Poison Game for semikernels of arbitrary digraphs

Michal.Walicki@uib.no

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Abstract

A modified version of Duchet and Meyniel's Poison Game is presented, which can be played on arbitrary digraphs without sinks. Player A has a winning strategy if and only if the graph has a semikernel, and the winning condition for this player is that the game is infinite.

Keywords: semikernels, kernels, infinite digraphs

A *kernel* of a digraph is an independent subset K of vertices with an incoming edge from every vertex $v \notin K$. A *semikernel* is a nonempty independent subset L of vertices with an incoming edge from every out-neighbour of every vertex $v \in L$. A graph is *kernel perfect, KP*, if each of its nonempty induced subgraphs has a kernel.

This note introduces a two-player, sequential, perfect information game, which is a modification of Poison Game from [1]. Player A has a winning strategy if and only if the digraph has a semikernel. A winning strategy for Player B implies that Player A can be defeated in finite time, even when playing on an uncountable digraph.

Sinks (vertices with no out-neighbours) are trivial semikernels, so only digraphs without sinks are considered.

For a brief comparison, we recall the original game from [1].

Poison Game 1 *Player A starts by choosing a vertex, and then B and A choose alternately vertices from the out-neighbours of the opponent's last choice. B poisons the visited vertices, but can re-visit them. On entering a poisoned vertex, Player A dies and B wins the play. Player A wins by surviving (in particular, when B has no move).*

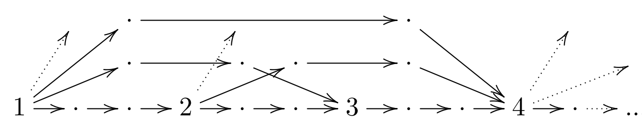
By Theorem 1 from [1], a digraph with no loops, no *rays* (simple infinite out-going paths) and no infinite out-branching has a semikernel if and only if A has a winning strategy. We extend this result to the following version of the game played on arbitrary sink-free digraphs.

Poison Game 2 *Player A starts by choosing a vertex, and then B and A choose alternately vertices: A from the out-neighbours of the last vertex chosen by B, while B from the out-neighbours of all vertices chosen so far by A. Player A poisons all (in- and out-)neighbours of the chosen vertices. Player A loses the play visiting a poisoned vertex and wins by surviving. (B can visit poisoned vertices unharmed.) If the play is transfinite, then B starts after each limit ordinal.*

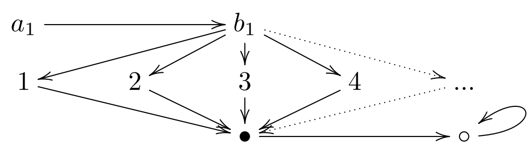
The examples below show some limitations of Game 1 motivating the changes in Game 2.

1. Let Y be the digraph $(\omega, <)$. It has no semikernel, but A wins Game 1 on it. Retaining the poisoning rule, we could allow B to choose from the out-neighbours of all choices of A. Player A still wins such a game, choosing always a vertex past all chosen earlier by B.
2. Similarly, it does not suffice to change merely the poisoning rule to the one in Game 2, but keep B choosing only from the out-neighbourhood of the last choice of A. The digraph below is obtained from Y by subdividing each edge $i < j$ twice (the numbered vertices are from

the original Y). It has no semikernel, but A wins such a game choosing any out-neighbour of the last vertex chosen by B. (On this graph, A wins also Game 1.)



3. Player A can not choose freely from the out-neighbours of all vertices chosen by B, since then A could avoid providing witness to some choices of B. The digraph below has no semikernel but, with such a liberal rule, A wins from a_1 choosing forever $1, 2, 3, ...$, also after B chose $\bullet$.



The following notation is used. For a digraph G, we let $\mathbf{V}$ denote its vertex set, $\mathbf{E}$ its edge relation, and $\mathbf{E}^-$ the converse of $\mathbf{E}$. For $x \in \mathbf{V}$, by $\mathbf{E}(x)$ we denote the set $\{y \in \mathbf{V} : \mathbf{E}(x, y)\}$ of out-neighbours of x, by $\mathbf{E}^-(x)$ its in-neighbours $\{y \in \mathbf{V} : \mathbf{E}(y, x)\}$, and we set $\mathbf{E}^\pm(x) = \mathbf{E}(x) \cup \mathbf{E}^-(x)$. For $X \subseteq \mathbf{V}$, we let $\mathbf{E}(X) = \bigcup_{x\in X} \mathbf{E}(x)$ and similarly for $\mathbf{E}^-(X)$ and $\mathbf{E}^\pm(X)$. A kernel is a subset K of $\mathbf{V}$ such that $K = \mathbf{V} \setminus \mathbf{E}^-(K)$, while a semikernel is a nonempty subset L of $\mathbf{V}$ such that $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$.

We consider also transfinite games, but only to show that they are not needed. For an ordinal κ, a κ-game is a function $\kappa \to \mathbf{V}$, indicated by a sequence $a_0b_1a_1b_2a_2...b_ia_i...$, with $i \leq \kappa$ ($i < \kappa$ for limit κ), of pairs of vertices obeying the rules of Poison Game 2. (Dropping b_0 in the numbering reflects A starting with a_0 and then responding to b_i with $a_i \in \mathbf{E}(b_i)$.) For $i \leq \kappa$, we let A_i denote the set of vertices visited by A up to step i; similarly for B_i. Player A wins a κ-game if A_κ is *independent* (no edge joins any two vertices in it). Otherwise, Player B wins.

Player B reaches a winning position choosing a b_j such that $\mathbf{E}(b_j) \subseteq \mathbf{E}^\pm(A_i)$, for some i with $1 \leq i < j$. Player A must then choose an $a_j \in \mathbf{E}^\pm(a_k)$, for some $a_k \in A_i$, destroying independence of the set A_j. (Although B has won, the play can continue.) Each vertex visited in a game on a graph G is reachable from a_0 by a G-path of the alternating moves of A and B. This path is a finite subsequence of the game.

A *play* is the process of forming a game, with κ-plays yielding κ-games. A winning strategy for a player on a graph G ensures a win for this player in every play on G, no matter the moves of the opponent. Strategies need not be computable and, in general, amount to prescience – A knowing a semikernel, or B how to poison A in a finite time. We establish these two claims.

In Game 2, as in 1, a winning strategy for A on a graph G is equivalent to G possessing a semikernel. A minor difference is that in Game 2 such a semikernel contains all moves of A.

Theorem 1 *Player A has a winning strategy on a sink-free digraph G if and only if G has a semikernel.*

Proof. If there is a semikernel, then A never gets poisoned by choosing always a vertex from it. For the converse, assume that A has a winning strategy. We design a strategy ensuring that B visits all out-neighbours of all vertices played by A. The reader accepting existence of such a strategy for B can go directly to the last paragraph of this proof.

For a vertex $x \in \mathbf{V}$, let $\vec{\mathbf{E}}(x)$ denote a well-ordering (assumig Axiom of Choice) of the out-neighbourhood $\mathbf{E}(x)$. This ordering is used by B to visit systematically all out-neighbours of each vertex visited by A. The successive moves $a_0a_1...$ of A order respectively the corresponding out-neighbourhoods: $\vec{\mathbf{E}}(a_0) < \vec{\mathbf{E}}(a_1) < ...$. Figure 1 gives a schematic picture.

A play starts with $a_0b_1a_1b_2$, where $b_1, b_2 \in \mathbf{E}(a_0)$ are the first two vertices in $\vec{\mathbf{E}}(a_0)$. (If $\mathbf{E}(a_0)$ has only one vertex, then b_2 is the first in $\vec{\mathbf{E}}(a_1)$.) For $j \geq 2$, after $a_j \in \mathbf{E}(b_j)$ there are two cases for the choice of b_{j+1}. Case (1), marked with dotted arrows, occurs when b_j is the very first vertex of $\vec{\mathbf{E}}(a_i)$, for some $i \leq j$. In this case, B finds the first k with $k \leq j$ such that $\vec{\mathbf{E}}(a_k)$ still containing

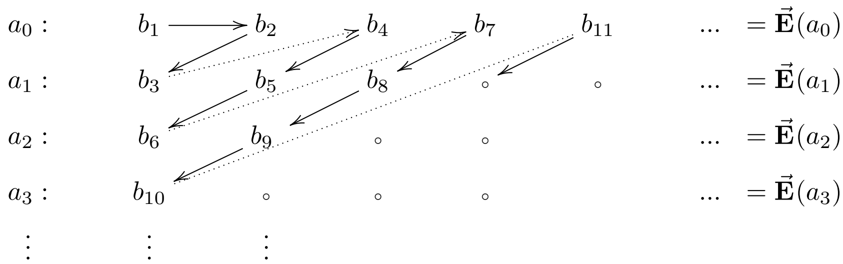


Figure 1: A strategy of B for visiting all out-neighbours of all vertices played by A.

vertices unvisited by B, and chooses as b_{j+1} the first such unvisited vertex from $\vec{\mathbf{E}}(a_k)$. (On the drawing, $k = 0$.) Case (2), marked by solid arrows, occurs when b_j is the n-th vertex of $\vec{\mathbf{E}}(a_i)$, for some $i \leq j$ and $n > 1$. Player B chooses then as b_{j+1} the first unvisited vertex of $\vec{\mathbf{E}}(a_{i+1})$. (If all vertices in $\mathbf{E}(a_{i+1})$ have been visited, then B goes to the first $\mathbf{E}(a_k)$, with $i < k \leq j$, having unvisited vertices. If all such $\mathbf{E}(a_k)$ have been visited in their entirety, then B restarts with $k = 0$ as in case (1).)

In either case, if the search for an unvisited vertex fails, then B has visited all vertices in $\bigcup_1^j \mathbf{E}(a_k)$ and can only make a move b_{j+1} repeating some earlier move. Player A can then repeat earlier answer, and the game can continue thus only repeating earlier moves. The union $\bigcup_1^j \mathbf{E}(a_k)$ of out-neighbourhoods of all vertices played by A is then a finite set, giving a special case of the general situation.

In general, for a countable graph, this strategy of B exhausts in ω steps the out-neighbourhood of each vertex played by A. For an uncountable digraph, the play continues in this way transfinitely. At step $\lambda + 1$, for each limit ordinal λ, Player B chooses as in case (1) the least vertex unvisited by B in $\vec{\mathbf{E}}(a_i)$, for the least i with $i < \lambda$ and $\mathbf{E}(a_i)$ still having such vertices. The play continues until B exhausts out-neighbourhoods of all choices of A.

For the (least) ordinal κ at which B has visited all out-neighbours of all vertices visited by A, we have $\mathbf{E}(A_\kappa) \subseteq B_\kappa \subseteq \mathbf{E}^-(A_\kappa)$. The second inclusion holds since A provided an out-neighbour for each vertex in B_κ. Since A survived, no vertex in A_κ is poisoned, $A_\kappa \cap \mathbf{E}^\pm(A_\kappa) = \emptyset$, which means that A_κ is independent. Thus A_κ is a semikernel. □

Consequently, the game is determined; on each sink-free digraph, exactly one of the players has a winning strategy. Player A wins moving always in a semikernel, if one exists. Otherwise B, searching exhaustively through the out-neighbours of all vertices visited by A, forces eventually A to choose a vertex with an edge joining it to some vertex chosen earlier by A.

If B follows this brute-force strategy on a countable graph, then the game ends in no more than ω steps. If the graph has no semikernel, then B wins after finitely many steps.

Fact 2 *If B has a winning strategy on a sink-free countable digraph, then B can win every play after finitely many steps.*

On uncountable graphs, this brute-force strategy of B may require uncountable plays. To show that such plays are not needed, we consider κ-plays with ordinals κ possibly smaller than the cardinality of $\mathbf{V}$. Such games are also determined. If B has a winning strategy, then A does not have it. The less obvious opposite implication holds also because A wins by not losing.

Fact 3 *For every digraph and ordinal κ, if B does not have a winning strategy for κ-plays, then A has it.*

Proof. If B has no winning strategy for κ-plays, then A can start with some a_0 after which B still does not have it. Any move b_1 of B can be then answered by A with some a_1, after which B still does not have a winning strategy. This ensures survival of A for ω steps without giving B any winning strategy also when making the first move after the limit. The same holds for all greater successor and limit ordinals until κ. □

One more fact is used.

Fact 4 *For each digraph and limit ordinal λ, if A has a strategy σ for winning κ-plays for all $\kappa < \lambda$, then σ is also a winning strategy for A in λ-plays.*

Proof. Every λ-play g_λ is the limit of its prefixes, κ-plays g_κ, for $\kappa < \lambda$. If B wins g_λ, in which A follows σ, then A_λ contains a pair $\{a_m, a_n\}$ with $a_n \in \mathbf{E}^{\pm}(a_m)$. Since $A_\lambda = \bigcup_{\kappa<\lambda} A_\kappa$, such a pair is contained in A_κ for some $\kappa < \lambda$. Hence B wins already g_κ, but A following σ wins all κ-plays. This contradiction establishes the fact. □

The three facts give the second main claim.

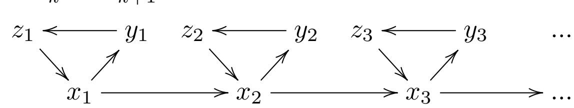
Theorem 5 *For each sink-free digraph, B has a winning strategy if and only if B has one for winning in finitely many steps.*

Proof. The if direction is obvious, so we show the other implication. For countable digraphs the claim is Fact 2, so consider an arbitrary uncountable digraph G. If B has a winning strategy for all plays on G, then B can use it to win every ω-play, which happens in finitely many steps. However, a winning strategy on G might possibly require B to play uncountably many moves. So suppose that B has a winning strategy for the uncountable plays on G, but does not have it for the countable ones. By Fact 3, A has then a winning strategy for the countable plays which, by Fact 4, gives also a winning strategy for A in ω_1-plays, for the first uncountable ordinal ω_1. This contradicts B having a winning strategy for all uncountable plays. Hence, if B has a winning strategy for the uncountable plays on G, then B has one also for the countable plays, in particular, for ω-plays. Thus B can ensure that no play on G continues past ω, winning each in finitely many steps. □

Summarizing, a sink-free digraph G of arbitrary cardinality has no semikernel if and only if A, starting from any vertex, gets poisoned by optimally playing B in finitely many steps. Dually, G has a semikernel if and only if A has a strategy for surviving ω-plays.

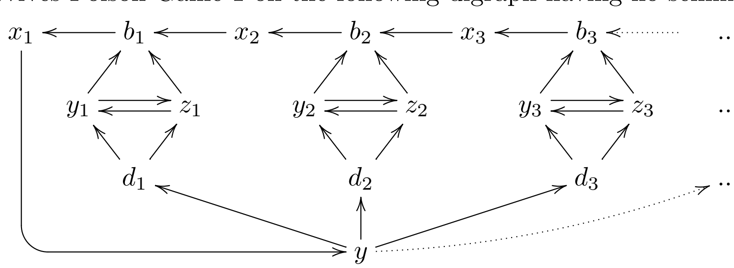
Some examples

(A) The following two examples from [1] show that Poison Game 1 is insufficient on digraphs with rays or infinite out-branching. A wins that game on the following digraph starting with any x_i and continuing always from x_k to x_{k+1}:



The digraph, however, has no semikernel. Player A looses Game 2 on it arriving at some x_i from which B poisons A playing y_i.

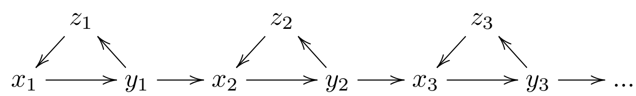
Likewise, A survives Poison Game 1 on the following digraph having no semikernel:



Player A starts with any d_i and never chooses b_i, playing $y_i \leftrightarrows z_i$ until B moves to b_i. After that, A reaches x_1 and then B, forced to play y, enables A to choose a fresh d_k. This strategy does not work in Game 2, since A choosing d_i poisons y_i, z_i and y. After B plays y_i or z_i, A must choose b_i. Then, B reaches x_1 after which A must play y poisoned by the initial choice of d_i.

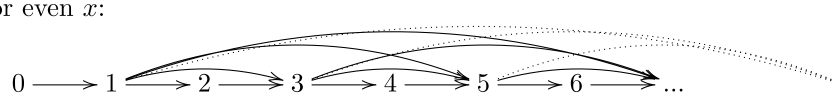
The reader is invited to verify that A loses Poison Game 2 no matter the starting position.

(B) The available results ensure typically not only kernel existence but kernel perfectness. (For instance, Richardson's Theorem from [2] states existence of kernels in digraphs with no odd cycles and either no rays or no infinite out-branching. Since these conditions are inherited by induced subgraphs, they actually ensure kernel perfectness.) No such result gives kernel existence in non-KP digraphs, e.g., in the following one, on which A has a simple winning strategy.



(C) A winning strategy for A on a given digraph can be used also when arbitrary new out-going edges or paths are added from vertices that only B can choose when A follows this strategy. Player A simply moves as before from such vertices with new out-going edges. For instance, A still wins following the previous strategy on the digraph from **(B)** extended with edges (y_j, z_i), for all $i < j$, which add infinitely many odd cycles to the graph.

Now, A wins trivially on a ray $0 \to 1 \to 2 \to ...$, starting from any vertex, e.g., from 0. The same strategy works then on the following digraph, modifying $(\omega, <)$ by retaining only one edge $(x, x+1)$ for even x:



A *variant* of this digraph is obtained by removing some even vertices. If we remove one, say 2, and A starts from 0, then B playing 1 forces A to choose some odd vertex i, after which B's choice of even $i+1$ defeats A, since $i+2$ is now poisoned. A wins, however, starting with an even vertex past the missing 2. If there is another missing even vertex after that, then A loses again. If only finitely many even vertices are missing, then A wins by starting from any even vertex after all missing ones. Put differently, if infinitely many even vertices are missing, then A loses every game: with B reaching easily an odd vertex just before a missing even one, A must move to an odd vertex, after which B's move to the next odd vertex just before a missing even one leads to poisoning A in the next move. Thus, this variant has no semikernel. Generally, a variant of this digraph has a semikernel if and only if at most finitely many even vertices are missing.

(D) Let digraph G have real numbers as vertices and edges from each vertex to all greater by at most 1, that is, $\mathbf{E} = \{(x, y) \colon x < y \leq x+1\}$. To A starting at 0, let B answer by 0.2. Now A must choose some $y \in \mathbf{E}(0.2)$ outside the interval $(0, 1]$, poisoned by the first move. Any such move $y \in (1, 1.2]$ poisons all vertices in $[1, 2]$ (and more), so B wins now by playing anywhere between the first move of A and of B. For instance, B playing 0.1 forces A to choose from $\mathbf{E}(0.1)$, that is, from the poisoned $(0.1, 1.1]$. This strategy for B works obviously no matter where A starts, and thus G has no semikernel.

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References

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