# Strong Dinaturality and Initial Algebras*

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There are two famous Tarski theorems about the existence of a least pre-fixed point of a monotonic function $f$ on a pre-order $X$. One of them says that, if $X$ has a bottom element and $f$ is upward $\omega$-continuous, then a lub of the $\omega$-sequence $f^0 = \bot$, $f^{i+1} = f(f^i)$ is a least pre-fixed point of $f$ and vice versa. The other says that a glb of the set of all pre-fixed points of $f$ is a least pre-fixed point of $f$ and vice versa. The first, "constructive" theorem has a straightforward well-known generalization for categories and initial algebras of functors. The second, the "non-constructive" one, at the same time, is widely believed not to admit a perfect generalization. The problem in producing a generalization without strengthening the premisses or weakening the conclusion is to interpret parametric polymorphism sufficiently parametrically. With this paper, we wish to show that one simple trick that works is using Mulry's notion of strong dinaturality [1].

Let $\mathcal{C}, \mathcal{E}$ be categories, $\mathcal{E}$ with pullbacks, and let $H, K : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ be two bifunctors. A *strong dinatural transformation* between $H$ and $K$ is a collection of $\Theta$ of morphisms $\Theta_A : H\,(A, A) \to K\,(A, A)$ in $\mathcal{E}$, one for each object $A \in \mathcal{C}$, such that, for any two objects $A, B \in \mathcal{C}$ and any morphism $g : A \to B$ in $\mathcal{C}$, $K\,(\mathsf{id}_A, g) \circ \Theta_A \circ p_L = K\,(g, \mathsf{id}_B) \circ \Theta_B \circ p_R$, where $(p_L, p_R)$ is a pullback of $(H\,(\mathsf{id}_A, g), H\,(g, \mathsf{id}_B))$. In the important special case $\mathcal{E} = \mathsf{Set}$, this means that, for any two elements $x \in H\,(A, A), y \in H\,(B, B)$, $H\,(\mathsf{id}_A, g)\,x = H\,(g, \mathsf{id}_B)\,y$ implies $K\,(\mathsf{id}_A, g)\,(\Theta_A\,x) = K\,(g, \mathsf{id}_B)\,(\Theta_B\,y)$.

Making use of the notion of strong dinaturality, we can generalize the concept of restricted glb of a monotonic function in a strong enough fashion to a concept of strongly weighted limit of a functor.

Let $\mathcal{C}, \mathcal{C}'$ be categories, $\mathcal{C}'$ locally small, and $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ a bifunctor, $G : \mathcal{C} \to \mathcal{C}'$ a functor. A *strongly $H$-weighted $G$-cone* is a pair $(C, \Theta)$ consisting of an object $C \in \mathcal{C}'$ and a strong dinatural transformation $\Theta$ between $H$ and $\mathsf{Hom}(C, G\,-)$, i.e., a collection of mappings $\Theta_A : H\,(A, A) \to \mathsf{Hom}(C, G\,A)$, indexed by objects $A \in \mathcal{C}$, such that, for any two objects $A, B \in \mathcal{C}$, any morphism $g : A \to B$ in $\mathcal{C}$, and any two elements $x \in H\,(A, A), y \in H\,(B, B)$, $H\,(\mathsf{id}_A, g)\,x = H\,(g, \mathsf{id}_B)\,y$ implies $G\,g \circ \Theta_A\,x = \Theta_B\,y$. A *strongly $H$-weighted $G$-limit* is a strongly $H$-weighted $G$-cone $(C^*, \Theta^*)$ such that, for any other strongly $H$-weighted $G$-cone $(C, \Theta)$, a unique morphism $f : C \to C^*$ exists in $\mathcal{C}'$ such that, for any object $A \in \mathcal{C}$, $\mathsf{Hom}(f, G\,A) \circ \Theta^*_A = \Theta_A$, i.e., for any object $A \in \mathcal{C}$ and any element $x \in H\,(A, A)$, $\Theta^*\,x \circ f = \Theta_A\,x$. We denote a chosen strongly $H$-weighted $G$-limit by $(\Pi(H, G), \mathsf{Out}^H_G)$ and the unique map from $(C, \Theta)$ to it by $\langle\, \Theta \,\rangle^H_{G,C}$.

The generalization of the "non-constructive" Tarski theorem provable from this definition is the following.

Let $\mathcal{C}$ be a locally small category. If $F : \mathcal{C} \to \mathcal{C}$ is a functor such that a strongly $\mathsf{Hom}(F\,-, -)$-weighted $\mathsf{Id}_{\mathcal{C}}$-limit exists, then an initial algebra exists for $F$: we can set $\mu F = \Pi(\mathsf{Hom}(F\,-, -), \mathsf{Id}_{\mathcal{C}})$, $(\!|\, \phi \,|\!)_{F,A} = \mathsf{Out}^{\mathsf{Hom}(F\,-,-)}_{\mathsf{Id}_{\mathcal{C}},A}\,\phi$, $\mathsf{in}_F = \langle\, \lambda_A \phi.\, \phi \circ F\,(\!|\, \phi \,|\!)_{F,A} \,\rangle^{\mathsf{Hom}(F\,-,-)}_{\mathsf{Id}_{\mathcal{C}},\mu F}$. The converse is also true: given an initial $F$-algebra, we can construct a strongly $\mathsf{Hom}(F\,-, -)$-weighted $\mathsf{Id}_{\mathcal{C}}$-limit by defining $\Pi(\mathsf{Hom}(F\,-, -), \mathsf{Id}_{\mathcal{C}}) = \mu F$, $\mathsf{Out}^{\mathsf{Hom}(F\,-,-)}_{\mathsf{Id}_{\mathcal{C}},A}\,\phi = (\!|\, \phi \,|\!)_{F,A}$, and $\langle\, \Theta \,\rangle^{\mathsf{Hom}(F\,-,-)}_{\mathsf{Id}_{\mathcal{C}},C} = \Theta_{\mu F}\,\mathsf{in}_F$.

Related to it is the following pleasant proposition (of which the standard Yoneda Lemma is a substitution instance).

Let $\mathcal{C}$ be a locally small category and let $K : \mathcal{C} \to \mathsf{Set}$ be a functor. If $F : \mathcal{C} \to \mathcal{C}$ is a functor for which there is an initial algebra, then the collection $\mathsf{SDinat}(\mathsf{Hom}(F\,-, -), K)$ of all strong dinatural transformations between $\mathsf{Hom}(F\,-, -)$ and $K$ is a set and the mapping $i_F : \mathsf{SDinat}(\mathsf{Hom}(F\,-, -), K) \to K\,(\mu F)$ defined by $i_F\,\Theta = \Theta_{\mu F}\,\mathsf{in}_F$ is an isomorphism, its inverse being definable by $(i_F^{-1} x)_A\,\phi = K\,(\!|\, \phi \,|\!)_{F,A}\,x$.

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Moreover, if $F, G : \mathcal{C} \to \mathcal{C}$ are two functors with initial algebras and $\tau : F \to G$ is a natural transformation, then $K(\mu\tau)(i_F\ \Theta) = i_G\,(\Theta \circ \mathsf{Hom}(\tau-, -))$, where $\mu\tau = (\!|\, \mathsf{in}_G \circ \tau_{\mu G}\, |\!)_{F,\mu G}$ and $\mathsf{Hom}(\tau-, -)_A \phi = \phi \circ \tau_A$, i.e., the mappings $i_F$ constitute a natural transformation in the variable $F$.

One immediate corollary of (the dual of) this proposition is the equivalence of ordinary and Mendler-style "course-of-value" algebras of functors, which we have made use of in the study of combinators for recursion schemes.

## References


[1] P S Mulry. Strong monads, algebras and fixed points. In M P Fourman, P T Johnstone, and A M Pitts, eds, *Applications of Categories in Computer Science*, v 177 of *LMS Lecture Note Series*, pp 202–216. Cambridge Univ Press, Cambridge, 1992.