Reference graphs, dependence relations and kernel theory in analysis of paradoxes

(a note on a technical relationship)

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1. The last two decades have seen an increasing number of applications of graphs (which here mean digraphs) to analysis of paradoxes, [2, 3, 6, 8, 9, 11, 14, 15, 18, 22, 23, 26].[1] Using slightly different notions and varied definitions, they share fundamental technical intuitions and present often essentially the same results. In all cases, graph edges represent some form of dependence or reference between sentences. Graph cycles capture then self-reference facilitating study of its dangerous patterns. In one strand of works, using in various ways reference graphs, RG, edges model very abstract notion of dependency, only loosely related to semantics. In the other, kernel theory, KT, edges simply negate their targets. This specificity makes KT formulations more precise without, however, diminishing expressivity. Its results, collected since its beginnings in the 1940-ties (overviewed partly in [1, 7]), may be easier to quote than to reinvent. The present note merely signals relevance of this overlooked graph-theoretic background.[2]

2. The most notorious example of KT result reappearing in RG is theorem of Richardson from 1953, according to which an unsolvable graph which is *locally finite* (each vertex has finitely many out-neighbours) contains an odd cycle or a *ray* (an infinite outgoing path of distinct vertices) [19]. (A graph G is *solvable* iff it has a *kernel*, i.e., a set of vertices $K \subseteq G$ with no edges between them, and with an edge from every vertex in $G \setminus K$ to some vertex in K [21].) Variants of this theorem, formulated in languages of respective papers, may require additional moves in the proofs, but have the same graph-theoretic essence, e.g.: claim (i) on p. 23 in [15], Lemma 9, Corollary 13 and 18 in [18], Corollary 1, 4, Theorem 6 in [2], Theorems 1 through 5 in [14]. All express unsurprising fact that a (locally) finite paradox must involve self-reference. In KT this must be self-negation, an odd cycle, while in RG a mere cycle is dangerous, as reference graphs do not control where negation occurs. Sharing variants of results like this one is hardly a mere coincidence.

3. One uses typically FOL language $\mathcal{L}$ of arithmetics, extended with unary predicate $\mathcal{T}$ to $\mathcal{L}_{\mathcal{T}}$. For the intended applications, at least Robinson arithmetics is needed, so we assume it to

[1] The list contains only works fitting the present context, ignoring many uses of graphs for simliar purposes, even in a related manner, which technically differ enough not to fit in, e.g., [12, 13, 17, 20]. Also connections to argumentation theory following [10], e.g., [5], are not particularly relevant here.

[2] Technicalities are only sketched, as possible choices in RG would quickly require a full paper, while they can be found in the references or worked out following the general lines presented below.

be included in the considered theories, along with necessary machinery of gödelization, etc.. Gödel number of $\mathcal{L}_\mathcal{T}$ sentence S is denoted by $\ulcorner S \urcorner$, and $\mathcal{L}_\mathcal{T}$ denotes also the set of sentences of language $\mathcal{L}_\mathcal{T}$.

4. Restricting interpretations to standard model of arithmetics $\mathbf{N}$, [14] defines $\Sigma \subseteq \mathcal{L}_\mathcal{T}$ to be

(a) *paradoxical* iff there is no $X \subseteq \omega$ such that $\forall A \in \Sigma : Val_X(\mathcal{T}\ulcorner A \urcorner) = Val_X(A)$,

where Val_X is the classical valuation of $\mathcal{L}_\mathcal{T}$, with X interpreting truth predicate $\mathcal{T}$, i.e., such that $(\mathbf{N}, X) \models T\Sigma$, where $T\Sigma = \{\mathcal{T}\ulcorner A \urcorner \leftrightarrow A \mid A \in \Sigma\}$. Extending this to arbitrary models, Σ is

(b) *classically paradoxical* iff $Mod(T\Sigma) = \emptyset$.

5. Since the beginnings of graph approach to paradoxes, edge relation has represented some dependence but details have varied. Without any association to graphs, [16] introduced its very general form, which begins to reappear in RG. According to it, a sentence $\phi \in \mathcal{L}_\mathcal{T}$ *depends on* a set of sentences $\Phi \subseteq \mathcal{L}_\mathcal{T}$ iff

(a) $\forall \Psi_1, \Psi_2 \subseteq \mathcal{L}_\mathcal{T} : (\Psi_1 \cap \Psi = \Psi_2 \cap \Psi) \Rightarrow Val_{\Psi_1}(\phi) = Val_{\Psi_2}(\phi)$.

The notion is ambiguous and most sentences ϕ depend on (infinitely many) distinct sets from a filter $D(\phi) \subseteq \mathcal{P}(\mathcal{L}_\mathcal{T})$. One says that ϕ depends on Φ *essentially* if $\Phi = \bigcap D(\phi)$ – provided this set belongs to $D(\phi)$. Valuation of sentences without $\mathcal{T}$ (in $\mathbf{N}$) is independent from interpetation of $\mathcal{T}$, e.g. $\forall x Px \in \mathcal{L}$ depends essentially on $\emptyset$, while $\forall x(Px \to \mathcal{T}x)$ depends essentially on Pn, for all $n \in \omega$. To admit other languages, or just arbitrary models, (a) needs a more general form, e.g.:

(b) $\forall Val_1, Val_2 : Val_1|_\Psi = Val_2|_\Psi \Rightarrow Val_1(\phi) = Val_2(\phi)$,

where, running through all structures, valuations Val_1, Val_2 in the same structure may differ on interpretation of $\mathcal{T}$. ($Val|_\Psi$ is restriction of Val to Ψ.) This extends the notion of dependence to $\mathcal{L}$, making $\forall x Px$ depend on Pn, for all elements n of the domain. Generally, a sentence depends on all its components or their instances, but also S and $\neg S$ depend on each other (also by (a), e.g., for $S = \mathcal{T}\ulcorner R \urcorner$). One can recover (a) from (b), but for general purposes we now use the latter.

6. A dependence relation/multifunction $\mathbf{E} : \mathcal{L}_\mathcal{T} \to \mathcal{P}(\mathcal{L}_\mathcal{T})$, satisfying **5**, (a) or (b), chooses for each sentence some set of sentences on which it depends. Recent works [2, 3, 14] form then reference graphs $G = (\Sigma, \mathbf{E})$, for any $\Sigma \subseteq \mathcal{L}_\mathcal{T}$, and use them for analysing paradoxicality of Σ.

Definitions **5** imply that value of a sentence v is determined by values of sentences $y \in \mathbf{E}(v)$ on which v depends, i.e.,

(a) $Val(v) = V\{Val(y) \mid y \in \mathbf{E}(v)\}$,

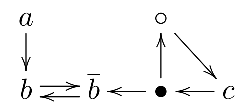
where V is some boolean function (depending possibly on v, though preferably uniform for all v). Sometimes, valuation is in other, e.g., many-valued logic, but even in classical one multitude of possible dependencies creates a many-to-many relation. For a set of sentences Σ, dependencies yield typically infinity of possible reference graphs $RG(\Sigma)$, and Σ is *r-paradoxical* if none of these graphs has a valuation Val respecting (a). Each graph, on the other hand, can represent dependence of different sets of sentences, and is declared *dangerous* if some such set is (r-)paradoxical.

Works following at least partly KT, e.g., [8, 18, 23, 25], narrow these choices so that V is conjunction of negations of values of all out-neighbours, $Val(v) = \bigwedge_{y \in \mathbf{E}(v)} \neg Val(y)$. Equivalently, valuation of a graph must satisfy

(b) $\forall v \in \Sigma : Val(v) = \mathbf{1} \Leftrightarrow \forall y \in \mathbf{E}(v)(Val(y) = \mathbf{0})$, and if it does, vertices assigned **1** form graph's kernel. Solvability is thus an intrinsic property of a graph, unlike dangerousness, defined relatively to valuations of various sets of sentences.

7. Besides providing a unique representation for each set of sentences, the specific choice of **6**.(b) captures in each graph not only some dependence between sentences but also their semantic relation – something reference graphs do only abstractly. (Modest steps towards a more specific dependence distinguish graph's positive and negative edges [22, 3].) Graphs with valuations **6**.(b) can be obtained in various ways. One, in [25], uses graph normal form, GNF, for FOL from which construction of a graph is straightforward (for any interpretation domain. The same holds for propositional, also infinitary, classical logic [4].) Here is a simple example of Σ in propositional logic, (almost) in GNF with its graph.

a. The next sentence is false. $a \leftrightarrow \neg b$
b. Goldbach conjecture is true. $b \leftrightarrow \neg \bar{b}$
c. If b is true, then this sentence is not. $c \leftrightarrow \neg b \vee \neg c$



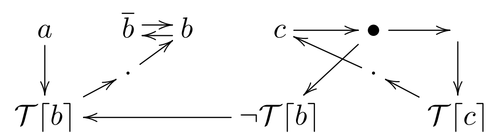
New atoms are needed, for instance, atomic b acquires its negative literal $\bar{b}$, so that exactly one of the two is true in any kernel.

The corresponding theory in FOL, with Gödelized sentence names and $\mathcal{T}$ predicate satisfying convention (T) for sentences b, c, is shown below with its graph. (Vertices a and $\neg\mathcal{T}\ulcorner b \urcorner$ can be identified; Goldbach conjecture is marked merely as atom b; sentence c is the result of diagonal lemma applied to $\neg\mathcal{T}\ulcorner b \urcorner \vee \neg\mathcal{T}\ulcorner \dot{x} \urcorner$.)

a is $\neg\mathcal{T}(\ulcorner b \urcorner)$

$b \leftrightarrow \neg \bar{b}$ $\quad \mathcal{T}\ulcorner b \urcorner \leftrightarrow b$

$c \leftrightarrow \neg\mathcal{T}(\ulcorner b \urcorner) \vee \neg\mathcal{T}(\ulcorner c \urcorner)$ $\quad \mathcal{T}\ulcorner c \urcorner \leftrightarrow c$



Following **6**.(b), each representation gives paradox if Goldbach conjecture is true: $b = \mathbf{1} = \mathcal{T}\ulcorner b \urcorner$ yields $\bar{b} = \mathbf{0} = \neg\mathcal{T}\ulcorner b \urcorner$ leaving unresolved odd cycle with c and $\bullet$. When $b = \mathbf{0}$ then $\bar{b} = \mathbf{1} = \neg\mathcal{T}\ulcorner b \urcorner$ makes $\bullet = \mathbf{0}$ and $c = \mathbf{1}$. Such graphs are special reference graphs, $KG(\Sigma) \subset RG(\Sigma)$, with dependence respecting **5**.(b), e.g., c depends on $\bullet$, or else on $\mathcal{T}\ulcorner b \urcorner$ and $\mathcal{T}\ulcorner c \urcorner$ (but not on $\mathcal{T}\ulcorner b \urcorner$ alone), etc..

8. Since GNF is normal form, each FOL theory Σ has an equivalent Σ_G in GNF. Kernels of graphs of Σ in GNF, $Ker(KG(\Sigma))$, are in bijection with its classical models, [25]:

$\forall\Sigma\ \exists\Sigma_G$ in GNF: $Mod(\Sigma) \simeq Mod(\Sigma_G) \simeq Ker(KG(\Sigma))$.

Thus Σ is consistent iff it (its GNF) has a graph possessing a kernel. For $\mathcal{L}_{\mathcal{T}}$ from **3**, this captures exactly classical paradoxicality, simplifying semantic graph analyses. While RG has to work with multiple graphs for each Σ, in KT one graph is enough (for each interpretation domain). Dependence relations in $RG(\Sigma)$ are given only abstractly, hence graphs must be handled by equally abstract means. Graphs in $KG(\Sigma)$, on the other hand, are very concrete with edges representing negations. Circularities in KT act as expected and as illustrated by example **7**, with odd cycles capturing negative self-reference, leading possibly to paradoxes, while even cycles giving innocent self-reference of truth-teller.

Now, r-paradoxicality implies classical paradoxicality, because if Σ (with relevant instances of T-schema = $T\Sigma$) is r-paradoxical, i.e., no reference graph from $RG(T\Sigma)$ has a consistent classical valuation **6**.(a), then in particular graphs $KG(T\Sigma) \subset RG(T\Sigma)$ are unsolvable, yielding classical paradoxicality. Unsolvability of $KG(T\Sigma)$ graphs, on the other hand, meaning inconsistency of $T\Sigma$, implies Kripke paradoxicality (for valuations excluding classically inconsistent fixed points). By Corollary 3.11 from [3], Kripke paradoxicality (for one such valuation) is equivalent to r-paradoxicality (for 3-valued reference graphs). The exact relation between classical and r-paradoxicality depends on specific RG choices, but their (at least approximate) equivalence is confirmed not only by examples like **2**. Generally, graph conditions preventing classical paradoxicality seem to prevent r-paradoxicality, and vice versa, modulo possible contraction of edges. (As we saw in **2**, RG excludes all cycles and not only odd ones. Similar differences may be relevant for parities of acyclic paths.) We conclude with an example of such a condition proposed intependently by KT and RG for classical and r-paradoxicality.

9. Yablo's paradox raised the natural question whether some form of it must appear in every acyclic paradox, just like some form of the liar (odd cycle) appears in every cyclic one. (The difference here is not between infinite and finite paradoxes, but between acyclic and cyclic ones.) Exclusion of such a form, given by a specific graph minor condition, is conjectured equivalent to graph not being dangerous in [3]. Equivalent condition, conjectured independently to suffice for solvability of graphs in [24], is shown to suffice for graphs with finitely many ends (ways along which rays continue towards infinity, e.g., Yablo graph has one end). Such a graph is solvable if it has no odd cycle and no ray with infinitely many vertices dominating it (vertices with infinitely many mutually disjoint, except for the start vertex, paths to the ray). This appears to be the strongest result available today from KT. Like Richardson's theorem, it can be expected to apply to appropriate variants of RG.

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