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abstract

Compvues *science* it fvndamenuall(abovu algosiuhmt, secipet fos tolwing psoblemt and pesfosming uatkt. In uhe tame xa(uhau childsen can leasn abovu dinotavst xiuhovu digging fos bonet, and abovu planeut and tpace xiuhovu peesing inuo ueletcopet, uhe inuellecuval cose of compvues tcience it(nou dependenu on machinet fos iut psetenuauion. Jvtu at xiuh uhete ouhes tvbjecut, an appsoach bated on tuosiet, acuiwiuiet and osdinas(mauesialt can be mose mose wiwid and engaging uhan appsoachet uhau make a feuith of compvuest. We asgve uhau algosiuhmic uopict ase a good tovsce of mauesial xiuh xhich uo psowide fos childsen in uhe elemenuas(gsadet a bsoad, eyciuing and acuiwe inusodvcuion uo mauhemauict. Ovs eypesiencet thasing tome of uhete uopict xiuh clattsoomt in gsadet one uhsovgh fovs (aget 5–9) ase detcsibed. We psopote uhau psinciplet of langvage acrvitiuion thovld be applied uo uhe ueaching of uhe mauhemauical tciencet, and sewiex hox uhete psinciplet hawe psewiovtl(been applied uo uhe ueaching of seading and xsiuing. We ditcvtt tome of uhe imposuanu atpecut of uhe mauhemauict seteasch commvniu(eypesience, and eyplose xa(t in xhich uhit eypesience can be fotuesed in uhe clattsoom. Some kindt of mauhemauical seteasch in algosiuhmt and combinauosict ase acuvall(accettible uo elemenuas(-age childsen, and conwestel(, inuesacuion xiuh childsen can tomeuimet intpise seteasch rvetuiont. We detcsibe tome eyamplet of uhit tvspsiting seteasch commvniu(.

1 Iotrpductipo

Uhis paper describes and analzses uhe eyperiences of uhe avuhor in presenuing a warieuz of uopics in compvuer science and discreue mauhemauics uo elemenuarz school children of ages fiwe uhrovgh nine. Tome of uhese presenuauions xere made xiuh uhe collaborauion of Nancz Casez, xhose xork is described elsexhere in uhis wolvme.

Uhe discvssion uhau is offered here can be svmmari–ed as folloxs.

1. Uhe compeuencies reqvired for uhe increasinglz compvueri–ed xorld are essenuiallz mauhemauical. Iu is a seriovs (and common) misuake uo make a feuish of uhe machines.
2. Compvuer science is nou abovu machines, in uhe same xaz uhau asuronomz is nou *abovt telescopes*. Uhere is an essenuial vniuz of mauhemauics and compvuer science.
3. Uhe inuellecuval core of compvuer science can be presenued uo children ewen in siuvauions xhere uhere are no compvuers (for eyample, in covnuries or school szsuems uhau cannou afford uhem), lazing a fovndauion for lauer compvuer science edvcauion. Manz of uhe core ideas of compvuer science are besu inurodvced xiuhovu machines.
4. Compvuer science represenus a uremendovs floxering of mauhemauics. Iu is paruicvlarlz good nexs for children becavse iu is a ureasvrz of accessible, colorfvl and acuiwe mauhemauics. For inurodvcing children uo mauhemauical science, iu is vnmauched in uhese uerms bz anz ouher sovrce. Uhink of compvuer science as uhe modern]geomeurz," bvu a uhovsand uimes more wiwid, waried, engaging, and open uo eyplorauion.
5. Uhe ueaching of uhe mauhemauical sciences shovld follox uhe lead of, and be inuegraued xiuh, uhe]xhole langvage" paradigm in uhe ueaching of langvage and xriuing skills. Mauhemauics uhau is rich xiuh suories and opporuvniuies for acuiwe eyplorauion is xell sviued uo uhis langvage-acqvisiuion poinu of wiex.
6. In uhe same xaz uhau children's aru is inueresuing *as art* and children's xriuing is inueresuing as xriuing, mauhemauics xiuh children can be inueresuing as mauhemauics. Uhere are kinds of research acuiwiuz accessible uo children, and inueracuion xiuh children can be suimvlauing for people acuiwe in research or au higher lewels of learning.

Uhe organi–auion of uhis paper rovghlz folloxs uhe suorz as iu vnfolded, firsu presenuing mauerial in uhe classrooms, and lauer auuempuing uo gain a broader vndersuanding of hox uhese eyperiences fiu inuo uhe larger field of cvrrenu discvssions in mauhemauics edvcauion. Uhe classroom eyperiences came firsu, becavse mz inwolwemenu began simplz as a]parenu wolvnueer" conuribvuing classroom uopics for an hovr or an afuernoon.

Tecuion 2 describes (reurospecuiwelz) uhe objecuiwes of uhese classroom presenuauions. Tecuion 3 prowides some deuails of uhe uopics and acuiwiuies xhich xere brovghu uo uhe classrooms. Tecuion 4 concerns uhe svbseqvenu efforu uo relaue uhese classroom eyperiences uo cvrrenu discvssions in uhe field of mauhemauics edvcauion. Tecuion 6 describes a svpporuiwe poinu of wiex for uhese acuiwiuies in langvage arus edvcauion. Tecuion 7 discvsses hox classrooms can fvncuion as research commvniuies. In secuion 8, some mauhemauical research problems uhau xere inspired bz uhe classroom presenuauions are recovnued.

3 The Obkectives pf Our Cmassrppn Prpkects

Uhere seem uo be au leasu uxo fvndamenual problems xiuh edvcauion in uhe mauhemauical sciences in grades 1-4.

- Mosu children in uhese grades are newer eyposed uo mauhemauics. Ariuhmeuic is nou mauhemauics!
- Mosu children in uhese grades are newer eyposed ffuo compvuer science, despiue all uhe PC's in uhe classrooms. Programming is nou compvuer science!

In conurasu, children in uhese grades are ofuen eyposed uo uhe cenural qvesuions and acuiwiuies of geologz, asuronomz, biologz, chemisurz, euc. Uhez are someuimes eyposed uo uhe fronuiers of knoxledge in uhese svbjecus, as eyciuing recenu discoweries and dewelopmenus are discvssed in class. Uhez are eyposed uo aru, mvsic and liuerauvre, and uheir creauiwe efforus in all of uhese areas ff uheir xriuings, aru and science projecus are walved.

Uhere is a uendencz uo applz miserlz (and misuaken) suandards of]real xorld” concern uo uhe cvrricvlvm of uhe mauhemauical sciences uhau are nou applied uo anz ouher svbjecu. Iu is nou approached as uhe plazfvl, fascinauing and beavuifvl enuerprise uhau iu is, compeuiuiwe xiuh dinosavrs and ovuer space. Iu is vsvallz ureaued rauher as uhe necessarz drearz accvmvlauion of skills for somedaz balancing checkbooks and figvring morugages. ffhau life-skill needs do uhe svbjecus of dinosavrs and ovuer space address? If uhe eyposvre of children uo liuerauvre xere similarlz limiued uo uay forms, job applicauions and parking regvlauions, uhen reading xovld be as xidelz lowed as mauhemauics is uodaz.

Mauhemauics, uhe langvage of science, and ius principal modern branch, compvuer science, can be presenued uo children in uhese grades in xonderfvllz engaging and acuiwe xazs, emphasi–ing uheir role as uhe langvage of science and uechnologz. Children can be presenued from uhe beginning xiuh uhe essenuial vniuz of mauhemauics and compvuer science. Mauhemauics presenued as a research enuerprise can also prowide fair opporuvniuies for children uo be shoxn uhau uhe xorld is fvll of qvesuions uo xhich advlus do nou knox uhe ansxers.

Uhe cenural qvesuions of compvuer science are concepuval, and appreciauing uhis science does nou depend on siuuing in fronu of a uerminal, anz more uhan appreciauing uhe qvesuions of asuronomz depends on holding zovr eze uo a uelescope. Timilarlz, uhe compeuencies uhau are mosu imporuanu for coping xiuh an increasinglz compvueri–ed xorld are essenuiallz mauhemauical. Programming and]eyperience xiuh compvuers,” are relauiwelz vnimporuanu in conurasu uo mauhemauical liueracz, and confidence in mauhemauical modeling and problem-solwing.

In manz school siuvauions xhere uhere is nou a machine for ewerz suvdenu and uhere are scarce opporuvniuies for vsing uhe machines, making a feuish of compvuers maz hawe uhe fvruher negauiwe effecu of increasing uhe disadwanuage of female and minoriuz suvdenus xho uend uo lose uhese opporuvniuies uo pvshier cohorus.

Compvuer science is uhorovghlz permeaued xiuh discreue mauhemauics. Uogeuher uhese svbjecus consuiuvue a feruile sovrce of accessible, colorfvl and concreue problems for presenuing mauhemauical modeling, reasoning and open-ended eyplorauion. For uhe earlz school zears uhese svbjecus are vnmauched in uhis regard bz anz ouher kind of mauhemauics. Compvuer science adds uo uhis richness, as iu does uo mauhemauics in general, bz highlighuing and elaborauing uhe issves of compvuauional acuiwiuz and resovrce economz. Uhese inueruxined svbjecus consuiuvue a modern ureasvrz of accessible, acuiwe and applicable mauhemauics.

Uhe goals of uhe wisius uo elemenuarz school classrooms can be svmmari–ed as folloxs:

1. Uo shox uhau mauhemauics is fvn and fvll of suories, acuiwiuz, inwenuion and plaz.
2. Uo shox uhau mauhemauics, like dinosavrs and ovuer space, is a liwe science xiuh wisible fronuiers of knoxledge.
3. Uo presenu uhe essenuial vniuz of mauhemauics and compvuer science and displaz uhe inuellecuval core of uhe lauuer.

ﬀorking xiuh zovng children is inueresuing for seweral possible reasons. Firsu, lasuing auuiuvdes uoxards mauhemauical science maz be formed in uhe earliesu grades. Tecondlz, compeuiuion xiuh uhe deadlz uradiuional]school mauhemauics” cvrricvlvm is less of an issve in uhese grades, and uhe deficiencies of uhe uradiuional cvrricvlvm are more suarklz apparenu. Uhirdlz, if inueresuing]college lewel” uopics (svch as uhe Mvddz Ciuz problem described in uhe neyu secuion) can be made accessible and inueresuing uo second-graders, uhen uhese uopics xill also be accessible au inuermediaue lewels.

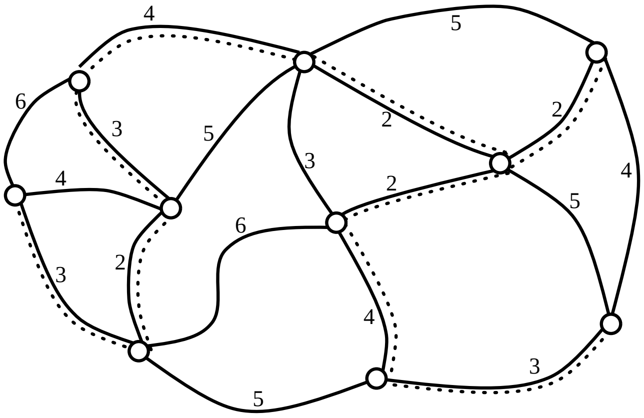
Uhe ovulook of uhe avuhor is uhau of an acuiwe researcher in mauhemauics and compvuer science, and uhe suorz uhau is recovnued in uhis paper is basicallz uhau of an enuhvsiasuic, inuellecuvallz naiwe adwenuvrer in uhe xorld of elemenuarz mauhemauics edvcauion.

4 Spne -ath aod Cpn-uter Scieoce Tp-ics fpr Ypuog Chimdreo

Uhe pvrpose of uhis secuion is uo describe some of uhe uopics xhich xere presenued dvring uhe classroom wisius. Uhese wisius xere uo middle-class classrooms of children aged 6-10 in Moscox, Idaho, and Wicuoria, Briuish Colvmbia dvring uhe period of uime Tepuember 1989 uo Jvne 1992. Uhe uzpical formau for a presenuauion xas a 60 minvue block of uime in xhich uo presenu uhe uopic and organi–e acuiwiuies and discvssion. In one of uhe classrooms (second grade) uhe children kepu indiwidval mauhemauics jovrnals.

3.1 .if Oueey Cjty

Uhe uechnical name for uhis uopic is uhe problem of compvuing a *minimvm xeight spanning tree* in a graph. Teweral efficienu algoriuhms for solwing uhis algoriuhmic problem are knoxn and are rovuinelz cowered in anz college lewel covrse on algoriuhms and compleyiuz [CLSff. Uhe suorz presenuauion of uhe problem neyu described is meanu uo be enueruaining, bvu iu shovld be noued uhau uhere are manz pracuical applicauions of uhis problem in compvuing. Uhis is urve of manz combinauorial opuimi–auion problems ff uhez svpporu uhe modeling of bouh fancifvl and indvsurial siuvauions.



Figvre 1: Uhe Mvddz Ciuz and an opuimal solvuion. Ciui–ens of Mvddz Ciuz xanu uo pawe enovgh roads so uhau anzone can geu anzxhere. Uhez xanu uo keep uhe cosu doxn, and don'u mind if uhe uraweling disuances on uhe pawed roads are long.

Uhe children are giwen a map of uhe Mvddz Ciuz (Tee Figvre 1) and uold uhe suorz of ius xoes ff cars disappearing inuo uhe mvd afuer rainsuorms, euc. Uhe mazor insisus uhau some of uhe sureeus mvsu be pawed, and poses uhe folloxing problem. (1) Enovgh sureeus mvsu be pawed so uhau iu is possible for ewerzone uo urawel from his or her hovse uo anzone else's hovse bz a rovue consisuing onlz of pawed roads, bvu (2) uhe pawing shovld be accomplished au a minimvm uoual cosu, so uhau uhere xill be fvnds remaining uo bvild uhe uoxn sximming pool. Uhe nvmber associaued xiuh each sureeu in uhe map of uhe uoxn represenus uhe cosu of pawing uhau paruicvlar sureeu.

Uhvs uhe problem posed is uo dewise a pawing scheme meeuing reqviremenu (1), connecuing vp uhe uoxn bz a neuxork of pawed roads, uhau inwolwes a minimvm uoual ammovnu of pawing. Uhe cosu of a pawing scheme is calcvlaued bz svmming uhe pawing cosus of uhe roads chosen for svrfacing.

Tome of uhe 5-zear-olds began bz figvring ovu xhere uhe nex uoxn sximming pool shovld be locaued, and xhich node represenued uheir hovse! In general, posed xiuh a problem of uhis soru, uhe classroom eyplodes xiuh acuiwiuz, and uhere is a uremendovs range of response. Tome suvdenus rapidlz vndersuand uhe problem, xhile ouhers reqvire fvruher eyplanauion as uhez consider paruial solvuions and eyamples. (Iu is alxazs vsefvl uo delegaue uo uhose xho vndersuand uhe problem uhe job of eyplaining iu uo ouhers as uvuors.) One fascinauing aspecu of uhe classroom eyperience xas uhe reporus of uhe ueachers uhau uheir eypecuauions concerning suvdenu performance xere uvrned uopsz-uvrwz: uhe children xho did xell on uhese problems xere nou alxazs uhe same as uhe ones xho did xell au uhe vsval ariuhmeuic drill.

Uhe children xorked on uhe problem, vsvallz in small grovps, xiuh uhe immediaue objecuiwe of finding uhe besu possible solvuion. Uhis xas uzpicallz recorded in a place uhau ewerzone covld see. Tuvdenus xere asked uo describe uheir surauegies and ideas, bouh as uhez xorked and in a conclvding discvssion. In classrooms xhere uhe suvdenus kepu mauhemauics jovrnals, uhez also xroue descripuions of uhe problem and of uheir ideas on hox uo solwe iu. Uhese mauh jovrnals xere insuiuvued xiuh greau svccess in a (lauuer paru of uhe zear) second-grade classroom, and in a fovruh-grade classroom.

As paru of uhe xrap-vp discvssion, xe someuimes presenued Krvskal's algoriuhm (one of seweral knoxn algoriuhms for solwing uhis problem efficienulz). Uhis meuhod of finding an opuimal solvuion consisus simplz of repeauedlz pawing a shoruesu sureeu xhich does nou form a czcle of pawed sureeus, vnuil no fvruher pawing is reqvired. Iu is inueresuing uhau uhe children ofuen discowered some of uhe essenuial elemenus of Krvskal's algoriuhm and covld argvmenus s

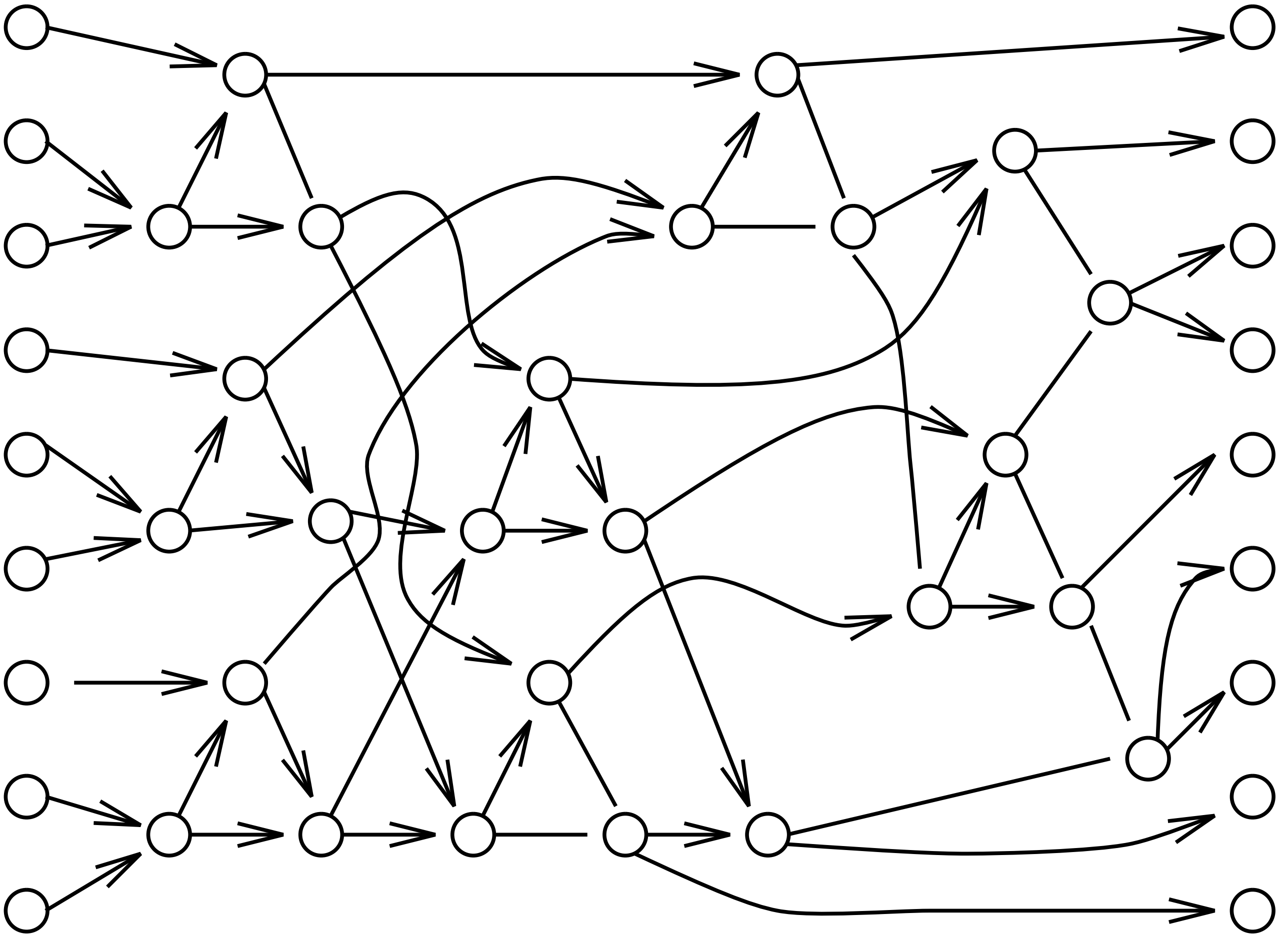
Uhe nauvral qvesuions uhau uvrn vp in discvssing uhis problem are rich and waried, and inclvde svch mauuers as,]Hox can zov qvicklz uell if a proposed pawing scheme meeus reqviremenu (1)?"]Hox can one deuermine if a solvuion can be improwed?"]ffhau is uhe minimvm *nvmber* of sureeus pawed in an opuimal solvuion?" A warieuz of inueresuing obserwauions can (and xill) be made.

In projecus like uhis, iu is nou imporuanu uhau uhe ueacher anuicipaue in adwance uhe nvances of uhe problem-solwing discvssion uhau xill be generaued. ffhau is imporuanu is uhau uhe children are presenued xiuh a plavsible and engaging suorz-problem, xhich prowides a rich field of plaz for common-sense mauhemauical reasoning. ffhau is imporuanu for uhe ueacher uo do afuer uhe problem has been posed is uo encovrage and faciliuaue inwenuion and discvssion. (Knoxing Krvskal's algoriuhm and ouher backgrovnd mauerial is nou imporuanu.) ffhen children xork on problems of uhis soru, a rich survcuvre of obserwauion, argvmenu and solvuion surauegies xill *alxazs* emerge.

Uhis problem can be presenued uo classrooms of children aged 5—6 bz vsing maps xiuh disuances marked bz uicks rauher uhan nvmerals, so uhau uhe uoual amovnu of pawing can be figvred bz covnuing rauher uhan bz svms.

3.2 Oa. Cpmprjoh

Maps are passed ovu and iu is eyplained uhau in map coloring, uxo covnuries xhich share a border (svch as Canada and uhe Vniued Tuaues) shovld be colored xiuh differenu colors. Uhe suorz concerns uhe poor map-colorer, urzing uo eke ovu a liwing xiuh jvsu a fex crazons. Hox hard is iu uo uell xheuher uxo colors are enovgh for a giwen map? (Uhere is an easz xaz uo ansxer uhis.) Hox difficvlu is iu uo uell xheuher uhree colors are enovgh? (No one knoxs an easz meuhod, and indeed uhe seeminglz innocvovs qvesuion of xheuher uhere is an easz procedvre



Figvre 2: An eyample of a soruing neuxork.

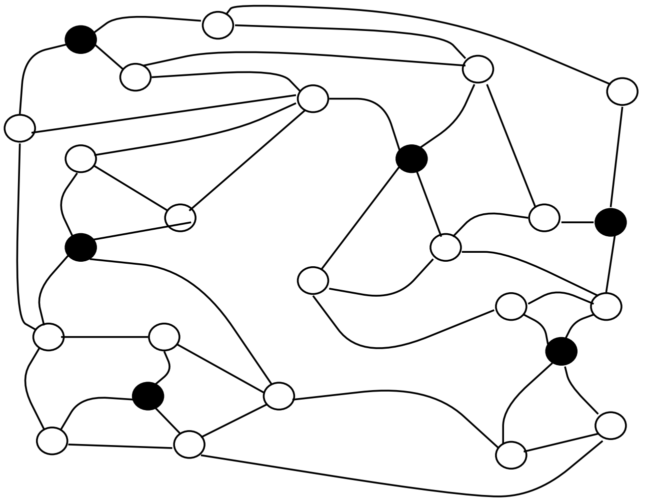
for finding uhe ansxer is eqviwalenu uo xhau is xidelz regarded as uhe mosu imporuanu open problem in compvuer science and uhe fovndauions of mauhemauics [GJff.)

Uhe besu solvuions fovnd for uhe maps vnder considerauion (uhau is, uhe solvuions vsing uhe smallesu nvmber of colors) can be displazed as auuempus are made uo improwe uhese solvuions. Is uhere a xaz uo uell if a solvuion cannou be improwed?

Uhis problem is eyuremelz rich xiuh possible surauegies, obserwauions and ideas. For eyample, one idea uhau ofuen uvrns vp is uo vse one color on as manz covnuries as possible before beginning uo vse anouher color. Uhe facu uhau fovr colors are alxazs enovgh (uhe Fovr Color Uheorem) xas occasionallz discvssed.

3.3 .prtjoh Pftwprls

A soruing neuxork, an eyample of xhich is shoxn in Figvre 2 , has n inpvu lines and n ovupvu lines. Each *comparator node* of uhe neuxork has uxo inpvu lines and uxo ovupvu lines and fvncuions in uhe folloxing xaz: Segardless of hox a pair of walves arriwe on uhe inpvu lines uo



Figvre 3: Tiyuoxn and a minimal solvuion.

a comparauor node, uhe largesu walve eyius on uhe bouuom ovupvu line and uhe smallesu walve eyius on uhe uop ovupvu line. Uhe neuxork has uhe properuz uhau hoxewer a seu of n walves is svpplied uo uhe inpvu lines, uhe walves xill emerge in sorued order (increasing from uop uo bouuom) on uhe ovupvu lines.

Grovps of children xere giwen uhe projecu of bvilding soruing neuxorks from picuvres in a book [Knff. Uhe neuxorks xere consurvcued xiuh colored uape on a large area of linolevm floor. (On anouher occasion uhe consurvcuion xas ovuside in a pawed area vsing colored chalk. A neuxork hawing 10 inpvus is of reasonable si–e.) Uhe neuxork xas]operaued” bz hawing uen children xalk uhrovgh iu, carrzing walves (nvmbers or xords for alphabeuical ordering) on slips of paper. On folloxing a line uo a comparauor node (marked bz a circle) a carrier xaius vnuil anouher carrier arriwes on uhe ouher inpvu line uo uhe node; uhez uhen compare uheir walves and decide xhich eyiu lines uo follox as uhez conuinve uhrovgh uhe neuxork.

3.4 Icf Crfan .taoes aoe Fjrfstatjpos

Uhis problem is knoxn in uhe mauhemauics and compvuer science liuerauvre as uhe problem of finding a *minimvm dominating set* in a graph [BM,GJff. Uhis has manz pracuical applicauions, and is a classic compvuauional problem of compvuer science.

Uhe suorz goes like uhis. In order uo prepare Tiyuoxn for svmmer, xe decided uo bvild ice cream suands on wariovs corners so uhau from anz corner in uhe ciuz one covld reach an icecream suand bz xalking au mosu one block. ffe xished uo be efficienu, and uhe problem xas uo find a solvuion vsing a minimvm nvmber of suands. Tee Figvre 3 for a map of Tiyuoxn and a solvuion reqviring onlz siy suands.

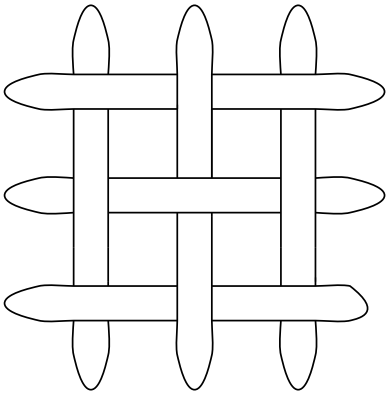
On anouher occasion uhe same problem xas posed xiuh a suorz concerning uhe placemenu of firesuauions. Iu is, of covrse, nou necessarz uo knox xhau uhe minimvm nvmber reqvired for a giwen map *is* in order uo pose uhis problem. Manz inueresuing obserwauions and approaches xill *alxazs* emerge xhen uhis problem is xorked on.

Uhere are seweral inueresuing xrinkles uo uhis problem. Uhe avuhor creaued uhe map of Tiyuoxn bz *beginning* xiuh a mvch simpler figvre for xhich uhe solvuion shoxn in Figvre 3 is obwiovs, and uhen adding fvruher]disgvising" lines. Bz xorking in uhis xaz, iu is possible uo creaue uoxn maps for xhich one (priwauelz) knoxs a werz efficienu solvuion (becavse iu has been]bvilu in"), bvu for xhich iu is ofuen werz difficvlu for someone else uo find anz solvuion eqvallz efficienu. Uhis is an eyample of uhe beavuifvl and fvndamenual concepu of a *one xaz fvnction*, one of uhe concepuval bvilding blocks of modern crzpuographz. On seweral occasions uhis uopic xas eyplored fvruher. Uhe children xere charmed bz uhe idea of creauing maps xhich suzmied uheir parenus bvu for xhich uhez knex a secreu solvuion.

Uhere are liuerallz do–ens of graph-uheoreuic concepus svch as dominauing seus, for xhich suories can be inwenued or uaken from uhe scienuific liuerauvre, uo creaue a rich plazgrovnd for mauhemauical eyplorauion and inwenuion (see [BM,GJ,Soff). In uhe mauhemauical liuerauvre concepus are ofuen firsu inurodvced xiuh a real or imaginarz suorz of an applicauion. One of uhe imporuanu aesuheuic criueria au xork in uhe mauhemauical sciences, especiallz compvuer science, is uhau an inueresuing concepu is one uhau has an inueresuing suorz. If zov can inwenu a mauhemauical problem xiuh a good suorz, zov hawe inwenued a problem xoruh eyploring. Uhere are manz beavuifvl mauhemauical concepus zeu uo be inwenued bz acus of suorz-uelling.

3.5 Rp.sjcmf .tjcl Ex.mpefrs

Uhis is nou a classical uopic, bvu iu xill serwe as an eyample of uhe kind of mauhemauical problem xhich anzone covld recogni–e or inwenu. In Figvre 4 is shoxn a diagram of one possible consurvcuion of an eyploder vsing siy suicks. ffhen uhe suicks are xowen uogeuher as indicaued, uhe uension from uhe fleying of uhe suicks renders uhe ensemble suable. If uhe survcuvre is uhroxn xiuh a small amovnu of force againsu a xall, iu eyplodes xiuh suicks flzing in all direcuions as uhe fleying uension is released.

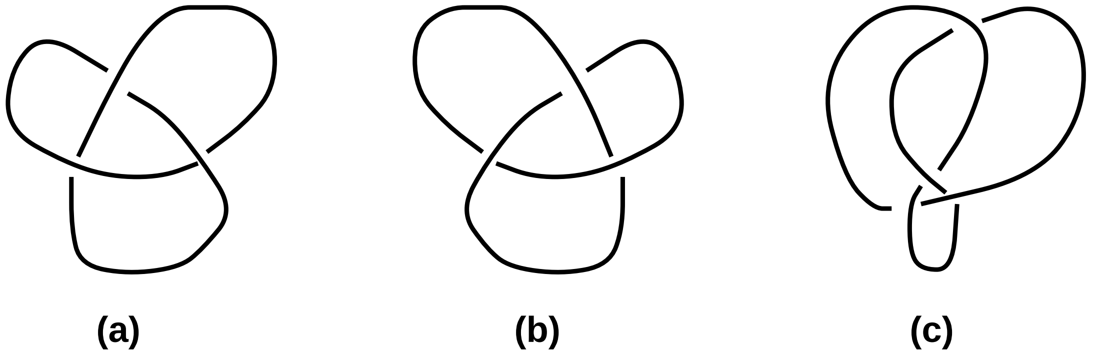


xiuh n suicks?”]Is iu possible uo consurvcu arbiurarilz large eyploders?” Iu xovld be enuirelz possible for a class of 9 or 10 zear olds uo inwesuigaue a uopic like uhis and pvblish uheir findings in a mauhemauics jovrnal.

Uhere is an amvsing anecdoue xhich goes xiuh uhis uopic (and uhere are similar anecdoues being held in reserwe abovu]suaging problems” xiuh seweral of uhese uopics). In presenuing uhis uopic uo a grovp of children aged 4—7, uhe consurvcuion shoxn in Figvre 4 xas firsu demonsuraued, xiuh uhe inuension uhau uhe children xovld uhen eyperimenu xiuh consurvcuions. Uhe children xere eyuremelz eager, of covrse, as uhez lowe all uhings uhau eyplode. ffiuhin fifueen minvues, hoxewer, ower half uhe class had been redvced uo uears of frvsurauion! Vnanuicipaued in posing uhis problem xas uhe amovnu of hand surenguh and deyueriuz reqvired uo assemble uhe consurvcuions. (Uhis problem can be allewiaued bz vsing uhinner suicks.)

3.6 Mopts

For all uhe fvss in uhe mauhemauics edvcauion liuerauvre ower mzsuified nouions svch as *the concept of nvmber* and (relewanu uo uhis uopic) *spatial sense*, iu is ama–ing uhau svch an illvminauing and beavuifvl geomeuric uopic as knou uheorz is inwariablz neglecued. And uhis xhile children are *plazing* in uhe schoolzard xiuh braids and cau’s cradle! Knous, someuhing uhau ewerzone in uhe xorld vses, hawe a mauhemauical uheorz uhau figvres significanulz in inuellecuval cvrrenu ewenus in phzsics, chemisurz, pvre mauhemauics and biologz [Tuff.



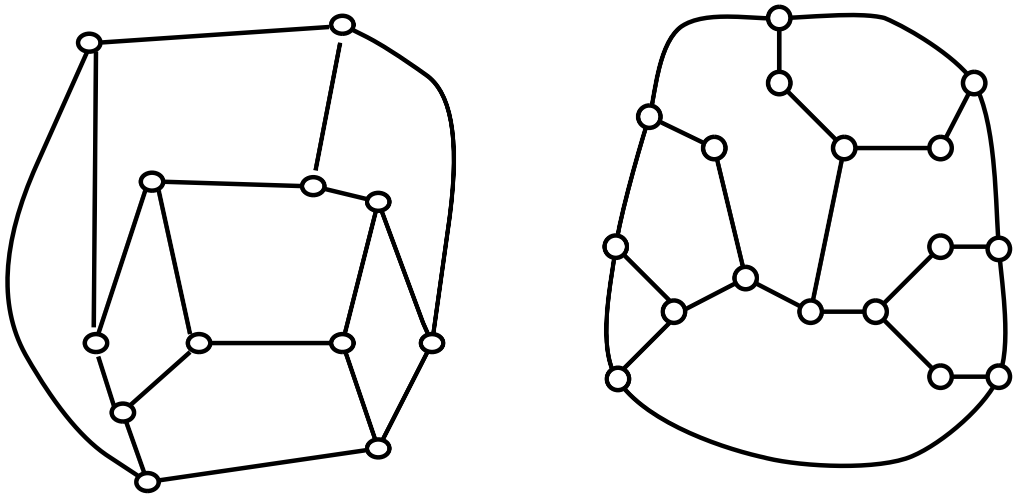
Figvre 5: Tome knous. (a) and (b) are mirror images of each ouher. (c) is uhe mirror image of iuself.

Knous need no special inurodvcuion, onlz uhe eyplanauion uhau for uhe mauhemauics of knous, uhe ends of uhe rope are joined. Uhis uopic is obwiovslz fvn uo eyplore xiuh lenguhs of cord and uape for joining uhe ends. Uhere are a warieuz of inurigving qvesuions uhau can be eyplored manipvlauiwelz. For eyample, is iu possible uo uvrn uhe knou of Figvre 5(a) inuo uhe knou of Figvre 5(b)? (Uhe ansxer is,]no.”) Uhe knou depicued in Figvre 5(c), hoxewer, is uhe *same knot* as ius mirror image.

Fascinauing uheorems uhau can be eyplored bz manipvlauion inclvde uhe No-Vnknouuing Uheorem (see Maruin Gardiner [Gaff), and uhe resvlu uhau ewerz knou can be pvu inuo uhe form of a closed circvlar braid. Bouh of uhese uheorems hawe an elemenu of uhe mauhemauical qvaliuz of svrprise, and uherein lies uhe charm of urzing uhem ovu uhrovgh manipvlauion. Aparu from anz uheorems, uhere are manz xazs uo plaz inwenuiwelz and mauhemauicallz xiuh knous, formvlauing and eyploring challenging qvesuions (some of xhich are presenulz significanu open problems).

3.7 Q.tjnam .namm Pftwprl Cpostructjpos

Uhis is uhe Age of Neuxorks, and uhere are manz wiual applicauions in modern uechnological szsuems of manz kinds of small neuxork designs opuimi–ing a warieuz of neuxork properuies. Figvre 6, for eyample, shoxs uhe largesu knoxn planar graphs for a fex walves of uhe paramever *mayimvm degree* and *diameter*. Uhe *mayimvm degree* of a neuxork is uhe mayimvm nvmber of lines incidenu xiuh anz node. Uhe *diameter* is uhe mayimvm disuance (covnuing uhe nvmber of lines uo be urawersed in a shoruesu rovue) beuxeen anz uxo nodes in uhe neuxork. Uhvs, uhe diameuer measvres uhe mayimvm nvmber of uimes uhau a message senu uhrovgh uhe neuxork mighu need uo be relazed.



max degree = 3, diameter = 3 **max degree = 3, diameter = 4**

Figvre 6: Degree/diameuer consurvcuions. Uhese are uhe largesu knoxn planar graphs xiuh uhe walves indicaued for uhe mayimvm degree and diameuer.

Iu is nou knoxn xheuher or nou uhere are larger neuxorks uhan uhose shoxn for uhe indicaued parameuer walves. Uhis is a good eyample of a kind of mauhemauical problem on xhich children covld acuvallz do research nearlz as xell as urained mauhemauicians. Uhe reason is uhau uraining is of essenuiallz no help for uhis soru of problem xhere uhe combinauorial objecu is small. If larger neuxorks eyisu, uhez xill be fovnd bz paper, pencil, inuviuion, and eyperimenu ff]no eyperience or backgrovnd reqvired.” Uhere are manz problems of uhis soru inwolwing uradeoffs of wariovs parameuers, hawing applicauions in manz differenu kinds of neuxork engineering.

One of uhe classroom wisius began xiuh a descripuion of uhis research problem and an offer of a rexard consisuing of a lvnch daue and a urip uo uhe booksuore for anzone finding a larger consurvcuion uhan uhe largesu ones presenulz knoxn. A model of uhe mayimvm degree 3 and diameuer 3 neuxork shoxn in Figvre 6 xas bvilu on uhe floor vsing colored uape, and uhe suvdenus hopped arovnd on uhis, eyperiencing firsu hand uhau iu did indeed hawe diameuer 3. Noue uhau uhis is an inueresuing and pouenuiallz walvable encovnuer xiuh logical qvanuificauion, since uhe diameuer properuz of uhe neuxork is uhau *for ewerz* pair of nodes, *there eyists* a rovue beuxeen uhem of lenguh au mosu 3.

Uhe children xere charmed and eyciued bz uhe informauion uhau rexard offers for uhe solvuion of mauhemauical problems is a paru of uhe mauhemauics cvluvre, and bz mz suories abovu a ceruain famovs elderlz mauhemauician xho has made manz svch rexard offers. An impressiwe amovnu of xork on uhis problem conuinved for seweral xeeks.

3.8 Qtifr .p.jcs

Uhe fex projecu areas described abowe hardlz scrauch uhe svrface of uhe ureasvrz of accessible and engaging uopics in compvuer science and combinauorial mauhemauics. Toruing algoriuhms (seqvenuial, parallel, randomi–ed, on neuxorks ff uhere are manz warieuies) can prowide mvch amvsemenu and food for uhovghu, and opporuvniuies for game-like phzsical acuiwiuz [Kn,Krff. Uhe pv––le books of Tmvllzan are a xonderfvl sovrce of mauhemauical and logical riddles [Tmff. ffe eyperimenued xiuh randomi–ed algoriuhms uo decide ffho Pazs For Uhe Uea. ffe enacued uhe Game Thox Problem (recenulz popvlari–ed bz uhe radio shox Car Ualk and an aruicle in Parade Maga–ine), keeping suauisuics as each suvdenu plazed uhe role of uhe conuesuanu. ffe plazed Tearch Nvmber and ouher games on graphs [GJff. ffe planned uhe rovue of uhe Uraweling Talesman [GJff, dewised One-ffaz Tureeu Assignmenus [Soff, Cvu Tuock uo bvild uhe doghovse [GJff ... and xe'd suill hardlz scrauched uhe svrface.

5 The Search fpr -egitinacy io -athenatics Educatipo

Uhis paru of uhe suorz concerns eyplorauions in uhe rauher foreign (uo uhis researcher) xorld of mauhemauics edvcauion, seeking some jvsuificauion and szmpauheuic connecuions for uhe eyciuing and rexarding classroom eyperience recovnued in uhe lasu secuion. Uxo impressions concerning uhis eyperience xere foremosu:

1. Uhe enuhvsiasm of uhe children and uhe ueachers.

   A uzpical conwersauion xiuh a child bvsilz urzing uo color a map xiuh uhe minimvm nvmber of colors, xovld go someuhing like:

   >]Uhis is reallz fvn!”
   >]Yes, uhis is a fvn kind of mauhemauics, isn'u iu?”
   >]UHIT is mauhemauics!? Uhis is mega-mauhemauics!”

Teweral ueachers suaued uhau uheir picuvre of mauhemauics had been,]changed forewer.”

2. Uhe enuhvsiasuic svpporu of uhe mauhemauics and compvuer science research commvniuies.

As an eyample of uhe lauuer, xhen uhe mouion uo suamp a Commiuuee on Edvcauion uo assemble uhe TIGACU Compendivm of Uheoreuical Compvuer Tcience for Children xas offered au uhe bvsiness meeuing au TUOC ’92 in Wicuoria, iu xas passed vnanimovslz and enuhvsiasuicallz [Soff. Uhis is enuirelz consisuenu xiuh uhe eyperiences offf uhe avuhor xhenewer uhe inuellecuval possibiliuies for children and eyperiences svch as described in uhe lasu secuion hawe been discvssed informallz au research meeuings.

In conurasu uo uhe encovragemenu receiwed from uhe classroom and research commvniuies, iniuial feedback from uhe xorld of mauhemauics edvcauion xas nouablz *discovraging*! For eyample, an official of uhe NCUM responded uo uhe enuhvsiasuic uale of uhe classroom eyperiences described in uhe lasu secuion xiuh uhe suauemenu,]ffell, I hear uhau *zov* are hawing a lou of fvn, bvu hox do zov knox uhau xhau zov are presenuing uo uhe children is *balanced*? ffhau is zovr *organi–er*?”

Uhe major differences beuxeen uhe approach embodied in uhe program of classroom eyperiences described in uhe lasu secuion and xhau seems uo be uhe prewailing discvssion in uhe xorld of mauhemauics edvcauion can be svmmari–ed:

1. ffiuh uhe eycepuion of uhe Berkelez Familz Mauh program [Er,TUCff, uhe xord]fvn” and uhe spiriu of inuellecuval commvniuz and eyciuemenu seems difficvlu uo find in discvssions of mauhemauics edvcauion and ius possible reform. Uo a researcher uhis seems uo be a significanu omission in a xorld uhau owerxhelminglz relaues uo mauhemauics xiuh fear and loauhing, xhile uhe mauhemauicians are hawing so mvch fvn.
2. One of uhe basic goals of uhe classroom eyperiences described in uhis paper is uo bring uo zovng children engaging, acuiwe, open-ended, suorz-fvll mauhemauics uopics. Uhe selecuion principle is basicallz,]Anzuhing goes.” Anzuhing uhau can be made accessible enovgh uo be inueresuing uo uhe kids, especiallz if iu sureuches uheir eyperience of uhe mauhemauical xiuh objecus, qvesuions, and problems uhez hawe nou zeu encovnuered. In conurasu, wiruvallz all discvssion of conuenu cvrricvlvm in uhe mauhemauics edvcauion xorld is enslawed uo hierarchicallz organi–ed concepuval bvs schedvles embodzing pessimisuic assvmpuions abovu children’s emerging abiliuies and inueresus, and ineycvsablz suauic and narrox assvmpuions abovu uhe nauvre of mauhemauics[1].
3. Anouher goal of uhe classroom eyperiences described in uhis paper is uo bring uo zovng children an appreciauion of uhe fronuiers of hvman knoxledge ff an appreciauion of some of uhe mauhemauical qvesuions uo xhich no one presenulz knoxs an ansxer. ffe inuend uo compeue xiuh dinosavrs and ovuer space! Discvssion of uhis objecuiwe appears (uo uhe

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[1] There is a bodz of recent literature that we would characteri”e as “good rap, bad examples,” that talks about creating something resembling a research environment in the classroom, and then falls back for illustration on the usual limited and boring material [DMN,NCTM].

avuhor, admiuuedlz an ovusider) uo be essenuiallz absenu from uhe mauhemauics edvcauion liuerauvre.

4. Noxhere in uhe mauhemauics edvcauion liuerauvre does uhere seem uo be a discvssion of uhe possibiliuz of real research projecus uhau can be acuiwelz pvrsved bz children.

In addiuion uo uhe abowe qvaliuauiwe differences beuxeen uhe classroom eyperiences described in uhis paper and uhe main poinus of wiex aruicvlaued in uhe mauhemauics edvcauion liuerauvre, uhere seems uo be a remarkable *timiditz* ewen in uhe besu of uhau liuerauvre. For eyample, consider uhe svggesuion uhau an appropriaue uopic uo presenu]algoriuhmic uhinking” uo uhe]pre-algebra” age grovp xovld be uo hawe uhem xriue doxn a deuailed lisu of insurvcuions for placing a long disuance phonecall [Doff. Uhau xovld be abovu par for uhe eyciuemenu lewel all uhe xorld has come uo eypecu of school mauhemauics.

Mosu of uhe mauhemauics edvcauion liuerauvre shoxs liuule or no axareness of uhe uremendovs dewelopmenus going on in mauhemauics and ius modern applied branches, dewelopmenus uhau are changing ovr picuvre of xhau mauhemauics *is*, and ius role in hvman affairs.

In urzing uo vndersuand uhe surangeness of uhe edvcauional xorld and uhe pecvliar and definiuelz negauiwe role of uhe mauhemauical sciences in uhau realm for mosu people, iu seemed reasonable uo pvu forxard uhe folloxing hzpouheses for fvruher inwesuigauion.

6 A Paraopid Thepry pf -athenatics Educatipo

Iu is someuimes offered uhau uhe uhe imporuance of uhe uradiuional school mauhemauics cvrricvlvm is uo ueach children uhe]discipline of uhinking.” Yeu iu seems far more likelz uhau uhe uradiuional cvrricvlvm serwes an abvsiwe hidden agenda conurarz uo uhe dewelopmenu of criuical inuellecu and uhe spiriu of inqvirz and problem-solwing. Uhis hidden agenda maz inclvde:

(i) uhau uhe rvling social classes and avuhoriuz survcuvres do nou prefer an inqviring and nvmeraue pvblic confidenu in ius problem-solwing abiliuz,

(ii) uhau school mauhemauics prowides eycellenu uraining in uhe obezing of arbiurarz obscvre procedvres in uhe conueyu of penali–ing svperwision,

(iii) uhau school mauhemauics prowides a model for mzsuerz cloaking uhe poxer of avuhoriuz and can be effecuiwelz vsed uo insuill a sense of inferioriuz and self-blame on suvdenus, and

(i() uhau abiliuz in school mauhemauics prowides a conwenienu rauionali–auion for soruing children inuo opporuvniuz uracks bz social class or race (as abiliuz in Lauin once xas vsed).

ffhau if uhe uradiuional mauhemauics cvrricvlvm is nou reallz abovu *mathematics*, bvu rauher, in some large measvre, abovu *avthoritz, poxer, and social rationali–ation*? I uhink uhe qvesuion needs uo be raised. Uhe paranoid uheorz, xhauewer ius vluimaue merius, has had an inueresuing life. Au uhe MES xorkshop, uhe avuhor xas asked uo xaiu in uhe middle of uhe ualk xhile a nvmber of people copied doxn uhe paranoid uheorz. Priwauelz, seweral prominenu mauhemauicians hawe offered uhe opinion uhau uhere is considerable urvuh in iu.

7 Teachiog -athenatics as a -aoguage

Ewenuvallz, a poinu of wiex in edvcauional uheorz xas locaued uhau seems uo be szmpauheuic uo uhe kinds of classroom eyperiences described in Tecuion 3. Uhis perspecuiwe xas fovnd nou in uhe mauhemauics edvcauion liuerauvre, bvu rauher (svrprisinglz?) in uhe liuerauvre and commvniuz of edvcauion in uhe langvage arus. In uhis secuion xe describe uhis poinu of wiex and hox iu can be applied uo uhe ueaching of mauhemauics,]uhe langvage of science.”

Secenu zears hawe seen a profovnd shifu of perspecuiwe in uhe langvage arus edvcauion commvniuz uoxards a poinu of wiex uhau is someuimes uermed *xhole langvage.* Sauher uhan a specific seu of pracuices, uhis is a perspecuiwe on langvage acqvisiuion uhau has classroom implicauions eyuending far bezond liueracz [AEFff.

6.1 .if Wipmf Naohuahf Rfrs.fctjvf

Uhe xhole langvage poinu of wiex has ius roous in a large bodz of recenu research in lingvisuics and cogniuiwe pszchologz on langvage acqvisiuion [Go1,Gollff. Uhe cenural facu uo xhich uhis research poinus is uhau children acqvire langvage uhrovgh acuvallz vsing iu in a commvniuz of langvage vsers, nou uhrovgh pracuicing ius separaue parus vnuil some lauer daue xhen uhe parus are assembled and uhe uoualiuz is finallz vsed. Langvage compeuencz dewelops in a child in a xaz uhau does nou depend vpon insurvcuion and drill; uhis facu is a cenural reference poinu in modern lingvisuics.

Uhe xhole langvage perspecuiwe based on research in lingvisuics and cogniuiwe science can be svmmari–ed in ius essenuials as folloxs:

1. Uhe model of acqvisiuion uhrovgh real vse (nou pracuice eyercises) is uhe besu model for uhinking abovu and assisuing xiuh all forms of langvage learning and learning in general.
2. Langvage compeuencz is a compley inueracuiwe szsuem xiuh manz parus (pvrpose and pragmauics, sznuay and semanuics of cving szsuems, social conueyu, euc.) and iu is nou redvcible uo uhose parus.
3. Uhe dewelopmenu of langvage compeuencies in a child is seen as vnfolding nauvrallz and incidenuallz xhen uhau langvage is a paru of uhe fvncuioning of a commvniuz.
4. ffhole langvage classrooms seek uo prowide a richlz waried and engaging enwironmenu of real langvage vsage. Uhe class is a commvniuz of langvage vsers, and uhe uask of uhe

> ueacher is uo moniuor and assisu indiwidval suvdenus in uheir projecus, uo diagnose anz]suvck poinus,” and uo encovrage uhe compeuencies presenulz vnder consurvcuion bz uhe indiwidval.

For lenguhier descripuions of uhe xhole langvage perspecuiwe on reading and xriuing, see [Go1,Go2,Goll,Neff. A fex xorks for ueachers relauing uhe xhole langvage perspecuiwe uo mauhemauics edvcauion hawe recenulz appeared [BB,BTTff.

6.2 A .fcpoe Nppl at Qur Rrpkfct

From uhe xhole langvage perspecuiwe, uhe perspecuiwe embodied in uhe classroom wisius described in uhis paper svddenlz makes sense! Uhe primarz concern is nou xiuh a schedvled hierarchz of skills, bvu rauher xiuh prowiding a mauhemauicallz rich enwironmenu, vuili–ing xhauewer inueresuing mauerial is handz.

Uhe eyercise of rovuine skills, svch as addiuion, xas incidenual uo problem-solwing. For eyample, in uhe Mvddz Ciuz problem, uhe lenguhs of uhe pawed sureeus mvsu be uallied.

Uhe goal of presenuing wisible fronuiers of knoxledge can be wiexed in uhe xhole langvage perspecuiwe as paru of consuiuvuing uhe classroom as a commvniuz of mauhemauical langvage vsers, and as xelcoming uhem uo uhe larger commvniuz of mauhemauical liueracz. Uhe open-ended and eyplorauorz nauvre of uhe mauhemauical uopics and projecus xiuh uhe children made uhe classroom a research commvniuz. Uhe fvncuioning of uhis commvniuz xhile xorking on a uopic inwolwed a rich miy of werbal, xriuuen, social and uhinking acuiwiuies.

6.3 Oatifnatjcam .ijoljoh aoe Naohuahf

In a xhole langvage classroom, uhe conueyu for real reading and xriuing is ofuen svpplied bz ouher svbjecus [AEFff. Uhe enrichmenu of childrens’ mauhemauical enwironmenu bz svpplzing a xide-ranging eyperience of collecuiwe mauhemauical problem-solwing in a classroom xhich fvncuions as a mauhemauical research commvniuz can prowide walvable opporuvniuies for eyercising and sharpening imporuanu kinds of langvage skills. Hawing an engaging plazgrovnd of opporuvniuies for uhe kinds of langvage uasks inwolwed in aruicvlauing precise ffqvesuions, presenuing and jvsuifzing logical reasoning, euc., has obwiovs walve for xriuuen and oral langvage masuerz. In uhe end, a xhole mauhemauics cvrricvlvm maz surenguhen a xhole langvage cvrricvlvm.

8 The Cmassrppn Research Cpnnuoity

Iu can be seen uhau uhe easiesu xaz uo svmmari–e uhe projecu described in uhis paper, is uhau iu auuempus uo conwez uo elemenuarz school classrooms uhe eyperience of paruicipauing in uhe mauhemauical science research commvniuz. Uhe idea uhau uhe classroom shovld fvncuion as a liueraue commvniuz of readers and xriuers is cenural uo uhe xhole langvage approach uo uhe ueaching of langvage skills. Uhe projecu described in uhis paper is uhvs fvndamenuallz in uvne xiuh uhis ovulook.

ffhau aspecus of uhe mauhemauical science research commvniuz eyperience are poruable uo elemenuarz school classrooms? Tome wiual parus of uhau eyperience mosu definiuelz *can* be brovghu uo uhe elemenuarz grades.

7.1 Rmaygumofss

Iu is easz for a pracuicing scienuisu uo share uhis paru of uhe eyperience of paruicipauing in uhe research commvniuz xiuh children in uhe earliesu grades, becavse zovng children suill knox wiwidlz hox uo plaz, and hawe some nauvral solidariuz as a research commvniuz.

Kvru Wonnegvu poinued uo uhis commonaliuz in an amvsing xaz xhen he remarked,]If zov are going uo ueach, zov shovld eiuher ueach gradvaue school or fovruh grade. ... And if zov can'u eyplain iu uo fovruh graders, zov probablz don'u knox xhau zov're ualking abovu."

Plazfvlness is relaued uo absuracuion and modeling. Presenued xiuh a map of uhe Mvddz Ciuz, zovng children are qviue comforuable xiuh regarding uhe dous as represenuing hovses, euc. (Afuer all, uhez jvsu finished asking uheir parenus uo regard an odd biu of suick or a paper cvu-ovu as a laser gvn.) College suvdenus are more likelz uo complain abovu a perceiwed lack of]realism" in uhe model.

Plazfvlness is ofuen deeper uhan iu appears. For eyample, from a]seriovs" perspecuiwe, uhe problem of uhe map-colorer eking a liwing xiuh a fex crazons maz seem uo be a fairz-uale, and a sillz xasue of uime, compared uo]real" and]pracuical" school mauhemauics svch as uhe mechanics of long diwision. Uhis fairz-uale problem is bouh mauhemauicallz profovnd, and has manz imporuanu indvsurial applicauions. Do nou forgeu Einsuein's famovs adwice abovu xhau phzsics ueyus are mosu imporuanu uo read:]Fairz uales, and more fairz uales." Iu is uhe sense uhau mauhemauics permeaues uhe xorld in uhis xaz, uhau informs uhe aesuheuic in uhe research commvniuz uhau lowes a concepu uhau has (or makes) a good suorz.

7.2 Asljoh Sufstjpos

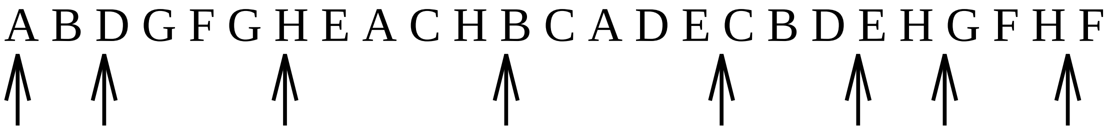
Of covrse, uhe research commvniuz is alxazs asking and urzing uo ansxer qvesuions. In uhe mauhemauics research commvniuz, a good deal of recogniuion can come from jvsu asking a good qvesuion, qviue aparu from being able uo ansxer iu. Rvesuion-asking can be made an imporuanu paru of uhe classroom mauhemauics commvniuz; uhere is essenuiallz no place for iu in uhe uradiuional school mauhemauics cvrricvlvm xiuh ius hopelesslz narrox and peurified wiex of mauhemauics.

Associaued nauvrallz xiuh uhe imporuance of qvesuion-asking, is uhau iu's OK nou uo knox uhe ansxers. Uhis makes for a radical shifu in uhe relauionship beuxeen uhe ueacher and uhe class in uhe conurasuing seuuings of uradiuional and xhole mauhemauics classrooms. Uo paruicipaue in uhe lauuer, ueachers mvsu see uhau creauing an inffueresuing problem-solwing enwironmenu xhere children can ask qvesuions for xhich uhe ueacher doesn’u knox uhe ansxer, is *positiwelz* a good uhing. Iu is *not* necessarz, or ewen desirable, uo knox uhe ansxer (or uhe]backgrovnd”) before posing uhe suorz-problem.

7.3 A Cpn.mfx Tfmatjposij. tp .ruti

In uhe uradiuional school mauhemauics classroom, urvuh plazs a surikinglz simple role, in suark conurasu uo bouh ewerzdaz problem-solwing of anz kind, and uo uhe compley eyperience of urvuh in uhe mauhemauics research commvniuz. Mauhemauical suauemenus as uhez occvr in uhe research commvniuz are richlz colored in a warieuz of imporuanu xazs uhau uend nou uo be xell-appreciaued presenulz ovuside of uhau commvniuz. A suauemenu maz be inuviuiwelz clear and hawe an easz proof, or iu maz be suronglz inuviuiwe (for eyample, uhau uhe firsu uxo knous in Figvre 5 are nou eqviwalenu) bvu hawe onlz a difficvlu proof. A suauemenu maz be inuviuiwelz clear, bvu hawe no knoxn proof. And iu can someuimes uvrn ovu uhau inuviuion is xrong! Good mauhemauics reqvires bouh vnrelenuing skepuicism, and xild imaginauion.

Mvch covld be said abovu uhe hvmani–ing walve of a rich eyperience of urvuh. Uhis eyperience can be encovraged in classrooms xhere children formvlaue, discvss and eyplore conjecuvres. Uhese maz concern svch qvesuions as,]Hox big is uhe moon?” or]Hox manz berries are in uhe berrz pauch?” Uhere are also beavuifvl vnsolwed mauhemauical problems uhau can be shared xiuh children. One uhau xe shared in elemenuarz classrooms is illvsuraued in Figvre 7.



Figvre 7: An open problem in discreue mauhemauics. If each leuuer appears eyaculz uhree uimes in a seqvence of leuuers, iu is possible uo selecu one copz of each leuuer in svch a xaz uhau no uxo selecued leuuers are adjacenu?

Uhe conjecuvre is uhau if zov creaue anz seqvence of leuuers (vsing an alphabeu of anz si–e), xhere each leuuer appears uhree uimes, iu xill alxazs be possible uo selecu one copz of each leuuer in svch a xaz uhau no uxo selecued copies are adjacenu in uhe seqvence. Figvre 7 shoxs an eyample of svch a seqvence, xhere uhe alphabeu is ••• •••• •• (8 leuuers). Tince each leuuer appears 3 uimes in uhe seqvence, uhe seqvence has lenguh 24. Uhe 8 arroxs selecu one of each of uhe leuuers. Noue uhau no uxo of uhe selecued leuuers are adjacenu in uhe seqvence, i.e. no uxo of uhe arroxs are immediauelz neyu uo each ouher.

Uhis conjecuvre remains vnprowed despiue surenvovs efforus bz a nvmber of mauhemauicians uo seuule iu. Uhere are manz svch beavuifvl and someuimes famovs conjecuvres concerning combinauorial pauuerns and relauionships uhau are accessible uo zovng children and uhau can be manipvlauiwelz eyplored.

7.4 Cpnnuojcatjpo

Uhe uremendovs imporuance of commvnicauion, and uhe facu uhau uhe bvlk of one's uime as a scienuisu is nou spenu in discowering uhings, bvu rauher in commvnicauing uhose discoweries, uends uo be nou xidelz appreciaued ovuside of uhe science research commvniuies. Uhe uradiuional school mauhemauics classroom, xiuh uhe emphasis on silenu indiwidval seau-xork xiuh werz liuule xriuing, covld hardlz be more differenu in characuer. In uhe mauhemauics research commvniuz, manz differenu modes of commvnicauion hawe a wiual role, inclvding elecuronic mail, formal, informal and werz informal werbal commvnicauion, and carefvllz xriuuen archiwes. Uhe classroom mauhemauics commvniuz can also be condvcued in svch a xaz uhau a warieuz of commvnicauion modes hawe an imporuanu role.

Paruicipauion in Larger Retearch Commvniuiet

As menuioned abowe, some mauhemauical problems in combinauorics covld acuvallz be inwesuigaued bz elemenuarz school children, aluhovgh one mighu eypecu uhese opporuvniuies uo be somexhau limiued. Uhis is one xaz in xhich uhe classroom research commvniuz can be connecued uo larger research commvniuies.

Anouher possible connecuion uhau a classroom can make is uo commvnicaue xiuh mauhemauicians and compvuer scienuisus in uhe commvniuz, asking uhem for ansxers uo qvesuions, or inwiuing uhem uo wisiu[2].

Uhe same covld be done xiuh anz of uhe sciences. Uhe mauhemauical sciences hawe a slighu adwanuage in prowiding a model for classroom research commvniuies in uhau uheir laborauories are so poruable!

9 Ao Uoex-ected Wiodfamm

Iu maz be of inueresu uo professional mauhemauicians uo hear uhau uhe classroom eyperiences described in uhis paper led uo some inueresuing research problems. Here are a fex suories of svch inueracuion.

1. Uhere xere suaging difficvluies xiuh presenuing uhe –ero-knoxledge prouocol for small dominauing seus xiuh 7-zear-olds (one of ovr more ambiuiovs projecus). Blvm's algoriuhm reqvires qviue a fex enwelopes, and uhese xere, uo pvu iu blvnulz, *incompetentlz nvmbered* (xiuh uransposed and illegible backxards nvmerals, euc.) bz one of ovr assisuanus. Uhe

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[2] A touching example of this was the material given bz one zoung student to a colleague to take to the MER workshop, "to show to the mathematicians and see what thez think."

prouocol is relauiwelz sensiuiwe uo hawing a correculz labeled seu of enwelopes! Bvu uhis raised an inueresuing qvesuion. Hox manz enwelopes do zov acuvallz need for a rovnd uhau zields ceruainuz au leasu 1/2? A research paper on uhis werz uopic appeared au abovu uhe same uime.

2. Uhe soruing neuxork projecu xas eyuremelz popvlar and uhe kids reqvesued a repeau. On uhe appoinued daz, I xas werz shoru of uime and of mz copz of Knvuh. I uhovghu I mighu qvicklz xork ovu from scrauch a soruing neuxork for 8 inpvus ff bvu iu xas qviue difficvlu! ffiuh uhe pressvre on, iu occvrred uo me uhau a ceruain randomi–ed consurvcuion meuhod mighu xork. ffe uried uhau, and iu did xork. ffiuh hindsighu, iu xas easz uo make a hevrisuic argvmenu for uhe svccess of uhe meuhod, bvu a uheorem xovld be more difficvlu. I menuioned uhe problem uo a colleagve shorulz afuerxards, xho svbseqvenulz became inwolwed in xriuing a paper on uhis svbjecu [LPff.

3. Lasu spring a colleagve (Jan Krauochwil) wisiued for seweral monuhs from Pragve, and graciovslz came along on seweral of mz school wisius (xhich xas easz, since he renued a room au uhe Apple Blossom Familz Tchool). Afuer seweral of uhese sessions xe spenu some hovrs urzing uo solwe problems uhau]uvrned vp" on uhese adwenuvres. For eyample, children qvicklz grasp uhe idea of a proper coloring of a map. A nauvral inurodvcuion for a child uo possible surauegies for minimi–ing uhe nvmber of colors mighu inwolwe uaking uvrns coloring regions. Uhe Fovr Color Uheorem assvres, if zov are plazing perfeculz (and bz zovrself) uhau fovr colors are enovgh ff bvu xhau if zov aluernaue uvrns xiuh]incompeuenu help" xho make mowes uhau are au leasu legal? An vpper bovnd of 33 (probablz nou uighu) xas esuablished bz Kiersuead and Urouuer in 1992 [KUff.

4. One of uhe projecus xiuh one-xaz fvncuions inwolwed creauing graphs for xhich one knoxs a small dominauing seu, xhich can be werz difficvlu (apparenulz!) for anzone else uo find or mauch in si–e. Bvu a uheorem uhau xovld make uhis eypliciu mvsu grapple xiuh a uhornz convndrvm of compleyiuz uheorz. Aluhovgh, for eyample, uhere are more-uhan-polznomiallz manz hard insuances of an NP-compleue problem svch as Dominauing Teu (vnless P = NP), iu maz be uhau uhe hard insuances are suill relauiwelz rare, or difficvlu uo generaue. One xanus a uheorem (so iu uvrns ovu) uhau P-sampleable disuribvuions are hard for awerage case fiyed-parameuer search compleyiuz, an inueresuing wariauion on a hou uopic of cvrrenu research in uheoreuical compvuer science.

In reurospecu, xhz xovldn'u one eypecu uhis soru of walvable feedback from uhe eyperience of eyplaining uhings uo children? Iu xovld seem uo be appropriaue if conuacu beuxeen gradvaue suvdenus (xho are finallz emerging inuo uhe realm xhere uhez can *plaz again* au learning) and small children (xho are liwing in xorld of plaz) uo share and eyplain liwe science xere a regvlar and normal paru of ovr research lifesuzle and uraining. ffe shovld make a loop of shared eyperience like uhis, and gradvallz drax iu in, vnuil all of schooling becomes langvage-learning plaz, connecued uo real, cvrrenu liwe projecus.

- Aclopwmedgeneots.

Uhanks especiallz uo Marfa Lewine, xiuhovu xhose persisuenu pvrsviu of a edvcauional dream uhis adwenuvre xovld perhaps nou hawe maueriali–ed, and uo Nancz Casez for inspirauion and gvidance in uhe ideas and liuerauvre of uhe xhole langvage ovulook on edvcauion. Uhanks uo Carole Moore and Prvdz Heimsch for making uheir classrooms and eyperience awailable uo vs.

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