The Star System Problem revived

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Abstract. In the Star System problem we are given a set system and asked if it corresponds to the multi-set of closed neighborhoods of some graph, i.e. given subsets $S_1, S_2, ..., S_n$ of an n-element set V does there exist a graph $G = (V, E)$ with $\{N[v] : v \in V\} = \{S_1, S_2, ..., S_n\}$. We study the computational complexity of this problem for H-free graphs. For example, we show that if the graph G is required to be C_k-free, i.e. not allowed to have an induced cycle of length k, then the problem is solvable in polynomial time for each $k \leq 4$, and is NP-complete for each $k \geq 5$.

1 History

The closed neighborhood of a vertex in a graph is sometimes called the "star" of the vertex. The "star system" of a graph is then the multi-set of closed neighborhoods of all the vertices of the graph. Note that the star system of a graph does not necessarily define the graph uniquely, for example $2K_k$, i.e. two disjoint copies of K_k, and $K_{k,k}$ minus a matching have the same star systems. Given a collection of sets, how hard is it to decide if this is the star system of some graph? An equivalent formulation is to decide if there is a permutation of the rows of a given 0-1 matrix that results in a symmetric matrix with all diagonal entries equal to 1.

This complexity question was first posed by Gert Sabidussi and Vera Sós in the mid-70s [5]. The first result by Babai [1] showed that this so-called Star System problem was at least as hard as the graph isomorphism problem and equivalent to deciding if a bipartite graph had an automorphism working along a matching, i.e. given a bipartite graph to decide if it has an automorphism of order 2 such that each vertex is adjacent to its image. In a related effort Lubiw [4] showed that deciding if an arbitrary graph had an automorphism of order 2 was NP-complete. Babai writes that between Lubiw's problem and the Star System problem he "did not believe there was a deeper relationship" [2]. It therefore came as

* * * Supported by Czech research grant 1M0545

a small surprise when Lalonde [3] showed that the Star System problem was NP-complete by a reduction from Lubiw's problem. Since that result was published in a 1981 Discrete Mathematics article, the Star System problem has lain dormant for over 25 years. In this paper we revive it. In particular, we will investigate, for a fixed graph H, what the complexity of the star system problem is if we ask wether the given collection of sets corresponds to the star system of a graph that does not contain H as an induced subgraph.

The paper is organized as follows. In the next section we give some preliminaries before continuing with polynomial-time solvable cases and NP-complete cases in two separate sections. We end by describing the consequences of our results for the graph automorphism problems studied by Babai, Lubiw and others.

2 Preliminaries

We use standard graph notation with $G = (V, E)$ being a simple loopless undirected graph with vertex set V and edge set E. We denote by $N[v]$ and $N(v)$ the closed and open neighborhoods of a vertex v, respectively, and by $\bar{G}$ the complement of a graph G having an edge uv iff $u \neq v$ and $uv \notin E(G)$. We also call $N[v]$ the star of v and say that v is the center of $N[v]$. An *automorphism* of a graph $G = (V, E)$ is an isomorphism $f : V \to V$ of the graph to itself, and it has order 2 if for every vertex x we have $f(f(x)) = x$, i.e. the image of its image is itself. For a graph $G = (V, E)$ we define the $|V|$-element multi-set $Stars(G) = \{N[v] : v \in V\}$. For a fixed graph H we say that a graph G is H-free if G does not contain an induced subgraph isomorphic to H.

The problem we will be investigating in this paper is the following:

H-free Star System Problem
Input: A set system $\mathcal{S}$ over a universe V
Question: Does there exist an H-free graph $G = (V, E)$ with $Stars(G) = \mathcal{S}$?

We will mainly be concerned with deciding for various fixed graphs H if the H-free Star System Problem is solvable in polynomial-time or if it is NP-complete. Note that the problem is in NP, since as a certificate we can use the graph G having $Stars(G) = \mathcal{S}$ and the 1-1 correspondence between the closed neighborhoods of G and the sets of the input set system S.

Let us also mention a related problem. Define the multiset $OpenStars(G) = \{N(v) : v \in V\}$. Note that a graph $G = (V, E)$ is H-free iff $\bar{G}$ is $\bar{H}$-free, and moreover if $Stars(G) = S_1, S_2, ..., S_n$ then $OpenStars(\bar{G}) = V \setminus S_1, V \setminus S_2, ..., V \setminus S_n$. The problem defined below is thus related to our problem by the following observation.

H-free Open-Star System Problem
Input: A set system $\mathcal{S}$ over a universe V
Question: Does there exist an H-free graph $G = (V, E)$ with $OpenStars = \mathcal{S}$?

Observation 1 *The H-free Star System Problem is polynomial-time solvable if and only if the $\bar{H}$-free Open-Star System Problem is polynomial-time solvable.*

3 Polynomial-time solvable cases

In this section we show that C_3-free and C_4-free Star System Problems are solvable in polynomial time.

Theorem 1. *The C_3-free Star System Problem is solvable in polynomial time.*

Proof. Let $\mathcal{S}$ be a set system on a ground set V. The crucial observation is that if $\mathcal{S}$ is a star system of a C_3-free graph $G = (V, E)$, then for every edge $uv \in E$ there are exactly two sets containing u and v. In fact, since $uv \in E$, we have that u and v should be in at least two stars, one of which is centered in u and one centered in v. Let S_u and S_v be these stars. If there is a third star S containing u and v, then the center of this star, $x \neq u, v$ is adjacent to u and v, and thus xuv forms a C_3 in G, which is a contradiction.

Let us assume that the system $\mathcal{S}$ is connected, i.e. for every two elements u and v there is a sequence of elements $u = u_1, u_2, \ldots, u_k = v$ such that for every $i \in \{1, \ldots, k-1\}$ there is a set $S \in \mathcal{S}$ containing u_i and u_{i+1}. (If $\mathcal{S}$ is not connected, then we apply our arguments for each connected component of $\mathcal{S}$.)

Assume that we have correctly guessed the star $S_v \in \mathcal{S}$ of a vertex v in some C_3-free graph G with $Stars(G) = \mathcal{S}$. Then each $x \in S_v$, $x \neq v$, is adjacent to v in G. Thus there is a unique star $S_x \neq S_v$ containing both v and x, and vertex x should be the center of S_x. Now every vertex y from

S_x should have a unique star containing x and y, and so on. Since $\mathcal{S}$ is connected, we thus have that after guessing the star for the first vertex v we can uniquely assign stars to the remaining vertices. There are at most n guesses to be made for the first vertex and we can in polynomial time check the correctness of the guess, i.e. check if the star system of the constructed graph corresponds to $\mathcal{S}$, to prove the theorem. □

Theorem 2. *The C_4-free Star System Problem is solvable in polynomial time.*

Proof. The proof is based on the following observation. Let $G = (V, E)$ be a C_4-free graph and let $a, b \in V$. Let $S_1, S_2, \ldots, S_t$ be the set of stars of G containing both a and b. Then

$$|\bigcap_{i=1}^{t} S_i| \leq t \text{ if } ab \in E \tag{1}$$

$$|\bigcap_{i=1}^{t} S_i| \geq t+2 \text{ if } ab \notin E \tag{2}$$

In fact, if $ab \in E$, then a and b have $t-2$ common neighbors. Every vertex $v \in \cap_{i=1}^{t} S_i \setminus \{a, b\}$ is adjacent to a and b, thus v is the center of the star S_i for some $i \in \{1, \ldots, t\}$ and (1) follows.

If $ab \notin E$, then a and b have t neighbors in common. Moreover, because G is C_4-free, these neighbors form a clique in G. Thus $\cap_{i=1}^{t} S_i$ contains all these t vertices plus the vertices a and b which yields (2).

Given a set system $\mathcal{S}$ on ground set V, the algorithm checking if $\mathcal{S}$ is a star system of some C_4-free graph is simple. First, if there is a pair $a, b \in V$ with $S_1, ..., S_t$ being the set of stars containing both vertices and

$$|\bigcap_{i=1}^{t} S_i| = t+1$$

we immediately conclude that the answer is no.

If there is no such pair, then we construct a graph $G = (V, E)$ with $ab \in E$ if and only if the sets of $\mathcal{S}$ containing both a and b satisfy (1). Finally, we check if $Stars(G) = \mathcal{S}$. If this is the case then the answer is yes, otherwise the answer is no.

□

4 NP-complete cases

In this section we show that the H-free Star System Problem is NP-complete for an infinite class of graphs H, and in particular for all cycles of at least 5 vertices. To state our result we need the following definition.

Definition 1. *For an arbitrary graph H we define $B(H)$ to be its bipartite neighborhood graph, i.e. a bipartite graph with both color classes having $|V(H)|$ vertices labelled by $V(H)$ and having an edge between a vertex labelled u in one color class and a vertex labelled v in the other color class iff $uv \in E(H)$.*

For example, for the cycle on 5 vertices C_5, we have $\bar{C_5} = C_5$ and $B(\bar{C_5}) = C_{10}$. Our main NP-completeness result is that the H-free Star System problem is NP-complete whenever $B(\bar{H})$ has a cycle or two vertices of degree larger than two in the same connected component. For a bipartite graph $G = (V, E)$ with color classes V_1, V_2 we say that an automorphism $f : V \to V$ is side-switching if $f(V_1) = V_2$ and $f(V_2) = V_1$. Consider the following three problems.

AUT-BIP-2SS
Input: A bipartite graph G
Question: Does G have an automorphism of order 2 that is side-switching?

AUT-BIP-2SS-NA
Input: A bipartite graph G
Question: Does G have an automorphism of order 2 that is side-switching where any vertex and its image are non-adjacent?

AUT-BIP-2-A
Input: A bipartite graph G
Question: Does G have an automorphism of order 2 where any vertex and its image are adjacent?

Lalonde [3] has shown that the AUT-BIP-2SS problem is NP-complete. Together with Sabidussi he also reduced AUT-BIP-2SS to AUT-BIP-2SS-NA. The proof of our main NP-completeness result has two parts. One being a modification of the reduction of Lalonde-Sabidussi, which will ensure that AUT-BIP-2SS-NA remains NP-complete for various restricted classes of bipartite graphs, and the other being Theorem 3 below.

Theorem 3. *If AUT-BIP-2SS-NA is NP-complete on bipartite $B(\bar{H})$-free graphs, then the H-free Star System Problem is NP-complete.*

Proof. We reduce the first problem, which takes as input a bipartite $B(\bar{H})$-free graph F, to the second, which takes as input a set system S. We may assume the two partition sides of F are of equal size, since otherwise an automorphism switching the two sides cannot exist. Let the vertices of one color class of F be $\{v_1, v_2, ..., v_n\}$ and of the other $\{w_1, w_2, ..., w_n\}$. The set system we construct will be $S = \{S_1, S_2, ..., S_n\}$ where $S_i = \{w_j : v_i w_j \notin E(F)\}$, i.e. the non-neighbors of v_i on the other side.

As already noted by Babai-1980 it is not hard to see that F is a Yes-instance of AUT-BIP-2SS-NA iff there exists a graph G with $Stars(G) = S$. Let us give the argument. The equivalence of those two problems is most naturally proved by noting that they are both equivalent to the question if the bipartite complement C_F of F, with $V(C_F) = V(F)$ and $E(C_F) = \{v_i w_j : v_i w_j \notin E(F)\}$, has an automorphism of order 2 such that every vertex *is adjacent* to its image, and thus also side-switching.

It remains to show that if $Stars(G) = S$ then G must be H-free. First note that if $Stars(G) = S$ then its bipartite closed neighborhood graph $C(G)$, which is constructed by adding to its bipartite neighborhood graph $B(G)$ all $|V(H)|$ edges between pairs of vertices having the same label, is isomorphic to C_F. We therefore have that $B(\bar{G}) = F$, in other words the bipartite neighborhood graph of the complement of G is isomorphic to F. Moreover, if H is an induced subgraph of G then clearly $B(\bar{H})$ is an induced subgraph of $B(\bar{G}) = F$ and thus since F is $B(\bar{H})$-free we must have G being H-free. □

Definition 2. *Let D_p be the class of bipartite graphs of girth larger than p where any two vertices of degree three or more have distance at least p.*

Theorem 4. *For any integer p the problem AUT-BIP-2SS-NA is NP-complete even when restricted to graphs in D_p.*

Proof. We reduce from the NP-complete AUT-BIP-2SS problem and adapt the construction given by Lalonde and Sabidussi [3] for our purposes.

Given a bipartite graph $G = (V, E)$ with color classes A and B we describe how to construct $H \in D_p$ with the property that G is a yes-instance of AUT-BIP-2SS iff H is a yes-instance of AUT-BIP-2SS-NA. Note firstly that we can assume G has no vertex v of degree 1 since if we remove each such v (simultaneously) and add a cycle of length $2k$, where

k is greater than the maximum cycle length in G, attached to the unique neighbor of v, then G has a side-switching automorphism of order 2 if and only if the new graph has one.

Let p' be the smallest even integer at least as large as p. Let H be the graph obtained by replacing each edge of G by two paths of length $p'+1$. Note that the inner vertices of these paths are then the only vertices of degree 2 in H. Moreover, we have $H \in D_p$ and the two color classes of H respect A and B.

If $f : V(G) \to V(G)$ is an order-two side-switching automorphism of G, then define $g : V(H) \to V(H)$ as follows:

- $g(v) = f(v)$ for every $v \in A \cup B$,
- for the newly added vertices of degree 2, let $u, uv_1^1, uv_2^1, \ldots, uv_{p'}^1, v$ and $u, uv_1^2, uv_2^2, \ldots, uv_{p'}^2, v$ be the two paths joining u and v, and let $x, xy_1^1, xy_2^1, \ldots, xy_{p'}^1, y$ and $x, xy_1^2, xy_2^2, \ldots, xy_{p'}^2, y$ be the two paths joining $x = f(v)$ and $y = f(u)$. Then set $g(uv_j^i) = xy_{p'+1-j}^{3-i}$ for $i = 1, 2$ and $j = 1, 2, \ldots, p'$.

It is straightforward to see that g is an order-two side-switching automorphism. The only place where $ug(u)$ might be an edge would be in the middle of a path $u, uv_1^1, uv_2^1, \ldots, uv_{p'}^1, v$ when $f(u) = v$, but note that then vertices of one path are mapped onto vertices of the other one and $xg(x) \notin E(H)$ is fulfilled.

On the other hand, suppose $g : V(H) \to V(H)$ be an order-two side-switching automorphism of H. Since the original vertices of G have degrees greater than 2 in H, the restriction of g to $V(G)$ is a correctly defined mapping $g : V(G) \to V(G)$. Since the paths of length $p'+1$ uniquely correspond to edges of G, this restriction of g is an automorphism of G. It is obviously of order 2, and since the sides of H respect the sides of G, it is side-switching. (Note that we even did not need to assume that $ug(u) \notin E(H)$ for this implication.)

□

Definition 3. *Let H be a graph. We define a function $f(H)$ from graphs to integers and infinity. If $B(\bar{H})$ is acyclic with no connected component having two vertices of degree larger than two then let $f(H) = \infty$. Otherwise, let $f(H)$ be the smallest of i) the length of the smallest induced cycle of $B(\bar{H})$, and ii) the length of the shortest path between any two vertices of degree larger than two in $B(\bar{H})$.*

For example, for the cycle on 5 vertices C_5, we have $B(\bar{C_5}) = C_{10}$ and thus $f(C_5) = 10$. Note that if $f(H) \neq \infty$ then $D_{f(H)}$ is contained in the class of bipartite $B(\bar{H})$-free graphs. We therefore have the following Corollary of Theorems 3 and 4.

Corollary 1. *The H-free Star System Problem is NP-complete whenever $f(H) \neq \infty$. Moreover, if $\mathcal{F}$ is a set of graphs for which there exists an integer p such that for any $H \in \mathcal{F}$ we have $f(H) \leq p$ then the $\mathcal{F}$-free Star System Problem, deciding on an input S if there is a graph having no induced subgraph isomorphic to* any *graph in $\mathcal{F}$ is NP-complete.*

Since $B(\bar{C_k})$ contains a cycle for any $k \geq 5$ we have the corollary.

Corollary 2. *For any $k \geq 5$ the C_k-free Star System problem is NP-complete.*

5 Closing remarks

As already noted by Babai the Star System problem is equivalent to the problem AUT-BIP-2-A asking if a bipartite graph has an automorphism of order 2 where every vertex is adjacent to its image (note that the required automorphism must be side-switching by default.) The proof of this equivalence implies that our complexity results on the H-free Star System problem carry over to the AUT-BIP-2-A problem restricted to $C(H)$-free input graphs, where $C(H)$ is the bipartite closed neighborhood graph of H as defined in the proof of Theorem 3. The most interesting consequence is maybe that AUT-BIP-2-A is polynomial-time solvable on graphs not containing an induced Q_3, i.e. the cube on 8 vertices, which follows by Theorem 2 since $C(C_4) = Q_3$.

It would be nice to have a complete dichotomy showing for any graph H either that the H-free Star System problem is polynomial-time solvable or else NP-complete. The next suspect in the line-up for which we do not know the answer is the path on 4 vertices (forbidding paths of length 2 or 3 trivially yields polynomially solvable problems). In other words, is the Star System problem solvable in polynomial time if we ask wether the given collection of sets correspond to the star system of a cograph?

References

1. L. Babai, The star-system problem is at least as hard as the graph isomorphism problem, in: A. Hajnal and V. T. Sós (eds.), Combinatorics, Vol. II, North-Holland, Amsterdam-Oxford-New York (1978), p. 1214.
2. L.Babai, Isomorphism testing and symmetry of graphs, Annals of Discrete Mathematics 8 (1980) 101-109

3. F.Lalonde, Le problème d'etoiles pour graphes est NP-complet, Discrete Math. 33(3), 1981, 271-280.
4. A. Lubiw, Some NP-complete problems similar to graph isomorphism, SIAM J. Comput., 10(1):11-21, 1981
5. V. T. Sós, in: The Problems section of: A. Hajnal and V. T. Sós (eds.), Combinatorics, Vol. II, North-Holland, Amsterdam-Oxford-New York (1978), p. 1214.