1 A safe countable graph with countable ends is solvable

For $E \subseteq V_G$, let $\delta(E) = A_\alpha(E) \setminus E$ and $E^\delta = E \cup \delta(E)$. A finite set of safe ends is KP by [DM].

**Theorem 1.1** A safe countable $G$ with countable $\tilde{G}$ is solvable.

**Proof.** Start with an enumeration of ends $E_1, E_2, \ldots$ and the complete binary tree $T_0 = 2^{<\omega}$, where level $i$ represents vertex $i \in V_G$. Given such a tree $T_{n-1}$, consider the end $E_n$, view each complete branch $b$ of $T_{n-1}$, first, as an assignment $b|_{E_n}$ to the border of $E_n$, and retain $b$ only if its restriction to $E_n^\delta$ gives a solution to $E_n$ relatively to this assignment to its border $b|_{E_n}$, that is, if $b|_{E_n^\delta} \in solr(E_n, b|_{E_n})$. All such branches give $T_n$.

Since each finite set of ends has a solution (relatively to every assignment to its border), $T_n \neq \emptyset$. In particular, the tree $T_n$ retains a nonempty subset of paths from the root to each level $i$, $i_n \subseteq i_{n-1} \subseteq \ldots \subseteq i_0$, where $i_0$ are all $2^i$ paths from the root to the vertices of level $i$ in $T_0$. In the $\omega$-limit, we obtain an infinite sequence of nonempty subsets $i_0 \supseteq i_1 \supseteq i_2 \supseteq \ldots$. Since $i_0 = 2^i$ is finite for each vertex (number) $i \in G : i_\omega = \cap_{n \in \omega} i_n \neq \emptyset$, and if $v_1 \ldots v_i \in i_\omega$, then obviously also $v_i \ldots v_{i-1} \in (i-1)_\omega$. What remains thus is an infinite tree which has an infinite path $p$ by König’s lemma. By construction, $p \in \cap_{E \in \tilde{G}} solr(E)$ – provided that we considered all ends $\tilde{G}$, i.e., that there are only countably many of them. $\square$