_i}]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle \\ = & p_i^{u\cdot\sigma} \circ \langle 1,\ [N]_\Delta\rangle \\ = & p_i^u \circ fst^{u,\sigma} \circ \langle 1,\ [N]_\Delta\rangle \\ = & p_i^u \\ = & [x^{u_i}]_\Delta \\ = & [[N/x^\sigma]x^{u_i}]_\Delta \quad \text{for } u_i \neq \sigma \end{array}$$

   (c) If $M = QP$ then
$$\begin{array}{ll} & [QP]_{\Delta\cdot x^\sigma} \circ \langle 1_u,\ [N]_\Delta\rangle \\ = & eval \circ \langle [Q]_{\Delta\cdot x^\sigma},\ [P]_{\Delta\cdot x^\sigma}\rangle \circ \langle 1_u,\ [N]_\Delta\rangle \\ = & eval \circ \langle [Q]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle,\ [P]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle\rangle \\ = & eval \circ \langle [[N/x^\sigma]Q]_\Delta,\ [[N/x^\sigma]P]_\Delta\rangle \\ = & [[N/x^\sigma]PQ]_\Delta \end{array}$$

   (d) If $M = \lambda y_\tau.P^\rho$ such that $y^\tau \in FV(N)$ and $y^\tau \neq x^\sigma$ then
$$\begin{array}{lll} & [[N/x^\tau.P]]_\Delta & \\ = & [\lambda y^\tau.[N/x^\sigma]P]_\Delta & \text{def. subst.} \\ = & \Lambda([[N/x^\sigma P)_{\Delta\cdot x^\tau} & \text{def. } [\ ]_\Delta \\ = & \Lambda([P])_{\Delta\cdot y^\tau\cdot x^\sigma} \circ \langle 1,\ [N]_{\Delta\cdot y^\tau}\rangle & \text{induction} \\ = & \Lambda([P]_{\Delta\cdot y^\tau\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta \circ fst\rangle) & \text{by part 1} \\ = & \Lambda([P]_{\Delta\cdot y^\tau\cdot x^\sigma}) \circ \langle 1,\ [N]_\Delta\rangle & \text{by universality, see below} \\ = & [\lambda x^\tau.P]_{\Delta\cdot x^\sigma} \circ \langle 1,\ [N]_\Delta\rangle & \end{array}$$

   The details of the universality argument are as follows:
   First we have
$$\begin{array}{ll} & eval \circ (\Lambda([P]_{\Delta\cdot y^\tau\cdot x^\sigma} \circ \langle 1_{u\cdot\tau},\ [N^\sigma]_\Delta \circ fst^{u,\tau}\rangle \times 1_\tau) \\ = & [P]_{\Delta\cdot y^\tau\cdot x^\sigma} \circ \langle 1_{u\cdot\tau},\ [N^\sigma]_\Delta \circ fst^{u,\tau}\rangle \\ = & [P]_{\Delta\cdot x^\sigma\cdot y^\tau} \circ (\langle 1_u,\ [N^\sigma]_\Delta\rangle \times 1_\tau) \qquad (AA) \end{array}$$

   (The last step is just a reordering of the variables.) Now

$$eval \circ (\Lambda([P]_{\Delta \cdot x^\sigma \cdot y^\tau}) \times 1_\tau) = [P]_{\Delta \cdot x^\sigma \cdot y^\tau}$$

by the definitions of $eval$ and Λ so we have

$$\begin{array}{ll} (AA) = & eval \circ (\Lambda([P]_{\Delta \cdot x^\sigma \cdot y^\tau}) \times 1_\tau) \circ (\langle 1_u,\ [N^\sigma]_\Delta \rangle) \times 1_\tau) \\ = & eval \circ ((\Lambda([P]_{\Delta \cdot x^\sigma \cdot y^\tau}) \circ \langle 1_u,\ [N^\sigma]_\Delta \rangle) \times 1_\tau) \end{array}$$

so, by universality, (the uniqueness condition)

$$\Lambda([P]_{\Delta \cdot y^\tau \cdot x^\sigma} \circ \langle 1_{u \cdot \tau},\ [N^\sigma]_\Delta \circ fst^{u,\tau} \rangle) = \Lambda([P]_{\Delta \cdot x^\sigma \cdot x^\tau}) \circ \langle 1_u,\ [N^\sigma]_\Delta \rangle.$$

We leave the remaining cases to the reader.

□

15 Substitution

The concept of substitution plays a major role in computer science.

In this section we take a first look at substitution. In particular, given an S-sorted signature, Σ, we look at the substitution of $\Sigma(X)$-terms for elements of Y in $\Sigma(Y)$-terms, where X and X are S-sorted sets "of variables". Theorem 4.38 is the primary tool.

Definition 15.160 Let Σ be an S-sorted signature and let X, and Y be S-sorted sets. Then by an *assignment of* $\Sigma(Y)$*-terms to* X we mean an S-sorted mapping $\alpha : X \to U(T_{\Sigma(Y)})$. By Theorem 4.38 there exists a unique homomorphism $\overline{\alpha} : \mathbf{T}_{\Sigma(X)} \to \mathbf{T}_{\Sigma(Y)}$ such that $U(\overline{\alpha}) \bullet \eta_X = \alpha$. For each $\Sigma(X)$-term e, we call $\overline{\alpha}(e)$ the *result of simultaneously, for all* $x \in X$ *substituting* $\alpha(x)$ *for* x *in* e. □

Our claim is that this is a formally defined version of what you would intuitively expect for "simultaneous substitution". Of course we can't prove this since your intuition is not available for formal inspection.

Why is the formal definition worthwhile? One reason is that it allows us to prove certain results once-and-for-all in a manner that is independent of our choice of Σ and the particular syntax that we use to write down the terms. Indeed, all we need is Theorem 4.38

It is advantageous to do this in a slightly more general setting than it is done above, namely rather than considering simultaneous substitution of $\alpha(x)$ for x in an expression e, we consider simultaneous substitution of $\alpha(x)$ for x in an assignment $\beta : Z \to U(\mathbf{T}_{\Sigma(X)})$:

Definition 15.161 Let $\alpha : X \to U(\mathbf{T}_{\Sigma(Y)})$ and $\beta : Z \to U(\mathbf{T}_{\Sigma(X)})$ be assignments, then by the *result of the simultaneous substitution of* $\alpha(x)$ *for* x *in* β, we mean the assignment

$$subst(\alpha, \beta) \ =_{def} \ U(\overline{\alpha}) \bullet \beta : Z \to U(\mathbf{T}_{\Sigma(X)})$$

$$\begin{tikzcd}
Z \arrow[dr, "\beta"] & \\
X \arrow[r, "\eta_X"] \arrow[dr, "\alpha"'] & U(\mathbf{T}_{\Sigma(X)}) \arrow[d, "U(\overline{\alpha})"] \\
 & U(\mathbf{T}_{\Sigma(Y)})
\end{tikzcd}$$

□

Lemma 15.162 *Let* $\alpha : X \to U(\mathbf{T}_{\Sigma(Y)})$ *and* $\beta : Z \to U(\mathbf{T}_{\Sigma(X)})$ *be assignments, then*

$$U(\overline{\alpha}) \bullet U(\overline{\beta}) = U(\overline{U(\overline{\alpha}) \bullet \beta}).$$

Proof We have $U(\alpha \bullet \beta) = U(\overline{\alpha}) \bullet U(\overline{\beta}) \bullet \eta_Z \ = \ U(\overline{\alpha}) \bullet \beta \ = \ U(\overline{U(\overline{\alpha}) \bullet \beta}) \bullet \eta_Z$. From which it follows, by Theorem 4.38 and Fact 4.31, that$U(\overline{\alpha}) \bullet U(\overline{\beta}) = U(\overline{U(\overline{\alpha}) \bullet \beta})$. □

Proposition 15.163 *Substitution is associative. That is, if* $\gamma : W \to U(\mathbf{T}_{\Sigma(X)})$, $\beta : X \to U(\mathbf{T}_{\Sigma(Y)})$, *and* $\alpha : Y \to U(\mathbf{T}_{\Sigma(Z)})$, *then*

$$subst(subst(\alpha,\ \beta),\ \gamma)\ =\ subst(\alpha,\ subst(\beta,\ \gamma)).$$

Proof We have

$$\begin{array}{lll} & subst(subst(\alpha,\beta),\gamma) & \\ = & U(\overline{subst(\alpha,\beta)}) \bullet \gamma & \text{def. } subst \\ = & U(\overline{U(\overline{\alpha}) \bullet \beta}) \bullet \gamma & \text{def. } subst \\ = & U(\overline{\alpha}) \bullet U(\overline{\beta}) \bullet \gamma & \text{lemma 15.162} \\ = & U(\overline{\alpha}) \bullet subst(\beta,\gamma) & \text{def. } subst \\ = & subst(\alpha, subst(\beta,\gamma)) & \text{def. } subst \end{array}$$

□

Proposition 15.164 *Let* $\alpha : X \to \mathbf{T}_{\Sigma(Y)}$ *be an assignments, then* η_X *and* η_Y *are also assignments and*

$$subst(\alpha, \eta_X) = \alpha = subst(\eta_Y, \alpha).$$

This says then that given an S-sorted signature Σ and a set V of S-sorted sets we can form a category of assignments, $\mathbf{Asn}_V$, with V as its set of objects, and for $X, Y \in V$, all triples $\langle X,\ \alpha,\ Y\rangle$ where $\alpha : X \to \mathbf{T}_{\Sigma(Y)}$ is an assignment, as the set of morphisms $\mathbf{Asn}_V(X,\ Y)$, with composition operation $\circ$, where, for morphisms $f = \langle X,\ \alpha,\ Y\rangle$ and $g = \langle Y,\ \beta,\ Z\rangle$, we take $g \circ f = \langle X,\ subst(\beta, \alpha),\ Z\rangle$, and where, for each $X \in V$, η_X is the identity morphism for X.

There are some special categories that we can construct in this manner which then lead rather naturally to the concept of an S-sorted Lawvere algebraic theory.

Given a set S, and S-sorted signature Σ, let

$$X = \langle x_i^s \mid i \in [\omega],\ s \in S\rangle.$$

Given $w \in S*$ define $X_w = \{x_1^{w_1}, x_2^{w_2}, \ldots, x_{|w|}^{w_{|w|}}\}$.

A *typed term* of rank $\langle w, s\rangle \in S^* \times S$ is a triple $\langle w, t, s\rangle$ where $w \in S^*$, $s \in S$, and $t \in T_\Sigma(X_w)_s$.

Given $w, u \in S^*$ and $t_i : w \to u_i$, $i = 1, \ldots, |u|$, we regard the tuple $\langle t_1, \ldots, t_{|u|}\rangle$ as a term of rank $\langle w, u\rangle$ and frequently write $\langle t_1, \ldots, t_{|u|}\rangle : w \to u$.

We identity $t : w \to s$ and $\langle t\rangle : w \to s$

Definition 15.165 Given S, Σ and X as above define the *(Lawvere Algebraic) theory*, $\mathbf{T}_\Sigma$ generated by Σ to be the $(S^* \times S)$-sorted algebra with carriers

$$\mathbf{T}_\Sigma(u, w) = \textit{all terms of rank } \langle w, u\rangle.$$

Together with operations

$$\circ_{u,v,w} : \mathbf{T}_\Sigma(v, w) \times \mathbf{T}_\Sigma(u, v) \to \mathbf{T}_\Sigma(u, w)$$

$$\langle\ , \ldots,\ \rangle : \mathbf{T}_\Sigma(w, u_1) \times \cdots \times \mathbf{T}_\Sigma(w, u_{|u|}) \to \mathbf{T}_\Sigma(u, w)$$

$$x_i^w \in \mathbf{T}_\Sigma(w, w_i)$$

.

Identify $t : w \to u$ with the S-sorted mapping

$$\begin{aligned}\bar{t} : X_u &\to T_\Sigma(X_w)\\ x_i^{u_i} &\mapsto t_i\end{aligned}$$

then for every $t : u \to v$, $t' : v \to w$

$$(\overline{t' \circ_{u,w,v} t})_i = subst(\bar{t}'_i, \bar{t})$$

$$\langle\ ,\ldots,\ \rangle \text{ is just tupling.}$$

$$x_i^w = \langle w,\ x^{w_i},\ w_i\rangle.$$

$\square$

Proposition 15.166 $\mathbf{T}_\Sigma$ *is a category with objects* S^**, morphisms* $\langle w,\ t,\ u\rangle$ *and composition in* $\circ$*. Furthermore,* $\mathbf{T}_\Sigma$ *has finite products:* $w \in S^*$ *is the product of* $w_1, \ldots, w_{|w|}$ *with projections* $x_i^w : w \to w_i$*. Note that the empty tuple is a morphism* $\langle\ \rangle : w \to \lambda$*.*

We can generalize the above into an equationally defined class of $S^* \times S$-sorted algebras with the indicated operations and obeying the following axioms:

$x_i^w \circ \langle \alpha_1, \ldots, \alpha_{|w|}\rangle = \alpha_i$

$\langle x_1^w \circ \beta, \ldots, x_{|w|}^w \circ \beta\rangle = \beta$

$\langle x\rangle = x$

$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$

$1_v \circ \alpha = \alpha = \alpha \circ 1_u$ for all $\alpha : u \to v$ where $1_w = \langle x_1^w, \ldots, x_{|w|}^w\rangle$.

The corresponding homomorphisms give us a category, $\mathbf{Th}_S$, of S-sorted algebraic theories.

16 T-algebras

In this section we introduce yet another approach to algebras. This approach is considerably more general than those discussed earlier.

We start with an example. Let Σ be a 1-sorted signature. Let T : **Set** $\rightarrow$ **Set** be the functor that takes each set A to the set,$T(A)$ where $T_\Sigma(A)$. the smallest set such that

$A \subseteq T(A)$.

if $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$ then (the string) $\sigma(a_1, \ldots, a_n) \in T(A)$.

Furthermore, if $f : A \rightarrow B$ in set, then

$$\begin{gathered} T(f) : T(A) \rightarrow T(B) \\ \sigma(a_1, \ldots, a_n) \mapsto \sigma(f(a_1), \ldots, f(a_n)). \end{gathered}$$

Then each Σ-algebra $\mathcal{A}$ with carrier A determines a mapping

$$\begin{gathered} F_{\mathcal{A}} : T(A) \rightarrow A \\ a \in A \mapsto a \\ \sigma(a_1, \ldots, a_n) \mapsto \sigma_{\mathcal{A}}(a_1, \ldots, a_n). \end{gathered}$$

Furthermore, given Σ-algebras $\mathcal{A}$ and $\mathcal{B}$ with respective carriers A and B, each homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$, viewed as a mapping $h : A \rightarrow B$, is such that the diagram

$$\begin{tikzcd} T(A) \arrow[r, "T(h)"] \arrow[d, "f_{\mathcal{A}}"'] & T(B) \arrow[d, "f_{\mathcal{B}}"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

commutes, and conversely, if $h : A \rightarrow B$ makes the above diagram commute then h is a homomorphism.

The above treatment of Σ-algebras is easily generalized as follows:

Definition 16.167 Let **C** be a category and let $T : \mathbf{C} \rightarrow \mathbf{C}$ be a functor (such a functor, with the same category as its source and target, is frequently called an *endofunctor*). Then a *T-algebra* is a pair $\langle A, \alpha \rangle$ where $A \in |\mathbf{C}|$ and $\alpha \in \mathbf{C}(T(A), A)$. The T-algebras in **C** form a category, $\mathbf{Talg_C}$ where a morphism $h : \langle A, \alpha \rangle \rightarrow \langle B, \beta \rangle$ is a morphism $h \in \mathbf{C}(A, B)$ such that the diagram

$$\begin{tikzcd}
T(A) \arrow[r, "T(h)"] \arrow[d, "\alpha"'] & T(B) \arrow[d, "\beta"] \\
A \arrow[r, "h"] & B
\end{tikzcd}$$

commutes. $\square$

17 Monads

Recall that a monoid, $M = \langle e, \cdot \rangle$, is a 1-sorted algebra with an associative binary operation, $\cdot$, and a distinguished constant, e which is a left and right identity for the operation $\cdot$. That is

$$\begin{array}{ll} (x \cdot y) \cdot z = x \cdot (y \cdot z) & \text{associativity} \\ e \cdot x = x & \text{left identity} x \cdot e = x \quad \text{right identity} \end{array}$$

We can view a given monoid $\langle M, e, \cdot \rangle$ as a object, M, of **Set** (i.e., as a set) equipped with two morphism $e : \mathbf{1} \to M$ and $\cdot : M \times M \to M$, the axioms for monoids then correspond the commutativity of the diagrams

$$\begin{tikzcd} M^3 \arrow[r, "1_M \times \cdot"] \arrow[d, "\cdot \times 1_M"'] & M^2 \arrow[d, "\cdot"] \\ M^2 \arrow[r, "\cdot"] & M \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{1} \times M \arrow[r, "1_M \times e"] \arrow[dr, "\cong"'] & M^2 \arrow[d] & M \times \mathbf{1} \arrow[l, "e \times 1_M"'] \arrow[dl, "\cong"] \\ & M & \end{tikzcd}$$

We now ask, how can we generalize the above ideas to an arbitrary category? One very useful generalization is the concept of a monad. Here the set M is replaced by an endofunctor $T : \mathbf{C} \to \mathbf{C}$ on some category $\mathbf{C}$ but otherwise the definition is very close to the above. Here are the details:

Definition 17.168 Let $\mathbf{C}$ be a category, then a *monad over* $\mathbf{C}$ is given by the following data

$T : \mathbf{C} \to \mathbf{C}$, an endofunctor. Define $T^0 = 1_{\mathbf{C}}$, $T^1 = T$, and $T^{n+1} = T \circ T^n$.

$\mu : T^2 \to T$, a natural transformation.

$\eta : 1_{\mathbf{C}} \to T$, a natural transformation.

satisfying the following conditions (i.e., such that the following diagrams commute):

$$
\begin{tikzcd}
T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1_{\mathbf{C}} \circ T \; = \; T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & T^2 \arrow[d] & T \; = \; T \circ 1_{\mathbf{C}} \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}
$$

□

Example 17.169 (From [1], page 72) Let $M = \langle M, e, \cdot \rangle$ be a monoid take

$$
\begin{gathered}
T : \mathbf{Set} \to \mathbf{Set} \\
A \mapsto M \times A \\
f \mapsto 1_M \times f.
\end{gathered}
$$

$$
\begin{gathered}
\eta_A : A \to M \times A \\
a \mapsto \langle e, a \rangle
\end{gathered}
$$

$$
\begin{gathered}
\mu_A : M \times (M \times A) \to M \times A \\
\langle m, \langle n, a \rangle \rangle \mapsto \langle m \cdot n, a \rangle.
\end{gathered}
$$

□

Example 17.170 (From [1], page 72) Let

$$
\begin{gathered}
T : \mathbf{Set} \to \mathbf{C} \\
A \mapsto \mathcal{P}(A) \\
f : A \to B \mapsto T(f) : \mathcal{P}(A) \to \mathcal{P}(B)
\end{gathered}
$$

where, for any set $X \in \mathcal{P}(A)$, $T(f)(X) = \{f(a) \mid a \in X\}$.

$$
\begin{gathered}
\eta_A : A \to \mathcal{P}(A) \\
\eta_A(a) = \{a\}.
\end{gathered}
$$

$$
\begin{gathered}
\mu_A : \mathcal{P}(\mathcal{P}(A)) \\
\mu_A(B) = \bigcup B.
\end{gathered}
$$

$\square$

Example 17.171 Let **Types** be a subcategory of **Set**, perhaps a category of finite sets. Let

$$\begin{gathered} List : \mathbf{Types} \to \mathbf{Types} \\ X \mapsto X^* \\ f : X \to Y \mapsto f^* : X^* \to Y^* \end{gathered}$$

where, for any list (string) $u = x_1 \cdots x_n \in X^*$, $f^*(u) = f(x_1) \cdots f(x_n) \in Y^*$.

$$\begin{gathered} \eta : 1_{\mathbf{Types}} \to List \\ \eta_X : X \to X^* \\ x \mapsto x \end{gathered}$$

i.e., the element $x \in X$ goes to the string $x : [1] \to X$ such that $x(1) = x$.

$$\begin{gathered} \mu : List \circ List \to List \\ \mu_X : (X^*)^* \to X^* \\ u = u_1 \cdots u_n \mapsto u_1 + u_2 + \cdots + u_n \\ f_1, \cdots f_n \mapsto f_1 + f_{f2} + \cdots + f_n \end{gathered}$$

Recall that the coproduct operation, $+$, on strings is concatenation.

The assertion that η and μ are natural transformations is just the assertion that the following two diagrams commute for each choice of X, Y and $g : X \to Y$ in **Type**.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "g"'] & X^* \arrow[d, "g^*"] \\ Y \arrow[r, "\eta_Y"] & Y^* \end{tikzcd}$$

$$\begin{tikzcd} (X^*)^* \arrow[r, "\mu_X"] \arrow[d, "g^{**}"'] & X^* \arrow[d, "g^*"] \\ (Y^*)^* \arrow[r, "\mu_Y"] & Y^* \end{tikzcd}$$

The assertion that we have a monad is, first of all, an assertion that the diagram

$$\begin{tikzcd}
((X^*)^*)^* \arrow[r, "\mu_{X^*}"] \arrow[d, "(\mu_X)^*"'] & (X^*)^* \arrow[d, "\mu_X"] \\
(X^*)^* \arrow[r, "\mu_X"] & X^*
\end{tikzcd}$$

commutes. Just what this means is perhaps best seen by looking at a specific example. So say that $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and let $u = [[[1\ 2\ 3]\ [4\ 5\ 6]]\ [[7\ 8\ 9]\ [10\ 11\ 12]]] \in ((X^*)^*)^*$ then

$$(\mu_X)^*(u) = [[1\ 2\ 3\ 4\ 5\ 6]\ [7\ 8\ 9\ 10\ 11\ 12]]$$

$$\mu_{X^*}(u) = [[1\ 2\ 3]\ [4\ 5\ 6]\ [7\ 8\ 9]\ [10\ 11\ 12]]$$

and

$$\mu_X(\mu_X)^*(u)) = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12] = \mu_X(\mu_{X^*}(u)).$$

The second part of the definition of monad is then the assertion that the diagrams

$$\begin{tikzcd}
X^* \arrow[r, "\eta_{X^*}"] \arrow[dr, "1_{X^*}"'] & (X^*)^* \arrow[d, "\mu_X"] \\
 & X^*
\end{tikzcd}$$

and

$$\begin{tikzcd}
X^* \arrow[r, "(\eta_X)^*"] \arrow[dr, "1_{X^*}"'] & (X^*)^* \arrow[d, "\mu_X"] \\
 & X^*
\end{tikzcd}$$

commute. To see what this means note that given the string $[a\ b\ c]$ that

$$\eta_{X^*}([a\ b\ c]) = [\ [a\ b\ c]\ ]$$

$$(\eta_X)^*([a\ b\ c]) = [\ [a]\ [b]\ [c]\ ]$$

$$\mu_X(\eta_{X^*}([a\ b\ c])) = [a\ b\ c] = \mu_X((\eta_X)^*([a\ b\ c])).$$

$\square$

18 Indexed Categories

The frequently used construct of an I-indexed sets of sets (also called an I-indexed family of sets), $S = \langle S_i \mid i \in I \rangle$, can be recast categorically as a functor $S : I \to \mathbf{Set}$ where the index set I is viewed as a discrete category. Given two I-indexed sets $S = \langle S_i \mid i \in I \rangle$ and $S' = \langle S'_i \mid i \in I \rangle$ a useful concept of a morphism $f : S \to S'$ is that of an I-indexed set of **Set**-morphism $f = \langle f_i : S_i \to S'_i \mid i \in I \rangle$. This can be recast categorically as a natural transformation from the functor S to the functor S'.

Having recast the notion of an indexed family of sets in the categorical framework we now ask, can we generalize it in a useful manner. The answer, not surprisingly, is yes, and we do it by replacing the category **Set** by the category **Cat** of all (small) categories, and/or by replacing the set I by an arbitrary category $\mathbf{C}$, and then defining a $\mathbf{C}$-indexed category as a functor $F : C^{op} \to \mathbf{Cat}$. The reason for taking $\mathbf{C}^{op}$ rather than $\mathbf{C}$ should become apparent from the examples.

Definition 18.172 Let **Cat** denote the category of (small) categories. Let $\mathbf{C}$ be a category. Then a $\mathbf{C}$-*indexed category* is a functor $F : \mathbf{C}^{op} \to \mathbf{Cat}$.

Given $f \in \mathbf{C}$ we will sometimes write f^* for Ff.

Let F and G be two $\mathbf{C}$-indexed categories, then a $\mathbf{C}$-*indexed functor* is a natural transformation $H : F \to G$.

Let $G : A \to B$ and $H : A \to B$ be two $\mathbf{C}$-indexed functors, then a $\mathbf{C}$-*indexed natural transformation*, τ, from G to H consists of a natural transformation $\tau(X) : G(X) \to H(X)$ for each object $X \in |\mathbf{C}|$ with the property that for every $f : X \to X'$ in $\mathbf{C}$

$$\tau(X) \circ G(f) = H(f) \circ \tau(X')$$

or, in a more overloaded notation, $\tau(X) \circ f^* = f^* \circ \tau(X')$. The composition used here is the horizontal composition of natural transformations. □

Example 18.173 Let K be the set of types for, say, some programming language. At any instant in the execution of a program in the language there is an execution state consisting of the current set of typed variables, the current set of possible values for such variables and an assignment of a value to each variable. Viewing K as a discrete category, we can describe the typed sets of variables and values and indexed categories $var : K \to \mathbf{Set}$ and $val : K \to Set$ and view the assignment of values to variables as an indexed functor $state : var \to val$.

We can go further and define a category of states, however the indexed natural transformations do not seem to be the desired morphisms. □

18.1 Grothendieck Constructions

Definition 18.174 Let $\mathbf{C}$ be a category and **Cat** be the category of all small categories. Let $F : \mathbf{C}^{op} \to \mathbf{Cat}$ be any functor. We can form the category $\mathbf{Gr}(F)$ termed the *(Grothendieck) cofibration of* F as follows. The class of objects of $\mathbf{Gr}(F)$ is the coproduct (disjoint sum) of the classes $|F(C)|$ for all $C \in \mathbf{C}$, i.e.

$$|\mathbf{Gr}(F)| = \coprod_{C \in \mathbf{C}} |F(C)| = \{\langle C, a \rangle \mid C \in |\mathbf{C}| \text{ and } a \in F(C)\}.$$

The morphisms of $\mathbf{Gr}(F)$ are all pairs

$$\langle f, g\rangle : \langle C, a\rangle \to \langle C', a'\rangle$$

such that $f : C \to C'$ is a morphism in $\mathbf{C}$ and $g : a \to (F(f))(a')$ is a morphism in $F(C)$. Note that $F(C)$ is a category and $F(f)$ is a functor in $\mathbf{Cat}$ so all the above is well-defined.

Composition of morphisms is defined as follows. Given

$$\langle f_0, g_0\rangle : \langle C_0, a_0\rangle \to \langle C_1, a_1\rangle, \quad \langle f_1, g_1\rangle : \langle C_1, a_1\rangle \to \langle C_2, a_2\rangle$$

define the composition

$$\langle f_2, g_2\rangle = \langle f_1, g_1\rangle\langle f_0, g_0\rangle : \langle C_0, a_0\rangle \to \langle C_2, a_2\rangle$$

where

$$f_2 = f_1 \circ f_2 : C_0 \to C_2,$$

and

$$g_2 = ((F(f_0))(g_1)) \circ g_0 : a_0 \to (F f_2)a_2 \text{ in } F(C_0)$$

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[rr, bend right, "g_2"'] & (F(f_0))(a_1) \arrow[r, "(F(f_0))(g_1)"] & (F(f_0))(((F(f_1))(a_2))
\end{tikzcd}$$

□

Proposition 18.175 *Let C be any category, $\mathbf{Cat}$ be the category of all small categories and $F : C^{op} \to \mathbf{Cat}$ be any functor. If C has an initial object and for each object $c \in |C|$ the category $F(c)$ has an initial object then the cofibration $\mathbf{Gr}(F)$ has an initial object.*

Proof It is easy to check that if $I \in |C|$ is initial in C and $i \in |F(I)|$ is initial in $F(I)$ then the pair $(I, i) \in |\mathbf{Gr}(F)|$ is initial in $\mathbf{Gr}(F)$. □

Example 18.176 The construction, from a set S, of the category $\mathbf{Sig}_S$ of all S-sorted signatures, can be represented as a functor $sig : \mathbf{Set}^{op} \to \mathbf{Cat}$. Let sig be such that it takes each set S to the corresponding category $\mathbf{Sig}_S$, and takes each mapping $f : S \to S'$ to the functor $\overline{f} : \mathbf{Sig}_{S'} \to \mathbf{Sig}_S$ (note the reversal) such that, for each $\Sigma' \in Obj(\mathbf{Sig}_{S'})$, $\overline{f}(\Sigma)$ is the S-sorted signature with $(\overline{f}(\Sigma))_{w,s} = \Sigma'_{f^*(w),f(s)}$, and for any morphism $g : \Sigma' \to \Sigma''$, $\overline{f}(g) : \overline{f}(\Sigma') \to \overline{f}(\Sigma'')$ such that $\overline{f}(g)_{w,s} = g_{f^*(w),f(s)}$. If we now perform the Grothendieck construction on sig the resulting category will be the category $\mathbf{Sig}$ defined as follows:

Definition 18.177 The objects of $\mathbf{Sig}$ are the pairs (S, Σ) with $\Sigma \in sig(S)$. A *signature morphism* $\theta : (S, \Sigma) \to (S', \Sigma')$ is a pair $\theta = (\theta_S, \theta_\Sigma)$ where θ_S is any mapping and $\theta_\Sigma : \Sigma \to (sig(\theta_S))(\Sigma')$ in $sig(S) = \mathbf{Sig}_S$. Composition is as given by the Grothendieck construction. □

□

Example 18.178 We now want to do something similar to produce a category, $\mathbf{Alg}$, of all algebras. The necessary functor is given by the following well-known result:

Fact 18.179 *Let* $\theta : (S,\ \Sigma) \to (S',\ \Sigma')$ *in* **Sig**, *then* θ *induces* an algebraic functor $\theta^\flat$: $Alg_{S',\Sigma'} \to Alg_{S,\Sigma}$ *such that for each* $A \in Alg_{S',\Sigma'}$,

$$(\theta^\flat(A))_s = A_{\theta_S(s)}$$

and for each $\sigma \in \Sigma_{w,s}$,

$$\sigma_{\theta^\flat(A)} = (\theta_{w,s}(\sigma))_A.$$

And, for any homomorphism, $h : A \to B$ *in* $Alg_{S',\Sigma'}$,

$$(\theta^\flat(h))_s = h_{\theta_S(s)}.$$

This gives us a functor

$$\begin{gathered}\flat : \mathbf{Sig}^{op} \to Cat \\ (S,\Sigma) \mapsto (S,\Sigma)^\flat = Alg_{S,\Sigma} \\ (\theta : (S,\Sigma) \to (S',\Sigma')) \mapsto \theta^\flat\end{gathered}$$

We can now use the Grothendieck construction to form a category, **ALG**, of all algebras. The objects of **ALG** are pairs $((S,\Sigma), A \in (S,\Sigma)^\flat)$, and a morphism, from $((S',\Sigma'),A)$ to $((S,\Sigma'),B)$ is a pair $(\theta : (S,\Sigma) \to (S',\Sigma'),\ h : A \to \theta^\flat(B))$ where θ is a signature morphism, and h is a homomorphism in $Alg_{S,\Sigma}$. □

While we defined the Grothendieck construction starting from a functor $F : \mathbf{C} \to \mathbf{Cat}$ there is an important special case for functors $F : \mathbf{C} \to \mathbf{Set}$ ($\subseteq \mathbf{Cat}$). The definition procedes as in the general case:

> An object in $\mathbf{Gr}(F)$ is a pair $\langle X, a\rangle$ with $X \in |\mathbf{C}|$ and with $a \in F(x)$.
>
> A morphism $\langle f, g\rangle : \langle X, a\rangle \to \langle Y, b\rangle$ can be defined as before, but, since $F(X)$ is a set (or, equivalently, a discrete categeory) the only choice for $g : a \to F(f)(b)$ is the identity morphism 1_a, that is, the definition reduces to the requirement that $F(f)(a) = b$.

There is also a dual construction. We start from a functor $F : \mathbf{C} \to \mathbf{Cat}$ (note: no ^{op}). An object in $\mathcal{G}(F)$ is again a pair $\langle X, a\rangle$ with $X \in |\mathbf{C}|$ and $a \in F(X)$, and a morphism $\langle f, g\rangle : \langle X, a\rangle \to \langle Y, b\rangle$ consists of a morphism $f : X \to Y$ in $\mathbf{C}$ together with a morphism $g : F(f)(a) \to b$ in $F(Y)$. We leave it to the reader to define the composition operation.

References

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