Abstract  Classical logic, of first or higher order, is extended with sentential quantifiers and operators. Semantics of the resulting language with self-reference, using semikernels of digraphs, is two-valued, but nonexplosive. Classical semantics arises as its special case for consistent theories. In a single layered metalanguage, marked by operators, intensional and semantic paradoxes occur via the same patterns. Unlike contradictions, they appear only at the metalevel and form locally coherent situations, without leading to explosion or affecting the object-level. Formally, they arise only from specific definitions of operators, but can be avoided, that is, the extended language is consistent. For so extended FOL language and nonexplosive semantics, a complete reasoning is obtained by extending LK with two rules for sentential quantifiers. Adding (cut) yields a complete system for the explosive semantics, specializing semikernels to kernels.

Keywords  sentential operators, semantic and intensional paradoxes, classical logic, paraconsistent semantics, (semi)kernels of digraphs

1 Introduction

Arithmetizing (Gödelizing) syntax enables one to develop metatheory and theory of syntax using object-level predicates as metapredicates. Predicate $T$ recovers then formula $\phi$ from Gödel code $⌜\phi⌝$, using convention (T), $T(⌜\phi⌝) \leftrightarrow \phi$. Since this can not hold for all codes so, trying to stay classical, the challenge is to investigate restrictions on (T) avoiding inconsistency.

Convention (T) provides only one example of problems caused by metapredicates modeled as predicates on arithmetized syntax. When such predicates reproduce some basic modal properties, elements of temporality, or just negation, diagonalization lemma yields paradoxes [9, 14, 19, 29] also without convention (T). Corresponding paradoxes do not arise when operator versions of such predicates are used, suggesting that operators may be less paradox prone. Advantages of operators, albeit only modal ones, are reviewed in [13]. The present paper develops such suggestions into a general formalization of sentential operators, not limited to modalities. Reflecting a different language model than that arising from predicates over arithmetized syntax, it offers also a different view of paradoxes and different ways to avoid them.

In this view, sentential operators (predicates applicable to sentences) mark a syntactic boundary between metalevel and object-level. This distinction is reflected by that between paradox and contradiction. Informally, it consists in the impossibility of evaluating the latter to true versus the impossibility of evaluating the former at all. The analysis of the liar arrives at a contradiction, but a problematic one, since concerning properties of the language, not of particular claims. It is a contradiction at the metalevel. Only coding such metastatements in object-language allows one to view paradoxes as mere contradictions.
Now, there is nothing pernicious in “the empirical assumption that the sentence engraved on Epimenides’ tomb is ‘The sentence engraved on Epimenides’ tomb is false.’ The problem lies not in the fact; it lies in our semantics” [15]. In models not distinguishing a statement from the fact of it being made or, as Frege might say, the content of an assertion from the act of asserting, expressing a contradiction can force one to deny some facts. We certainly do not want to do that. Karen can claim John to say only true things while John is accusing Karen of always lying. A brief reflection unveils the incompatibility, but this does not preclude the event. In this sense, paradoxes are parts of the world – perhaps, at the metalevel of linguistic discourse, but that is part of the world, too. This also distinguishes them from contradictions, which can not occur empirically. One can not simultaneously both say $S$ and not say $S$. Accepting both is not acceptable, even to dialetheists who embrace only some contradictions, like the liar who both lies and tells the truth. This, however, is a contradiction at the metalevel and nothing more follows from liar’s informal analysis. Paradox is a contradiction that may occur empirically, in the sense of only implying a contradiction but leaving the world unaffected. Even if the liar lies and does not lie, snow does not become not-white. Opting thus for classical logic, with no contradictions in the world, we admit them as consequences of paradoxes. But conclusions drawn from paradoxical expressions affect only these expressions and, unlike contradictions, do not effect any explosion. Such a disparate treatment of the two is enabled only by separating metalanguage from object-language.

This is roughly the view of paradoxes, and their interaction with the classical world, arising from the presented logic of sentential operators, LSO. (We consider only semantic and intensional paradoxes, so our general references to paradoxes concern implicitly only these.) Keeping the two language levels apart, LSO asks for patterns of metalevel expressions and atomic claims that avoid paradoxes, instead of for extensions of convention (T) to a larger class of numbers.

Section 2 introduces an extension of any classical, first or higher order, language with sentential quantifiers and operators and Section 3 presents the corresponding semantic notions. Any classical language $L$ is first extended with quantification over sentences, s-quantification, that is not propositional (ranging only over truth-values) but substitutional with the substitution class containing all sentences of the extended language. This extension does not introduce any paradoxes, as it does not even increase expressive power of the language. Then, as indicated, we do not internalize metalanguage, but use instead sentential operators (sentential predicates or s-predicates are synonyms), obtaining the full language $L^+$. Operators distinguish metalevel from object-level. Unrestricted s-quantification extends to the full language $L^+$ which remains consistent, that is, paradoxes can be avoided, but can now occur. Just like informally they occur due to some maliciously formulated claims, they appear formally due to unfortunate atomic claims, valuations of s-predicates. Reading $K(S)$ as Karen saying sentence $S$, $Kl = \forall \phi (\overline{\phi}(K\phi \rightarrow \neg \phi)$ states that every sentence Karen says is false. $K(Kl)$ is not paradoxical, as Karen can also say some true things. However, Karen saying only $Kl$ is paradoxical, implying $Kl \land \neg Kl$, and this is caused only by what Karen is saying, valuation of the s-predicate $K$.

S-predicates need not be truth-functional and can treat arguments purely syntactically, but do not aim at any deeper analysis of intensional phenomena. LSO is an intensional logic only in so far as s-predicates can be opaque, failing to preserve logical equivalence of arguments. However, it neither provides any intensional semantics nor considers the status of propositions or propositional attitudes. Propositions appear at most as mere sentences, while examples blur easily borders between “says $\phi$”, “thinks $\phi$”, “implies $\phi$”, etc.. The significant distinction is that between statements with and without s-predicates, between the metalevel and the object-level. Modalities, attitudes or intensions can be handled by further axiomatization of s-predicates, as shown by examples.

Paradoxes of semantical and intensional character arise and are treated in LSO in the same way. The liar, saying only one sentence “Every sentence I am saying now is false”, is not significantly different from Karen not believing any of her beliefs nor from a club whose members are all people not belonging to any club. Problems are caused by the same patterns involving typically vicious circularity, captured precisely by semantics utilizing graphs. Truth of sentence $\forall \phi \phi$ requires truth of each instance, in particular, of this very sentence. It is false due to existence of other false sentences, but the graph semantics substantiates such dependencies and allows to handle all related circularity and impredicativity, marginalizing for instance the issue of (un)groundedness. Vicious
circles, represented by unresolvable odd cycles in the language graphs, are the prime reason for paradoxes, but also Yablo exemplifies our general concept, commented briefly in Section 6.

Another central feature of our semantics is that interpretation of consistent theories, coinciding with the classical semantics, arises as a special case of paraconsistent one. Relations between the two are addressed in Section 4. Informally, Karen’s exclusive claim of always lying does not entail much beyond the contradiction that she is lying and not lying. LSO reflects this limited consequence. Nothing follows about what John may be saying, nor about whiteness of snow. Semantics of such local coherence, admitting nonsense but circumscribing its effects, utilizes the graph-theoretic generalization of kernels (providing the classical, explosive semantics) to semikernels.

These close connections between explosive and paraconsistent semantics are reflected in reasoning system LSO, extending UK with two rules for s-quantifiers. S-predicates and s-quantification bring flavour of higher order, but the operator form of the former and substitutional interpretation of the latter allow LSO to be sound and complete for the paraconsistent semantics of FOL+.

Explosive – one might want to say, classical – semantics is reflected by reasoning in LSO extended with (cut). It conjoins a specific contradiction, implied by a paradox, with all statements, implied by any contradiction. In this way also (cut) reflects a difference between contradictions and paradoxes, or between object-level, where it is admissible, and metalevel, where it makes paradoxes explode. The logic exemplifies thus nontransitive systems, studied e.g. in [8,32] but, unlike there, (cut) in LSO merely turns the paraconsistent logic into explosive one, as shown in Section 4.1.

Although LSO allows thus paradoxes to occur, providing the means for their analysis as well as for functioning in their presence, it does not imply any paradoxes. If somebody causes paradox, it can be mostly ignored by others and at the object-level, but as long as nobody makes unreasonable claims, the language remains consistent. Central theorems ensure consistency of the whole language, relatively to the choice of the atomic claims.

Section 5 gives a series of examples analyzing some cases from the literature and signalling some extensions of LSO. Concluding Section 6 summarizes the main threads, while appendix in Sections 7 and 8 contains all proofs and some lesser technicalities.

2 Reasoning about sentences

A classical (propositional, first or higher order) language $L$ is extended in two steps. First, $L^\Phi$ is obtained by adding sentential variables, $s$-variables $\Phi$, which can be $s$-quantified in the usual manner, so that $\forall \phi \forall x(A(x) \lor \phi)$ is a sentence. To this we add operators (or $s$-predicates), applicable to sentences, so that $\forall x \forall \phi(A(x) \lor \phi \lor P(\phi))$ is a sentence. In the resulting language $L^+$, the substitution class for the interpretation of $s$-quantifiers is unrestricted comprising all sentences of $L^+$. In the following grammar, these extensions of FOL to FOL+ are given by the underlined productions:

$$T_X := X \mid \text{Const} \mid \text{Func}(T_X, \ldots, T_X)$$

$$A_X^+ := P_1(T_X, \ldots, T_X) \mid C \mid P_2(F_X^+, \ldots, F_X^+, T_X, \ldots, T_X)$$

$$F_X^+ := A_X^+ \mid \neg F_X^+ \mid F_X^+ \land F_X^+ \mid \forall X F_X^+ \mid \Phi \mid \forall \phi F_X^+$$

Sentential constants $C$ are used only in some examples. $s$-predicates $P_2$ can also have terms $T_X$ as arguments, but these are handled in the expected way, mostly, without explicit mention. Not assuming any semantic restrictions, $s$-predicates treat their arguments purely syntactically, acting possibly as metapredicates in a theory of syntax, although the grammar above restricts their application to formulas and terms. In this paper, we restrict them even further, essentially to sentential operators, defining their semantics only in contexts where their arguments have no free variables, primarily, when applied to sentences. Like all formulas, atoms are divided into

(a) $L$ atoms, $A_X$, e.g., $A(t)$ for $A \in P_1$ and $t \in T_X$, and

(b) metalevel $s$-atoms, $A_X^+ = A_X^+ \setminus A_X$, e.g., $C \in C$ or $R(S, t)$, for $R \in P_2$ and $S \in S^+, t \in T_X$.

For a set $M$, by $T_M$ we denote the free algebra of terms over $M$, by $S_M/S_M^+$ all $L/L^+$ sentences over $T_M$, and by $S/S^+$ all $L/L^+$ sentences. Superscript $°$ marks the metalevel, added to the object-level $L$ and yielding the resulting extension $°+\setminus L$, e.g., $L^° = L^+ \setminus L$, $S_M^° = S_M^+ \setminus S_M$, etc.
Reasoning system LSO for FOL\(^+\), given below, extends LK with two rules for s-quantifiers. The basic syntax uses only \(\{\land, \neg, \forall\}\), with other connectives and \(\exists\), and rules for them, defined in the classical way. Sequents, written \(\Gamma \Rightarrow \Delta\), are formed from countable sets \(\Gamma \cup \Delta\) of \(F\_X\) formulas. \(\Gamma \vdash \Delta\) denotes provability of \(\Gamma \Rightarrow \Delta\) in LSO.

\[(\text{Ax})\quad \Gamma \vdash \Delta\quad \text{for } \Gamma \cap \Delta \neq \emptyset\]

\[(\neg L)\quad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\]

\[(\forall L)\quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta}\]

\[(\forall R)\quad \frac{\Gamma \vdash \Delta, A \land B}{\Gamma \vdash \Delta, A, B}\]

\[(\land L)\quad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\]

\[(\land R)\quad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \land B}\]

\[(\forall \_L)\quad \frac{F(t), \Gamma, \forall x F(x) \vdash \Delta}{\Gamma, \forall x F(x) \vdash \Delta}\quad \text{legal } t/x \text{ in } F\]

\[(\forall \_R)\quad \frac{\Gamma \vdash \Delta, F(y)}{\Gamma \vdash \Delta, \forall x F(x)}\quad \text{fresh } y\]

\[(\forall \_L^+)\quad \frac{F(S), \Gamma, \forall \phi F(\phi) \vdash \Delta}{\Gamma, \forall \phi F(\phi) \vdash \Delta}\quad \text{any } S \in S^+\]

\[(\forall \_R^+)\quad \frac{\Gamma \vdash \Delta, F(S)}{\Gamma \vdash \Delta, \forall \phi F(\phi)}\quad \text{for all } S \in S^+\]

Even with the infinitary rule \((\forall \_R^+)\), signalling missing compactness, the system is hardly any challenge for the intuition.\(^1\)

Infinite sequents allow to handle some cases of infinite axiomatizations. An example are infinite sets of premisses in typical situations of making only finitely many claims. This case has a finite representation using s-predicate for syntactic equality of sentences, s-equality, S \(\equiv Q\), which requires only a trivial check. We consider practically only sequents with no free s-variables, but they can be useful for handling s-equalities. For instance, ‘Karen saying only \(\phi\)’ is expressible as \(\forall x \phi = \{\forall \psi(K \psi \Rightarrow \psi \equiv \phi)\}. The following rules suffice.

\[(\text{ref})\quad \frac{S \equiv Q, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\quad \text{(rep) } \frac{A(S), A(Q), S \equiv Q, \Gamma \vdash \Delta}{A(Q), S \equiv Q, \Gamma \vdash \Delta}\quad \text{(neq) } \frac{\Gamma \vdash \Delta, Q \equiv S}{\Gamma \vdash \Delta, Q \neq S}\]

Claims like ‘for each sentence except \(S_1, \ldots, S_n : X\) are finitely expressible as \(\forall \sigma(\sigma \neq S_1, \ldots, \sigma \neq S_n \Rightarrow D(\sigma))\), allowing sometimes to establish \(\forall \phi D(\phi)\) by finite case analysis, instead of \((\forall \_R^+)\):

\[(\forall \_R)\quad \frac{\Gamma \vdash \Delta, D(S_1) \ldots \Gamma \vdash \Delta, D(S_n)}{\Gamma \vdash \Delta, \forall \phi D(\phi)}\quad \text{fresh } \sigma \in \Phi.\]

Properties of LSO, like soundness/completeness, the interaction of its object-level and metalevel, the role of (cut), will be introduced along the corresponding semantic notions.

### 3 Semantics

We keep the presentation focused on FOL, but semantic definitions and results of this section work equally for higher order classical logics. Informally, an interpretation of object-level sentences \(S\) in a structure \(M\) is extended to \(S^+\) by interpreting s-quantifiers substitutionally:

\[
M \models \forall \phi F(\phi) \iff \forall S \in S^+: M \models F(S).
\]

(3.1)

The right side has instances like \(F(\forall \phi F(\phi))\) or \(F(F(\forall \phi F(\phi)) \land \forall \phi F(\phi))\), apparently involving the definienndum. Such circularities are handled by recasting semantics in terms of graph kernels. Saying “graph” we mean a digraph \(G = (V_G, E_G)\), with \(E_G \subseteq V_G \times V_G\), and \(E_G\) denoting the converse of \(E_G\). The subscript \(\_G\) is dropped when an arbitrary or fixed graph is addressed. For a binary relation \(E\), we let \(E(x) = \{y \mid E(x, y)\}\) and extend function applications pointwise to sets, \(E(X) = \bigcup_{x \in X} E(x)\). A kernel (or solution, introduced in [33]) of \(G\) is a \(K \subseteq V\), which is

(a) independent, i.e., \(E^- (K) \subseteq V \setminus K\), and

(b) absorbing, i.e., \(V \setminus K \subseteq E^- (K)\),

in short, such that \(E^- (K) = V \setminus K\). Equivalently, it is an assignment \(\kappa \in 2^Y\), such that

\[
\forall x \in V : \kappa(x) = 1 \iff \forall y \in E(x) : \kappa(y) = 0.
\]

(3.2)

\(^1\) Instead of this infinitary rule we can use the counterpart of \((\forall \_R)\), with s-variable and s-quantifier replacing the object-level items, provided that semantics addresses not only the actual language, but all its extensions.
Intuitively, each edge marks negation of its target, and branching stands for conjunction of such negations. Given (3.2), the set \( \{ x \in V \mid \kappa(x) = 1 \} \) satisfies (a) and (b), while if \( K \) satisfies (a), (b) then \( \kappa \in 2^V \) given by \( \kappa(x) = 1 \Leftrightarrow x \in K \) satisfies (3.2). We therefore do not distinguish the two and by \( \text{sol}(G) \) denote the set of kernels or such assignments. Graph \( G \) is solvable if \( \text{sol}(G) \neq \emptyset \).

The equation \( E^-(K) = V \setminus K \) means that kernel \( K \) covers the whole graph, where a subset \( L \subseteq V \) covers vertices \( L \cup E^-(L) \), denoted by \( E^-[L] \). A valuation is coherent on vertices for which it satisfies (3.2), so a kernel represents a coherent valuation of all sentences. Our semantics is two-valued but admits paraconsistency, that is, only locally coherent valuations with no extension to the whole language. In the absence of a kernel, containing all sentences required to be true, a relevant part of the graph may still be covered by a semikernel, [21], namely, an \( L \subseteq V \) such that

\[
E(L) \subseteq E^-(L) \subseteq V \setminus L.
\]

The set of semikernels of \( G \) is denoted by \( SK(G) \). A kernel is a semikernel covering the whole graph. In a kernel, falsehood of every vertex in \( V \setminus K \) is justified by an edge it has to some (true) vertex in \( K \). In a semikernel \( L \), such a justification is required only for vertices which are out-neighbours of \( L \) and must have an edge back to \( L, E(L) \subseteq E^-(L) \). The inclusion \( E^-(L) \subseteq V \setminus L \) makes \( L \) independent. A semikernel \( L \) satisfies equivalence (3.2) for all \( x \in E^-[L] \). Thus it represents a coherent situation, in the sense that all statements denied by any true one (in \( L \)) are false (in \( E^-(L) \)), while every false statement denies some true one. Although locally consistent in this sense, such a coherent situation can entail inconsistency.\(^2\)

Every graph possesses a semikernel, since \( \emptyset \) satisfies trivially (3.3). But semikernels of interest are nontrivial, also in graphs not possessing any kernel.

**Example 3.4** The propositions to the left can be represented by the graph to the right.

\[
a_1 : \text{This and the next sentence are false.} \\
b_1 : \text{The next sentence is false.} \\
b_2 : \text{The previous sentence is false.} \\
a_2 : \text{This and the previous sentence are false.}
\]

The graph has no kernel, witnessing to the involved paradox, but has two semikernels: \( \{b_1\} \) and \( \{b_2\} \) covering, respectively, \( \{a_1,b_1,b_2\} \) and \( \{b_1,b_2,a_2\} \), which can be seen as coherent subdiscourses, where truth-values can be assigned consistently, i.e., respecting (3.2). \( \square \)

3.1 Language graphs

Semantics of \( \mathcal{L}^+ \) is defined by (semi)kernels of language graphs. One such graph is constructed for each \( \mathcal{L} \)-domain, that is, a set \( M \) with an interpretation of \( \mathcal{L} \)-terms \( T_M \) but not of the predicate symbols. Graph’s vertices are sentences \( S^+_M \) and outgoing edges amount to conjunction of the negations of their targets. A sink is a vertex with no outgoing edges and \( \overline{X} \) stands for \( \neg X \).

**Definition 3.5** The language graph \( G_M(\mathcal{L}^+) \), for a language \( \mathcal{L}^+ \) and domain \( M \), is given by:

1. Vertices \( V = S^+_M \cup \text{ AUX } \), with some auxiliary vertices \( \text{ AUX } \), explained below.
2. Each atomic sentence \( A \in A^+_M \), except s-equality, has a 2-cycle to its negation: \( A \equiv \overline{A} \).
3. For each \( S \in S^+_M \), s-equality atom \( S \equiv S \) is a sink; for each syntactically distinct \( S, Q \in S^+_M \), vertex \( Q \equiv S \) has an edge to the sink \( Q \equiv S \).
4. Each nonatomic sentence \( S \in S^+_M \) is the root of the subgraph \( G_M(S) \):

\(^2\) The branch of argumentation theory arising from [10] shares only its origins in a similar reading of digraph (semi)kernels. Also links to reference graphs, used in [5, 26] for paradox analysis, are inessential.
\[ \mathcal{G}_M(S) = \text{root with edges to the root of:} \]

\[ \begin{align*}
(a) & \quad \neg F & \rightarrow & \mathcal{G}_M(F), \\
(b) & \quad F_1 \land F_2 & \rightarrow & \mathcal{G}_M(\neg F_i), \text{ for } i \in \{1,2\}, \\
(c) & \quad \forall x Fx & \rightarrow & \mathcal{G}_M(\neg F(m)), \text{ for each } m \in M, \\
(d) & \quad \forall \phi F\phi & \rightarrow & \mathcal{G}_M(\neg F(S)), \text{ for each } S \in S^+. 
\end{align*} \]

The presentation uses FOL, but when \( \mathcal{L} \) is second or higher order, the only difference is the domain \( M \), containing required sets, with object quantifier(s) in point 4.(c) being those from \( \mathcal{L} \).

Keeping \( \mathcal{L}^+ \) implicit, we write usually \( \mathcal{G}_M \) instead of \( \mathcal{G}_M(\mathcal{L}^+) \), and by \( \mathcal{G}_T \) denote the class of all language graphs, for a language relevant in any actual context. We drop also \( M \) when it is inessential, and write \( \mathcal{G} \) for \( \mathcal{G}_M(\mathcal{L}^+) \).

AUX vertices are also sentences \( S^+_M \), with which they can be identified, but keeping them separate gives a clearer structure, without affecting the results.\(^3\) For \( S \in S \), the subgraph \( \mathcal{G}(S) \) is actually a tree \( T(S) \), reminding of \( S \)'s parse tree but, primarily, reflecting the semantics of the formula constructors (\( \neg, \land, \lor \)) in terms of kernels. Out-branching represents conjunction (or universal quantification), and each edge negation of its target. The 2-cycles at atoms force, in any kernel, choice of one element from each pair, giving valuations of atoms; sinks are true by (3.2).

The universal and existential quantifiers give rise to the following branchings to instantiations of the quantified variables by all elements \( a, b, c, \ldots \) of the domain, and of \( s \)-variables by all \( S^+ \). (A double edge \( x \rightarrow y \rightarrow z \), where \( x \) has no other out-neighbours and \( y \) no other neighbours, can be contracted by removing \( y \) and identifying \( x = z \), Fact 7.2. This is done for 3-pattern to the right.)

\[
\begin{align*}
\forall \phi D(\phi) & \quad \rightarrow \quad \exists x D(x) \\
D(S_1) & \quad \rightarrow \quad D(S_2) \\
\quad \downarrow & \quad \quad \quad \quad \quad \downarrow \\
D(\forall \phi D(\phi)) & \quad \rightarrow \quad D(a) \\
\quad \quad \quad \quad \quad \downarrow & \quad \quad \quad \quad \quad \downarrow \\
\quad \quad \quad \quad \quad \exists x \neg D(x) & \quad \rightarrow \quad \exists x \neg D(x) \\
\ldots & \quad \rightarrow \quad \ldots \\
D(S_1) & \quad \rightarrow \quad D(S_2) \\
\quad \downarrow & \quad \quad \downarrow \\
D(\forall \phi D(\phi)) & \quad \rightarrow \quad D(a) \\
\quad \quad \downarrow & \quad \quad \downarrow \\
\quad \quad \exists x \neg D(x) & \quad \rightarrow \quad \exists x \neg D(x) \\
\ldots & \quad \rightarrow \quad \ldots \\
D(a) & \quad \rightarrow \quad D(b) \\
\quad \downarrow & \quad \quad \downarrow \\
D(c) & \quad \rightarrow \quad \ldots \\
\end{align*}
\]

\[ (3.6) \]

Quantifier prefix is converted to the graph by successively performing such branchings and instantiations, until no quantified variables remain.

Representation of propositional connectives follows the same pattern: conjunction by branching and negation by an edge. Assuming a sentence in prenex form with the matrix in DNF, the quantifier prefix ends with DNF-feet, one for each instantiation of the quantified variables. For example, DNF matrix \( D(\phi, x) = (\neg \phi \land \neg Q(x)) \lor (\neg P(\phi) \land R) \), where \( R \) is a ground atom, gives one DNF-foot for each instantiation of \( \phi \) and \( x \), e.g., by \( S \in S^+ \) and \( a \in M \) in \( \mathcal{G}_M(D(S,a)) \):

\[
\begin{align*}
& \quad \quad \downarrow \\
D(S,a) & \quad \rightarrow \quad \mathcal{C}_{D(S,a)} \\
\quad \quad \quad \quad \downarrow \\
C_1(S,a) & \quad \rightarrow \quad \quad \mathcal{C}_{D(S,a)} \\
\quad \quad \quad \quad \quad \downarrow \\
S & \quad \rightarrow \quad Q(a) \\
\quad \quad \quad \quad \quad \downarrow \\
\neg S & \quad \rightarrow \quad Q(a) \\
\end{align*}
\]

\[ (3.7) \]

The auxiliary vertex \( \mathcal{C}_{D(S,a)} \) is the sentence \( \neg C_1(S,a) \land \neg C_2(S,a) \in S^+_M \), while the auxiliary \( C_1(S,a) \) is the sentence \( \neg S \land \neg Q(a) \). For \( K \in \text{sol}(\mathcal{G}_M(D(S,a))) : D(S,a) \in K \iff \mathcal{C}_{D(S,a)} \notin K \iff C_1(S,a) \in K \lor C_2(S,a) \in K \iff \{\neg S, \neg Q(a)\} \subseteq K \lor \{\neg P(S), R\} \subseteq K \), reflecting the expected \( D(S,a) = 1 \iff (S = 0 = Q(a)) \lor (P(S) = 0 \land R = 1) \).

Subgraph \( \mathcal{G}_M(\mathcal{L}) \) described so far captures language \( \mathcal{L} \). For each \( \mathcal{L} \)-sentence \( A \), its subgraph \( \mathcal{G}_M(A) \) sketched above is a tree except that, instead of leaves, there are atoms with 2-cycles. Exactly one element from each cycle can be in any kernel and every such a tree has exactly one kernel for every selection from these cycles. Inclusion of \( P(S) \) and \( R \) from (3.7) in a kernel \( K \)

\(^3\) Identifying vertices with identical out-neighbourhood does not change the (semi)kernels in any essential way, as intuition suggests and Fact 7.3 shows. AUX vertices can be taken as serving the presentation only.
forces, by independence, \( P(S) \) and \( \overline{R} \) out of it. This, in turn, forces \( C_2(S, a) \in K \) by absorption, so that \( \overline{\exists_i \overline{S} \in S} \not\in K \) and \( D(S, a) \in K \). This implication from \( \{ P(S), R \} \subseteq K \) to \( D(S, a) \in K \) reflects the implication from \( \neg P(S) \land R \) to \( D(S, a) \). Kernel \( K \) of \( G_M(L) \) represents exactly the satisfied formulas under valuation of atoms given by the selection from atomic 2-cycles, i.e., by \( K|_{A_M} \).

There is thus a bijection mapping a FOL structure \((M, \rho), \rho \in 2^{A_M} \), to the language graph with its kernel \((G_M(L), K_\rho)\), where \( A \in K_\rho \iff \rho(A) = 1 \) for \( L_M \)-atoms \( A \). Then also, for all \( S \in S_M \),
\[
(M, \rho) \models S \iff S \in K_\rho \tag{3.8}
\]
and this correspondence underlies the generalization of FOL semantics in what follows. A kernel for a language graph determines boolean values of all sentences, amounting to absence of paradoxes.

The full graph \( G_M(L^+) \) has, besides the forest \( G_M(L) \) described above, also subgraph \( G_M(L^0) \) containing subgraphs \( G_M(A) \) for sentences with \( s \)-quantifiers, \( A \in S_M^0 \). In such a \( G_M(A) \), each \( S \in S^+ \) substituted into \( A \) for \( s \)-variable \( \phi \) in a sentential position, i.e., not in the scope of any \( s \)-predicate, like \( \phi \) in \( C_1 = \neg \phi \land \neg R_1 \), becomes a leaf of a tree (3.7), but with a double edge to the root of the subgraph \( G_M(S) \). (The double edge can be contracted, as explained before (3.6), but keeping the two separate gives a more intuitive picture.) In particular, sentence \( A \) is also substituted for \( s \)-variable \( \phi \), and the resulting leaf (of a DNF-foot (3.7)) has a double edge to the root of this very \( G_M(A) \). Every \( S \in S^+ \), substituted for \( \phi \) in \( A \), either occurs on some path from the root \( A \) as an internal node or not. In the former case, the leaf \( S \) is called an internal atom of \( G_M(A) \), and has a double edge back to its occurrence in \( G_M(A) \) (possibly forming a cycle). In the latter case, when \( S \) occurs in \( G_M(A) \) only as a leaf, it is its external atom, \( ext(G_M(A)) \), and has a double edge to the root of its separate \( G_M(S) \). In this case, if \( S \) itself is \( s \)-quantified, its subgraph instantiates its \( s \)-variables by all sentences, in particular by \( A \), giving paths back to the root of \( G_M(A) \). All \( s \)-quantified sentences among \( S^0 \) form thus one strongly connected component of \( G_M(L^+) \). Their leaves instantiated with sentences \( S_M \) have double edges to the subgraph \( G_M(L) \), but there are no edges returning thence to \( G_M(L^0) \).

Such double edges, yielding cycles and connecting distinct sentence subgraphs, arise only from sentences substituted for \( s \)-variables in sentential positions. Sentences substituted into nominal positions, i.e., into the scope of some \( s \)-predicate, form atoms with 2-cycles to their duals, like \( P(S) \models P(S) \) in (3.7), arising from substituting \( S \) into \( P(\phi) \).

\begin{example}
Let \( S_1, S_2, \ldots \) stand for all \( S^+ \), except the leftmost two in each graph sketched below: \( G(A_\forall) \), for the sentence \( A_\forall = \forall \phi, \phi \), and \( G(A_3) \), for the sentence \( A_3 = \neg \forall \phi, \neg \phi: \)

The drawings indicate only the essential aspects, ignoring other edges and cycles.

In the left graph \( G(A_\forall) \), the two vertices \( A_\forall \) (as well as \( \forall \phi, \phi \) and \( A_\forall \)) could be identified. Any \( S_i \in S^+ \) valued at 0 yields \( \overline{S_i} = 1 \) and \( \forall \phi, \phi = 0 \), but even if all \( S_i = 1 \), the mere cycles involving \( A_\forall \) and \( \overline{A_\forall} \) force \( \forall \phi, \phi = 0 \). To obtain a kernel, the odd cycle via \( \overline{A_\forall} \) must be broken, i.e., some of its vertices must have an out-neighbour = 1. If all \( S_i = 0 \), this still happens when both \( \overline{A_\forall} = 0 \) = 0, making \( A_\forall = \overline{A_\forall} = 0 = \forall \phi, \phi \). Thus, \( \forall \phi, \phi \) is a counterexample to its own truth.

A dual situation occurs in \( G(A_3) \), where kernel requires breaking the odd cycle via \( \bullet \) and \( \overline{A_3} \). This happens if any \( S_i = 1 \), but even if they are all 0, the odd cycle via \( \overline{A_3} \) and even one via \( A_3 \) force \( \exists \phi, \phi = 1 \). The only way to break the odd cycle is then by \( \overline{A_3} = 0 \), which requires \( A_3 = 1 \), complying with \( \bullet = 0 = \overline{\bullet} \) and \( \exists \phi, \phi = 1 \), which provides thus a witness to its own truth.
\end{example}

\footnote{These 2-cycles are formed only for atoms with the outermost \( s \)-predicate. Substituting \( S \) into \( P(\phi, Q(\phi)) \) yields atom \( P(S, Q(S)) \) with edges to its dual \( P(S, Q(S)) \). The inner \( Q(S) \) does not obtain any edges to its dual \( Q(S) \), which happens only for the atom \( Q(S) \) that occurs in sentential position.}
This completes the description of language graphs. Semantics utilizes (semi)kernels, representing sets of true sentences (under valuations of atoms determined by these subsets). We begin with the kernel semantics, \( \models_e \), where subscript suggests its classical character, in that kernels cover whole graphs, determining values of all sentences. (Later, it will be also related to (cut).)

An \( \mathcal{L}^+ \)-sequent \( \Gamma \Rightarrow \Delta \) is \( c \)-valid, \( \Gamma \models_e \Delta \), iff in every language graph \( \mathcal{G}_M \in \mathcal{LG}(\mathcal{L}^+) \) every kernel satisfies it, where kernel \( K \) satisfies the sequent if some \( D \in \Delta \) is true, i.e., \( D \in K \), or some \( G \in \Gamma \) is false, i.e., \( G \in \mathcal{E}^+(K) = \bigvee \setminus K \). (This is generalized to free \( L \)-variables \( \bigvee \) \( (\Gamma, \Delta) \), by considering all valuations \( \alpha \in M^{\bigvee(\Gamma, \Delta)} \) of free object-variables: \( \alpha(\Gamma) = \{ \alpha(S) \mid S \in \Gamma \} \) and \( \alpha(S) \) denotes insertion of \( \alpha(v) \) for each free object variable \( v \) of \( S \). For free \( s \)-variables, all instances with \( \mathcal{S}^+ \) must hold.)

\[
\Gamma \models_e \Delta \Leftrightarrow \forall \mathcal{G}_M \in \mathcal{LG} \forall K \in \text{sol}(\mathcal{G}_M) \forall \alpha \in M^{\bigvee(\Gamma, \Delta)} ; \alpha(\Gamma) \cap \mathcal{E}^-(K) \neq \emptyset \lor \alpha(\Delta) \cap K \neq \emptyset.
\] (3.10)

For \( \Gamma \subseteq \mathcal{S}^+ \), kernel models are pairs \( (\mathcal{G}, K) \in \mathcal{LG}(\mathcal{L}^+) \times \text{sol}(\mathcal{G}) \) satisfying \( \Rightarrow F \), for all \( F \in \Gamma \).

The reader may justly wonder if the complexity of language graphs does not hide unavoidable paradoxes. Section 3.2 shows now that, for language \( \mathcal{L}^\Phi \) with \( s \)-quantifiers but no \( s \)-predicates, language graphs not only have kernels, but have unique one for every valuation of \( L \) atoms. Section 3.3 shows then solvability also of graphs for the full language \( \mathcal{L}^+ \). However, paradoxes become then possible, for which kernel semantic (3.10) is generalized to semikernels.

### 3.2 Sentential quantifiers and solvability of \( \mathcal{G}(\mathcal{L}^\Phi) \)

In \( \mathcal{L}^\Phi \), extending object-language \( L \) with \( s \)-quantifiers but no \( s \)-predicates, \( s \)-variables occur only in sentential positions. The only atoms are \( L \)-atoms \( A \) (and possibly \( C \). When \( A = \emptyset \), the language \( \emptyset^\Phi \) is that of quantified boolean sentences, QBs.) Given a domain \( M \) and \( \rho \in 2^{\mathcal{A}_M} \), all \( \mathcal{L}^\Phi \) sentences obtain unique values by a unique extension to a kernel \( \tilde{\rho} \) of the corresponding graph \( \mathcal{G}_M(\mathcal{L}^\Phi) \).

**Theorem 3.11** For each \( \mathcal{G}_M(\mathcal{L}^\Phi) \) and \( \rho \in 2^{\mathcal{A}_M} \) there is a unique \( \tilde{\rho} \in \text{sol}(\mathcal{G}_M(\mathcal{L}^\Phi)) \) with \( \tilde{\rho}|_{\mathcal{A}_M} = \rho \).

Proofs are given in Appendix, but we comment briefly that the proof of this theorem relies on the lemma below, showing that for any solution of \( \mathcal{G}_M(S) \) – denoting, for \( S \in \mathcal{S}^\Phi_M \setminus \mathcal{S}_M \), vertices of \( \mathcal{G}_M(S) \) without those in its DNF-foot – depends on the valuation of \( \mathcal{A}_M \), but not of external atoms \( \text{ext}(\mathcal{G}_M(S)) \), as the second part of the lemma states. In a way, DNF-foot determines a boolean function, and the value of \( S \) depends on this function (and valuation of \( \mathcal{A}_M \)), rather than on the values of external atoms, which span all possibilities. Valuation of \( \text{ext}(\mathcal{G}_M(S)) \) affects, of course, values in DNF-foots, where they occur. For either \( A \) from Example 3.9, the lemma implies that valuation of \( \mathcal{G}^-(A) \), i.e., the root vertex with its marked cycles, is independent from valuation of all external atoms among \( S_1, S_2 \), etc.

**Lemma 3.12** For every \( \mathcal{G}_M(\mathcal{L}^\Phi) \) and sentence \( A \in \mathcal{S}^\Phi_M \), each valuation \( \rho \) of atoms \( \mathcal{A}_M \) and external atoms of \( \mathcal{G}_M(A) \), \( \rho \in 2^{\mathcal{S}^\Phi_M \setminus \text{ext}(\mathcal{G}_M(A))} \), has a unique extension to \( \rho_A \in \text{sol}(\mathcal{G}_M(A)) \). Valuation of atoms, \( \rho|_{\mathcal{A}_M} \), determines restriction of \( \rho_A \) to \( \mathcal{G}_M^-(A) \).

Valuation of sentences \( \mathcal{S}^\Phi_M \setminus \mathcal{S}_M \) does not have any standard definition, which is merely suggested by (3.1). By Theorem 3.11, such a valuation \( \tilde{\rho} \) is determined by \( \rho \in 2^{\mathcal{A}_M} \), just as is valuation of \( \mathcal{S}_M \). Existence and uniqueness of \( \tilde{\rho} \) ensure well-definedness of (3.1), given by the following.

**Definition 3.13** An \( \mathcal{L}^\Phi \)-sentence \( A \) is true in an \( L \)-domain \( M \) under valuation \( \rho \in 2^{\mathcal{A}_M} \), \( (M, \rho) \models A \), iff \( \tilde{\rho}|_A = 1 \) for the unique solution \( \tilde{\rho} \in \text{sol}(\mathcal{G}_M(\mathcal{L}^\Phi)) \) with \( \tilde{\rho}|_{\mathcal{S}_M} = \rho \).

This gives a well-defined class \( \text{Mod}(\Gamma) = \{ (M, \rho) \mid \forall A \in \Gamma : (M, \rho) \models A \} = \bigcap_{A \in \Gamma} \text{Mod}(A) \) of \( L \)-structures modelling any theory \( \Gamma \subseteq \mathcal{S}^\Phi \). The bijection (3.8) between FOL structures and graphs with kernels, mapping \( (M, \rho) \) to \( (\mathcal{G}_M, K_\rho) \), extends to FOL\(^\Phi \) by mapping \( (M, \rho) \) to \( (\mathcal{G}_M, \tilde{\rho}) \).

The hardly unexpected but significant Theorem 3.11 implies that a classical language \( L \) remains free from paradoxes, under every valuation of atoms, when extended with quantification over all sentences to \( \mathcal{L}^\Phi \). In fact, by the following theorem, \( \mathcal{L}^\Phi \) has the same expressive power as \( L^e \).
Theorem 3.14 For every \( \Gamma \subseteq \mathcal{L}^\phi \) there is a \( \Gamma^- \subseteq \mathcal{L} \) with \( \text{Mod}(\Gamma) = \text{Mod}(\Gamma^-) \).

In particular, quantification over all sentences in FOL, extending FOL apparently as far as possibility of self-reference, reduces to propositional quantification.

3.3 Sentential predicates and solvability of \( \mathcal{G}(\mathcal{L}^+) \)

Predicates applied to sentences provide only fresh atoms, so one might think that everything works unchanged. It does, if only such predicates are introduced without sentential quantifiers. The language graph which is then, as for the object-language, a forest only with new s-atoms, is uniquely solvable for every valuation of atoms. However, combination of s-predicates with s-quantifiers changes things dramatically. For instance, blind ascriptions of truth, called also infinitary conjunctions, namely claims like “All Ks are true”, for \( K \in \text{P}^2 \), become expressible as \( \forall\phi(K\phi \rightarrow \phi) \).

Technically, a more significant novelty is the dependence of valuations of s-predicates on their argument sentences, not only boolean values of these sentences, and the possibility of violating semantic equivalence of arguments. Consequently, only even cycles can be broken, without breaking the corresponding odd ones, leading to paradoxes. Unlike valuations of \( \mathcal{L} \)-atoms in a domain \( M \), determining a unique solution of the graph \( \mathcal{G}_M(\mathcal{L}^\phi) \), some valuations of s-atoms can make language graph \( \mathcal{G}_M(\mathcal{L}^+) \) unsolvable, as illustrated by the example below.

A small comment seems appropriate. The usual representation of the liar as \( L \leftrightarrow \neg L \), or as \( L \leftrightarrow \neg T(L) \) with \( (T) \forall\phi(T(\phi) \leftrightarrow \phi) \), is a straightforward contradiction. Section 4 will distinguish between that and paradox. Self-reference arises in LSO via s-quantification, so we recast “This sentence is false” as “The only sentence I am saying now is false”.

Example 3.15 The liar Karen says only that everything she is saying now is false, \( K!Kl \), where \( Kl = \forall\phi(K\phi \rightarrow \neg \phi) \). Semikernel \( L = \{K(Kl)\} \cup \{K(S) \mid S \neq Kl\} \), capturing this situation, can not be extended to any kernel, because \( K(Kl) = 1 \) makes \( K(Kl) = 0 \), while each \( X \land K(X) = 0 \), for \( X \neq Kl \), due to \( K(X) = 1 \). This leaves the unresolved odd cycle \( Kl \rightarrow Kl \land K(Kl) \):

\[
\begin{array}{c}
Kl \\
Kl \land K(Kl) \\
X \land K(X) \\
K(Kl) \land X \\
Kl \land X \\
\end{array}
\]

\[
\begin{array}{c}
K(Kl) \\
S \\
S \land K(S) \\
K(S) \land K(K(S)) \\
K(K(S)) \land S \\
\end{array}
\]

Although FOL+ can thus express some paradoxes, none are implied. They appear, as in the example, only due to unfortunate valuations of s-atoms and extension \( \mathcal{L}^+ \) of a classical language \( \mathcal{L} \) remains consistent.

Theorem 3.16 Every language graph \( \mathcal{G}_M(\mathcal{L}^+) \) has a kernel.

To see this, consulting the proof of Lemma 3.12 may be needed, if the following summary does not suffice. The proof of the lemma relies on the fact that evaluation of the root of each DNF-foot \( D[\phi] \) is a boolean function of \( \phi \)’s value: either constant, \( D[\phi] = D[\neg \phi] \), or yielding complementary results on complementary instances, \( D[\neg \phi] = \neg D[\phi] \). Unlike this, DNF-foot \( \neg K(\phi) \lor \neg \phi \) depends on the sentence substituted for \( \phi \), not only on its boolean value. Consequently, while the cycles created by the former come only in pairs with opposite parities, the latter can leave unresolved odd cycles, as seen in Example 3.15. The proof of Lemma 3.12 goes unchanged for a graph \( \mathcal{G}_M(\mathcal{L}^+) \)

\[\text{Their role for truth-theory has been discussed at least since Quine’s [25]. When syntax is arithmetized, they become problematic due to the complications in controlling interaction with the restrictions on convention (T), e.g., [12, 23]. A paradox in LSO, in turn, requires a sentence or s-variable to occur in both a sentential and a nominal position, exemplified also by such blind ascriptions.}\]
if each s-predicate $P(\phi_1...\phi_n)$ is a boolean function of each of its s-arguments, for instance, if for each $1 \leq i \leq n$ and all sentences $\{\phi_1...\phi_n\} \setminus \{\phi_i\}$, one of the two situations occur:

$$
\begin{align*}
\text{either } & \forall \phi_i \in S^+_T : P(\phi_1...\phi_i...\phi_n) \iff P(\phi_1...\phi_i...\phi_n) \\
\text{or } & \forall \phi_i \in S^+_T : P(\phi_1...\phi_i...\phi_n) \iff \neg P(\phi_1...\phi_i...\phi_n).
\end{align*}
$$

This condition, depriving s-predicates of their syntactic character, is violated in Example 3.15, where $K(\neg Kl) = \neg K(\neg Kl)$, while $K(X) = \neg K(X)$ for all $X \notin Kl$. Paradoxes appear thus only when things said (under s-predicates) go too far against the semantics of negation.

Global semantics (3.10), covering by kernels the whole language graph, should be admitted, but Karen should be allowed to say whatever she likes. Such a possibility, allowing also locally coherent statements implying contradictions, can be accommodated by semikernels. A semikernel extending $L$ from Example 3.15, allowing Karen to be the liar, can cover also other facts, even the whole object-level, maintaining internal coherence. But since contradiction follows, $L \vdash Kl \land \neg Kl$, it cannot be extended to include also truth value of $Kl$. The fact $KlKl$ can thus occur but can not be included in a kernel, an overall consistent logical space.

An $L^+$-sequent $\Gamma \Rightarrow \Delta$ is valid, $\Gamma \models \Delta$, if in every language graph $G_M \in LGr(L^+)$ every relevant situation satisfies it. A situation is a semikernel $L$, it is relevant if it covers $\Gamma \cup \Delta$, i.e., $\Gamma \cup \Delta \subseteq E^+([L] = E^-(\Gamma) \cup L$, and it satisfies the sequent if some $D \in \Delta \cap L$ or some $G \in \Gamma \cap E^+(L)$. (As before, this is generalized to valuations $\alpha \in M^{\langle \Gamma, \Delta \rangle}$ of free $L$-variables $\forall \langle \Gamma, \Delta \rangle$.) The following replaces kernels from (3.10) by semikernels, adding the covering condition in the last line.

$$
\Gamma \models \Delta \iff \forall G_M \in LGr(L^+) \forall L \in SK(G_M) : L \models \Gamma \Rightarrow \Delta, \quad \text{where}
$$

$$
\begin{align*}
L \models \Gamma \Rightarrow \Delta \iff & \forall \alpha \in M^{\langle \Gamma, \Delta \rangle} : \\
& \alpha(\Gamma \cup \Delta) \subseteq E^-[L] \iff (\alpha(\Gamma) \cap E^-(\Gamma) \neq \emptyset) \vee (\alpha(\Delta) \cap L \neq \emptyset).
\end{align*}
$$

The semikernel models allow conundrums among $S^0$ affect their own truth, yielding paradoxes or sentences with undetermined values, even when object-language and s-atoms are fully interpreted. Nevertheless, each valuation $\rho$ of $L$-atoms determines interpretation of the object-language, independent from possible paradoxes in the following sense. By Theorem 3.11, the subgraph $G(L)$ (and $G(L^0)$) has a kernel — reflecting simply the standard interpretation of $L$ under $\rho$. Due to absence of edges from $G(L)$ to $G(L^0)$, this kernel is a semikernel of $G(L^0)$, independent from valuation of metalevel sentences $S^0$ and from possible nonexistence of a kernel of $G(L^0)$ extending $\rho$.

Metastatements in $S^0$ do not affect facts also in the sense that John saying $S$ excludes John not saying $S$, but allows John to say not-$S$ and Karen to say anything. Contradicting anybody, even facts, does not affect the object-level, only limits the shared situation to the things agreeable with one’s claims, precluding extension of the respective semikernel to a kernel.

Semantics is thus nonexplosive, admitting seeds of inconsistency in semikernels which can not be extended to kernels, but is two-valued: each semikernel determines a unique boolean value of each sentence, perhaps vacuously, by not covering it. Semikernel models of a theory $\Gamma$ need not cover the whole language, and may exist even if $\Gamma$ implies a contradiction. Even then consequences of $\Gamma$ are not arbitrary, relying on its semikernel models. Syntactic and semantic analyses of FOL$^+$ theories are not separated by any gap. LSO without (cut) provides a sound and complete reasoning.

**Theorem 3.19** $\Gamma \vdash \Delta \Rightarrow \Gamma \models \Delta$, for countable $\Gamma \cup \Delta \subseteq FOL^+$.

## 4 Paracoherence and (cut)

Although Karen can say whatever she likes, even $K(S \land \neg S)$, LSO is not dialetheic, as there is no semikernel containing both $S$ and $\neg S$, that is, satisfying $S \land \neg S$. Its derivability from some assumptions signals the impossibility of combining them with any coherent valuation of $S$. Turning this into a definition, we call $S \subseteq S^+$ a contradiction, $S \subseteq C$, if it is not contained in any semikernel, i.e., $S \not\subseteq L$ for every language graph $G^+$ and $L \in SK(G^+)$. It is a tautology, $S \subseteq T$, if it is contained in every semikernel covering it, $S \subseteq E^-[L] \Rightarrow S \subseteq L$. By Theorem 3.19, $C = \{S \subseteq S^+ \mid S \not\vdash \emptyset\}$ and $T = \{S \subseteq S^+ \mid \forall S_i : S \not\vdash S_i\}$. For a single sentence $S$, $\{S\} \subseteq C$ is abbreviated by $S \subseteq C$. 

Although not dialectic, LSO is paraconsistent being nonexplosive. Semikernels admit contradictions outside the covered set, and semantics (3.18) is local, checking satisfaction of \( \Gamma \Rightarrow \Delta \) only in semikernels covering \( \Gamma \cup \Delta \). A paradox – apparently meaningful statements which, at a closer analysis (expanding the context), lead to a contradiction – is represented by a set of sentences contained in a semikernel which can not be extended to a kernel. Statements implying a contradiction can thus form locally coherent situations that need not imply everything.

If Karen claims to be always lying we can prove that she does not, \( K(Kl) \nvdash \neg Kl \), but no paradox follows yet. If this is everything she says, then we can also prove that she is always lying, and \( K!Kl \vdash Kl \land \neg Kl \) witnesses to a paradox. (Example in Section 5.1 contains the details.)

Now, a contradiction entails every sentence \( S \), i.e., \( Kl \land \neg Kl \vdash S \), reflecting the fact that it does not belong to any semikernel. However, \( K!Kl \nvdash S \) for some \( S \), as can be gathered from the infinite branch of the following derivation:

\[
\begin{align*}
&\vdash S_1 \vdash Kl, K!Kl \vdash S, K(S), S_1 \vdash Kl \\
&\vdash S_1 \vdash Kl, K!Kl \vdash S, K(S) (\text{neq} \ S_1 \neq Kl)
\end{align*}
\]

\[
\begin{align*}
(K(S_1) \rightarrow S_1 \vdash Kl, K!Kl \vdash S, K(S))
\end{align*}
\]

\[
\begin{align*}
(K(S) \rightarrow S \vdash Kl, K!Kl \vdash S) \\
\end{align*}
\]

\[
\begin{align*}
(K!Kl \vdash S, K(Kl))
\end{align*}
\]

The branch keeps substituting all sentences into \( \forall \phi (K\phi \rightarrow \phi \equiv Kl) \). In the limit, \( K(S_i) \) for each \( S_i \neq Kl \) appears to the right of \( \vdash \). If \( S \) is not provable in disjunction with \( K(S) \) or other \( K(S_i) \), e.g., if \( S = J(S_0) \), the following countermodel results, reflecting \( K!Kl \nvdash S \):

\[\vdash \neg S, K(Kl), \forall \phi (K\phi \rightarrow \phi \equiv Kl) \cup \neg K(S_i) \mid S_i \neq Kl. \tag{4.1}\]

In the same way, \( K!Kl \nvdash \emptyset \). Its derivation would copy the attempted \( K!Kl \vdash S \) above, removing all \( S \) and yielding the infinite branch with the countermodel \( Z \setminus \{\neg S\} \).

This brings forth the difference, signalled from the start, between a contradiction, entailing every sentence and belonging to no semikernel, and a “half-contradiction” like \( K!Kl \) which entails some contradiction, but not every sentence, and can be contained in a semikernel. \( S \) is an \textit{s-contradiction} (a paradox) if \( S \vdash C \) for some \( C \in C \), but there is a semikernel \( L \supseteq S \). An \textit{s-contradiction} involves necessarily \textit{s-predicates} and the contradiction it entails involves these \textit{s-predicates}. It does not entail most other contradictions nor any contingent object-level sentences.

Since \textit{s-contradiction} entails some contradiction, like \( K!Kl \vdash \neg Kl \land Kl \), while contradiction entails every sentence \( S \), so that \( \neg Kl \land Kl \vdash S \), using (cut) would yield \( K!Kl \vdash S \). However, semikernel in (4.1) provides a countermodel \( Z \neq K!Kl \Rightarrow S \), so (cut) is not sound. It is trivially admissible for object-language, as long as only LK is used, but changes the semantics for the whole LSO. The contradiction \( Kl \land \neg Kl \), following from Karen’s statement, is not ‘discovered’ in \( Z \). A semikernel that is not a kernel represents a limited context which is only locally consistent, without taking into account the whole language. \( Z \) allows thus Karen to say only \( Kl \), but inquiry into the truth of what she says, i.e., \( Kl \) or \( \neg Kl \), expands this context to the point where the paradox – the impossibility of a valuation of \( Kl \) coherent with \( Z \) – is discovered. Even that does not prevent John from saying (or not) \( S_0 \), captured by a semikernel extending \( Z \) with \( J(S_0) \) (or \( \neg J(S_0) \)).

Provability \( K!Kl \vdash Kl \land \neg Kl \) does not imply nonexistence of a semikernel containing \( K!Kl \), as \( K!Kl \vdash \emptyset \) would do, but nonexistence of such a semikernel covering also \( Kl \land \neg Kl \). As its graph is \( (Kl \land \neg Kl) \vdash \neg Kl \neg Kl K!l \... \), this means that no semikernel containing \( K!Kl \) can contain \( Kl \) or \( \neg Kl \). Semikernels containing \( K!Kl \) have a special relation to this particular contradiction, like the informal reasoning concluding that the liar lies if he does not lie, but nothing more. Most contradictions are still not derivable from \( K!Kl \), e.g., \( K!Kl \nvdash J(S_0) \land \neg J(S_0) \).

To derive everything from \( K!Kl \), via the contradiction it entails, (cut) is needed. It makes derived contradictions explode, bringing us back to the kernel semantics (3.10). Extension of LSO with (cut), \( \vdash_c \), is sound and complete for it.

**Fact 4.2** For a countable \( \Gamma \cup \Delta \subseteq FOL^+ \), \( \Gamma \vdash_c \Delta \) iff \( \Gamma \vdash \Delta \).
Making thus contradictions explode seems the only contribution of \((\text{cut})\). Paradoxes discovered using \((\text{cut})\) can be diagnosed also without it, since if LSO with \((\text{cut})\) derives any contradiction from a theory, already LSO does. By the following theorem, LSO derives than a contradiction of the specific form \(\bot_Q = \bigvee_{S \in Q}(S \land \neg S)\), for a finite set of sentences \(Q\), denoted by \(Q \in S^+\).

**Theorem 4.3** For a countable \(\Gamma \subset \text{FOL}^+\): \((\forall Q \in S^+: \Gamma \not\vdash \bot_Q) \Rightarrow (\forall \bot \in \mathbb{C} : \Gamma \not\vdash_{\text{c}} \bot)\).

### 4.1 Nontransitivity

Unsoundness of \((\text{cut})\) for semikernel semantics (3.18) arises from the subtle element of (ir)relevance: vacuous satisfaction of a formula by a semikernel not covering it. If \(\{X\} \in \mathbb{C}\) then \(\Gamma \vdash X\) either if \(\Gamma \vdash 0\), i.e., \(\Gamma \in \mathbb{C}\), or if \(\Gamma \notin \mathbb{C}\) but every semikernel containing \(\Gamma\) satisfies \(\Gamma \Rightarrow X\) vacuously, by not covering \(X\). In the latter case, semikernels containing \(\Gamma\) may cover other sentences, enabling \(\Gamma \not\vdash \Delta\). Unsoundness of \((\text{cut})\) is limited to such cases, when countermodels to the conclusion satisfy premise(s) only vacuously.

**Fact 4.4** (a1) \(\Gamma \models \Delta, A\) and (a2) \(\Gamma, A \models \Delta\) and (c) \(\Gamma \not\models \Delta\) iff there is a semikernel \(L \not\models \Gamma \Rightarrow \Delta\) but none such can be extended to a semikernel \(L' \supset L\) with \(A \in \mathbb{E}^-[L']\).

**Proof.** \(\Rightarrow\) By \((\text{cut})\) \(\models\) \(\Delta, A\) and \(\Gamma \not\models \Delta\) means that there is a semikernel \(L \not\models \Gamma \Rightarrow \Delta\), i.e., \(\Gamma \in \mathbb{C}\) and \(\Delta \subseteq \mathbb{E}^-(L)\). Let \(L\) be any such. If \(L\) can be extended to \(L'\) covering \(A\), then \(\Gamma \subseteq L'\) and \(\Delta \subseteq \mathbb{E}^-(L')\), while either (ii) \(A \in \mathbb{E}^-(L')\), contradicting (a1) since \(L' \models \Gamma \Rightarrow \Delta, A\), or (ii) \(A \in \mathbb{L}'\), contradicting (a2) as \(L' \not\models \Gamma, A \Rightarrow \Delta\). Thus, \(L\) can not be extended to cover \(A\).

\(\Leftarrow\) A semikernel \(L \not\models \Gamma \Rightarrow \Delta\) establishes (c). Let \(L'\) be any semikernel covering \(\Gamma, \Delta\) and \(A\). By assumption \(L' \models \Gamma \Rightarrow \Delta\) (otherwise it could not cover \(A\)), establishing both (a1) and (a2).

### Possibility to extend a semikernel \(L \not\models \Gamma \Rightarrow \Delta\) to a semikernel covering \(A, \Delta\) and \(A\).

### Inadmissibility of \((\text{cut})\)

Inadmissibility of \((\text{cut})\) in LSO for semikernel semantics (3.18) reminds of nontransitive consequence, studied e.g., in [8, 31, 32]. That approach is fundamentally different, addressing only transparent truth predicate applicable to terms internalizing syntax. Still, the role of \((\text{cut})\), or rather of its absence, seems sufficiently similar in both to warrant a closer comment.

Relation \(\models_{\text{ST}}\) from [31], appearing most relevant for the comparison, holds for \(1 \models_{\text{ST}} \frac{1}{2} \) and \(\frac{1}{2} \models_{\text{ST}} 0\), with \(\frac{1}{2}\) representing paradox. To prevent \(1 \models_{\text{ST}} 0\), transitivity is blocked precisely when the cut formula is a paradox (relatively to the context). Fact 3.17 in [31] excludes \((\text{cut})\) between \(\Gamma \Rightarrow \Delta, A\) and \(\Gamma \Rightarrow \Delta, A\) when \(A\) evaluates to \(\frac{1}{2}\) in the models falsifying \(\Gamma \Rightarrow \Delta\). This is reflected, if not exactly repeated, by our Fact 4.4, barring transitivity to conclusions falsifiable only by models which display their latent paradox under extensions covering \(A\). Conveying very similar informal messages, the two facts differ due to different models of paradox.

This, however, is a fundamental difference between 3-valued semantics in [31, 32] and our 2-valued semantics. It might seem that a semikernel, not covering the whole graph, assigns a third value to all uncovered sentences. This impression is more wrong than right. Semikernels do handle inconsistency by leaving it out, but in each relevant semikernel \(L\), covering the actual sequent, every sentence of the sequent is either true (in \(L\)) or false (in \(\mathbb{E}^-(L)\)). Kernels are only special cases that leave nothing out. Since they exist for language graphs, all sentences can obtain truth values. Paradox is not any third value but a failure to assign any of the two. Atomic claims are either true or false, while the unfortunate third value of paradox, or rather the unfortunate fact of not being amenable to evaluation, arises only from confused (compounds of) sentences. Semikernels allow Karen to say that she is always lying and even to say only that. That a paradox results is as unfortunate a consequence as is the contradiction that she is both telling the truth and not,
which falls out of the reasonable discourse, out of any model. It could be seen as a third value, but it seems more adequate to see it as the impossibility of extending a bivalent valuation, given by a semikernel containing Karen’s statements, to one determining also their truth.

These differences in the semantics come clearly forth in the reasoning systems. 3-sided sequent systems in [31], reflecting 3-valued semantics, come in two variants, disjunctive and conjunctive, which can be related to the 2-sided ones in the expected ways, but extend considerably their expressiveness. We do not dispute their merits, but limit the comparison to the two-sided system for $=^{st}_+$. First, LSO restricted to the mere truth predicate is a trivial extension of LK admitting, besides unrestricted (cut), insertion of $T$ around any sentence. ST reasoning from [32] almost coincides with the so restricted LSO except that, using $=^{st}_++$ with 3-valued models, ST does not admit (cut). Restrictions on (cut) are very similar in ST and LSO, guarding against applications over paradoxes. However, while ST needs such restrictions for reasoning with truth predicate, responsible for paradoxes in internalized syntax, the mere truth predicate does not cause any paradoxes in LSO and restriction on (cut) is used here for reasoning with arbitrary $s$-predicates. Limited to the mere object-language, LSO is just LK admitting (cut).

Perhaps the most significant difference emerges as the result of admitting unrestricted (cut). As noted above, since \[ 1 =^{st}_+ 1 \text{ and } \frac{1}{2} =^{st}_+ 0, \] (cut) trivializes $=^{st}_+$ yielding \[ 1 =^{st}_+ 0. \] In LSO, (cut) does not trivialize the logic but only inconsistent theories, turning the paraconsistent logic of semikernels into the explosive one of kernels. Paradoxical statements of Karen imply specific contradictions, but do not affect John or object-level. Unrestricted (cut), letting paradox entail everything, destroys this bond of relevance, bringing LSO back to classical, explosive logic.

Given so diverging technical contexts, restricted transitivity via paradox is indeed a striking similarity between the two approaches. Still, differences in the scope of (cut)’s applicability and dramatically different consequences of lifting its restrictions suggest that the significance of this similarity may be smaller than its apparent appeal.

5 Examples and future work

Analysis of some examples illustrates the features of paradoxes and their modelling in LSO, suggesting also some directions for possible further work.

5.1 Stating a paradox is possible, even if not evaluating its truth-value

Karen saying that she always lies, $K(\neg \phi)$ with $K(\neg \phi) = \forall \phi (K\neg \phi \rightarrow \neg \phi)$, tells sometimes truth:

\[
\begin{array}{c}
\forall \phi (K\phi \rightarrow \neg \phi), K(\neg \phi) \vdash K(\neg \phi) \\
\neg K(\neg \phi), \neg K(\neg \phi) \vdash \neg K(\neg \phi), K(\neg \phi) \vdash K(\neg \phi) \\
K(\neg \phi) \rightarrow \neg K(\neg \phi), \neg K(\neg \phi) \vdash \neg \forall \phi (K\phi \rightarrow \neg \phi) \\
\end{array}
\]

(5.1)

As noted by Prior, [24], Karen must then also sometimes lie:

\[
\begin{array}{c}
(5.1) \\
\vdash K(\neg \phi), K(\neg \phi) \vdash K(\neg \phi) \\
\vdash \forall \phi (K\phi \rightarrow \phi), K(\neg \phi) \rightarrow K(\neg \phi) \\

\end{array}
\]

The resulting Prior’s theorem

\[
K(\forall \phi (K\phi \rightarrow \neg \phi)) \rightarrow (\exists \phi (K\phi \land \phi) \land \exists \phi (K\phi \land \neg \phi)),
\]

(5.2)
is so far no paradox, as Karen can also say other things. If she does not, what follows is not that she is saying two things, one true and one false, but a contradiction, signalling a paradox seen in Example 3.15. On the one hand, (5.1) gives $K(Kl) \vdash \neg Kl$. To obtain $Kl$ requires capturing that she says nothing else, which amounts to the infinite set of negated atoms $L = \{ \overline{K(S)} \mid S \neq Kl \}$.

\[
\begin{align*}
(5.1) & \\
\vdash & \\
\overline{L},\overline{K(Kl)} \vdash \neg Kl & \\
\overline{L} & ,\overline{K(Kl)},\overline{K(S)},S \vdash \overline{K(S)} \vdash \neg(K(S)) \in L & \\
\overline{L} & ,\overline{K(Kl)} \vdash \neg S & \text{for each } S \neq Kl & (\forall_R^+) \\
\overline{L},\overline{K(Kl)} & \vdash \forall\phi(\overline{K(\phi)} \rightarrow \neg \phi) & \\
\end{align*}
\]

S-equality allows a finite expression and proof of this fact, using $(\forall_R^\sigma)$ instead of $(\forall_R^+)$:

\[
\begin{align*}
(5.1) & \\
\vdash & \\
\overline{K(Kl)} \vdash \neg Kl & \\
\overline{K(Kl)},\overline{K(Kl)},\overline{Kl} & \vdash \overline{K(Kl)},\overline{Kl},\overline{Kl} & (\forall_R^\sigma) \overline{K(Kl)[\sigma/\phi]} & \\
\overline{K(Kl)} \rightarrow \forall\phi(\overline{K(\phi)} \rightarrow \neg \phi) & \\
\end{align*}
\]

The semikernel $L = \overline{L} \cup \{ \overline{K(Kl)} \}$ represents Karen saying only that she is lying. The paradox amounts to the provability of contradiction, $K Kl \vdash Kl \land \neg Kl$, and the fact that semikernel $L$ can not be extended to any covering $Kl$, i.e., making either $Kl$ or $\neg Kl$ true. Karen’s paradoxical statement makes evaluation of its truth-value impossible. This semikernel does not, however, validate any other statements. It does not follow that John is saying something or that snow is not white. Most other facts, and contradictions, remain unprovable.

5.2 Statements do not affect facts, but tautologies can make a difference

Just like the fact of making a statement should not be confused with the statement made, thinking a thought should be distinguished from the thought’s content, not to mention its truth. When thoughts are confused with their truth-values, thinking may seem to force something outside one’s thoughts. The following is quoted with inessential modifications after [1]:

If $K$ is thinking only “Everything I am thinking now is false iff X is false”, then $X$ is true.

We take $X$ as an arbitrary atom, while Asher, reading it as “Everything Tarski is thinking is false”, notes: “By reasoning that is valid in the simple theory of types, we conclude that Tarski was not able to think that snow is white, a bizarre and unwanted consequence of a logic for belief”.

The relevant part of the graph is shown below. $K$ is thinking $A = Kl \leftrightarrow \neg X$, where $Kl = \forall\phi(\overline{K(\phi)} \rightarrow \neg \phi)$ is as in Example 3.15; each $S \land K(S)$ has edges to $K(S)$ and $\overline{S}$, as shown for $A \land K(A)$.

Each combination $2^{|K(A) \cup X|}$ of literals over $K(A)$ and $X$, forms a semikernel and since $\{K(A), \overline{X}\} \nmid X$, so $K(A) \nmid X$. LSO proves something about relations between truth-values of $A$ and $X$, e.g., $A, K(A) \vdash X$, but this isn’t as exciting as $K$’s thought limiting Tarski’s.

If $A$ is the only thought of $K$, $K A = \{K(A), \forall\psi(\overline{K(\psi)} \rightarrow \psi \equiv A)\}$, then $K(S) = 0 = s \land K(S)$ for each $S \neq A$, leaving $A \land K(A)$ undetermined. In LSO still $K! A \nmid X$, as $\{K(A), \overline{X}\}$ can be extended to a semikernel containing $Kl A$. Something new follows now about the relation between truth of $A$ and $X$. Of the two cycles via $A \rightarrow Kl \rightarrow A \land K(A) \rightarrow A$, the one via $\sigma_2$ is even and the other, via $\sigma_1$,
[5.3 Metalanguage: paradoxes and indeterminate statements]

Non-explosive paradoxes are one feature distinguishing metalanguage from object-language. Another such feature are sentences that remain indeterminate in spite of fully interpreted object-language and s-atoms. John saying only that he always tells the truth, the framed $\text{Jt} = \forall \phi (\text{J} \phi \rightarrow \phi)$, is the truth-teller. Each $\pi^{\wedge \text{Jt}}$, for $X \neq \text{Jt}$, is false due to John not saying $X$, while $\text{J(Jt)} = 0$ leaves 2-cycle $\pi^{\wedge \text{Jt}} \models \text{Jt}$ with one solution $\text{Jt} = 1$ and the other $\text{Jt} = 0$.

Thus, unlike for the object-language $\mathcal{L}$, or $\mathcal{L}^+$, valuation of $\mathcal{L}^+$-sentences need not determine unique values of all $\mathcal{L}^+$-sentences. Considering this fact a flaw, as sometimes happens, seems due to the inability or unwillingness to distinguish metalanguage from object-language. In LSO it appears simply as another, besides paradoxes, feature distinguishing the two. The difference between such innocent self-reference of the truth-teller and vicious circularity of paradoxes, reflecting informal indeterminacy of the former versus impossibility of valuating the latter, is captured simply by even versus odd cycles in language graphs.

The unproblematic status of the truth-teller amounts to the informal observation that it says nothing. Making no real claim, its truth or falsity makes no difference. A difference appears if he says also something else, because then, no matter what else it is, the 2-cycle $\text{Jt} \models \pi^{\wedge \text{Jt}} = 1$ and $\text{Jt} = 0$. The mere claim of telling only truth implies consistency of this claim being false.\footnote{Buridan’s early proposal, that each statement claims its own truth in addition to whatever it may be saying, provides thus a ‘solution’ to the liar and similar paradoxes by making them false – but for the price of consistency of all statements being false. Earlier, Bradwardine maintained falsity of paradoxes without this overgeneralization, taking only some – self-negating – statements as claiming also their truth, \cite{Bradwardine, Buridan}. However, which statements are}
5.4 Discussive logic

Semikernels make statements of distinct persons independent from each other. All can occur simultaneously, even if they contradict each other. In this sense LSO follows the tradition of Jaškowski’s discussive logic. Contradictory statements can confront each other in our metalanguage, distinguishing it from the object-language, where contradictions yield explosion. In this sense, Karen saying \(X\) and Karen not saying \(X\) is an objective contradiction, yielding explosion, unlike Karen saying \(X\) and \(\neg X\), which remains the problem of Karen’s or her interlocutors.

The drawing below gives a part of the graph \(G^+\) for such a discussive example, with John saying that Karen always lies, \(J\), and Karen saying that John always tells the truth, \(K\). The two interact via the assumed truth of the framed statements.

The situation in LSO. We do not worry about distinction between ‘believes’ vs. ‘assumes’, central

5.5 Modal logics

The authors of [6] consider the following situation, with a peculiar anaphoric self-reference.

(1) Ann believes that Bob assumes that (2) Ann believes Bob’s assumption is wrong.

The question whether \(^*\) Ann believes Bob’s assumption to be wrong, is answered by the following informal reasoning. (We insert (a), (b), (c) to mark the assumptions used later.)

If so, then in Ann’s view, Bob’s assumption, namely “Ann believes that Bob’s assumption is wrong”, is right. But then Ann does not believe that Bob’s assumption is wrong (a), which contradicts our starting supposition. This leaves the other possibility (b), that Ann does not believe that Bob’s assumption is wrong. If this is so, then in Ann’s view, Bob’s assumption, namely “Ann believes that Bob’s assumption is wrong”, is wrong (c). But then Ann does believe that Bob’s assumption is wrong, so we again get a contradiction.

Whether statements form a paradox or not depends often on representation, and authors build an impressive machinery to ensure that these do. The following is only one possible way of capturing the situation in LSO. We do not worry about distinction between ‘believes’ vs. ‘assumes’, central

so self-negating is intractable and/or contingent, as suggested by the examples in Section 5.2 and following 5.4, or as witnessed by many-valued approaches trying to identify faulty statements.
in [6], but denote Ann’s thoughts by $A$, Bob’s by $B$ and let $\sigma$ be what Ann believes to be Bob’s assumption (2). This yields the following representation:

1. $AB\sigma$

2. $\sigma \iff A(B\sigma \land \neg \sigma) \iff AB\sigma \land A\neg \sigma$.

As in normal modal logic, $A$ distributes over conjunction and implication, (d). Equivalence (2), available to the agents, has any number of $A$s (or $B$s), in particular, $A(\sigma \leftrightarrow A(B\sigma \land \neg \sigma))$. Adding the assumptions from the informal arguments yields, for all $\phi, \psi$:

(a) $AA\phi \leftrightarrow A\phi$
(b) $A\phi \leftrightarrow \neg A\neg \phi$

(c) $\neg A\phi \rightarrow A\neg \phi$, for relevant $\phi$
(d) $A(\phi \land \psi) \leftrightarrow (A\phi \land A\psi)$ and $A(\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow A\psi)$

The question whether (*) Ann believes Bob’s assumption to be wrong, asking apparently whether $A\sigma$ or $\neg A\sigma$ (or $A\neg \sigma$), since $\sigma$ is what $B$ assumes, asks equally whether $\neg \sigma$ or $\sigma$, as $\sigma$ states exactly that Ann believes Bob’s assumption to be wrong. Taking the former, $LSO$ proves $Ax \vdash A\sigma \land \neg A\sigma$ from $Ax = \{(1), (2), (a), (b), (c), (d)\}$, but we spare the reader the involved intricacies. In spite of appearances, this paradox concerns only a single complex belief of Ann. No axioms about Bob are needed and their irrelevance becomes apparent when we note that (1) simplifies (2) to a version of knower’s paradox, $\sigma \leftrightarrow A\neg \sigma$. Subtleties of the analyses in [6], intended for applications to multiagent games, result in a particular modal logic, while $LSO$ is a general schema admitting various specializations and helping to unveil uniform patterns. Granting the ingenuity of the scenario, there seems to be nothing specifically intensional or modal about the paradox, at least, represented as above. Reading $A\phi$ as ‘$A$ claims $\phi$', the scenario becomes

1. Ann claims that Bob claims that (2) Ann claims that Bob’s claim is false, and paradox of a semantic character arises, and is proven, in the same way.

In general, modal paradoxes have natural representation in $LSO$, as modal operators are specific s-predicates, with axiomatic theories specializing $LSO$. ($LSO$ might need adjustments to handle such axiomatizations. To some extent this can be done in a structural way, e.g., as in [20], but one can also simply add modal axioms to the antecedents of sequents, as we did in the example above. Necessitation rule requires then infinity of such premises, with all nestings of (operators corresponding to) box around axioms, but $LSO$ admits that.) Relating modal logic to $LSO$ would take at least another paper, so we only comment one example, utilizing the difference between language graphs and Kripke frames, reflected here by that between sentences and propositions.

Karen saying only $\phi$, that is $K\phi \land \forall \psi(K\psi \rightarrow \psi \equiv \phi)$, enters Kaplan’s formula from [15], $\forall \phi \forall \psi(K\psi \leftrightarrow \psi \equiv \phi)$, written in the form

(A) $\forall \psi(K\psi \land \forall \psi(K\psi \rightarrow \psi \equiv \phi))$.

The important difference is that Kaplan uses $\equiv$ as equality of propositions, instead of our syntactic equality $\equiv$. With s-quantifiers ranging also over propositions, viewed as arbitrary subsets of possible worlds, a cardinality argument excludes an operator $K$ satisfying (A), making it false rather than paradoxical. However, (A) says that for every $\phi$ it is possible for $K$ to say $\phi$ and nothing else, which seems quite plausible in limited situations. According to [15], logic should not rule it out. Attempts to save (A) (e.g., by taking as propositions only some subsets, [18], or by restricting principle of universal instantiation, [3]) leave the issue open.

The cardinality argument does not affect language graphs, where $\Diamond$ and $K$ act on sentences and not any subsets. (A) is trivially satisfied by semikernel $L$ containing $\forall \phi \Diamond (K!\phi)$ and atom $\Diamond (K!S_i)$, for each $S_i \in S^+$. Ensuring modal content of $\Diamond$ and $K$, by closing $L$ under appropriate modal axioms, does not change the situation and keeps (A) satisfiable, as the problem is due to the model of propositions and not modalities.

5.6 Definitional extensions and convention (T)

As every kernel is a semikernel, the explosive kernel semantics is a special case of the paraconsistent semantics of semikernels, corresponding to consistent theories and languages. Theorem 3.19 can help checking whether we are in such a desirable situation, and so can condition (3.17). Its applications, however, can be hampered by its semantic character and syntactic conditions doing
the same would be desirable. A modest example of such a condition is a *definitional extension*, extending any language \( \mathcal{L}^+ \) to \( \mathcal{L}^P \) with a fresh \( s \)-predicate symbol \( P \), by a sentence of the form

\[
\forall \phi (P(\phi) \leftrightarrow \exists \psi F(\phi, \psi)),
\]

(5.3)

where \( \exists \psi F(\phi, \psi) \) is an \( \mathcal{L}^+ \)-formula, with free variables \( \phi \) among those of the left side \( P(\phi) \). Any extension of \( \mathcal{L}^+ \) to \( \mathcal{L}^P \) introduces into each graph \( G(\mathcal{L}^+) \) all complex sentences with \( P(\phi) \) and atomic 2-cycles \( P(S) \leftrightarrow P(S) \), for every sentence \( S \) of \( \mathcal{L}^P \). For the special extension (5.3), \( G(\mathcal{L}^+) \) can be extended to \( G(\mathcal{L}^P) \) by drawing from \( P(S) \) a double edge to its defining sentence \( \exists \psi F(S, \psi) \), instead of an edge to \( P(S) \). Every kernel of such a graph determines a kernel of the graph with atomic 2-cycles where each \( P(S) \) and \( \exists \psi F(S, \psi) \) obtain the same values, and vice versa.

Definitional extension preserves solvability by Theorem 5.4 below, according to which any solution of any graph \( G_M(\mathcal{L}^+) \) can be expanded to a solution of \( G_M(\mathcal{L}^P) \). The proof amounts to elimination of symbol(s) \( P \), replacing each \( P(S) \) by its definiens \( \exists \psi F[S, \psi] \). This operation, trivial in FOL, has to be performed recursively (e.g., \( P(P(S)) \) needs repeated replacements) on a cyclic graph and is given in Appendix 7.5.

**Theorem 5.4** For every \( \Gamma \subseteq \mathcal{L}^+ \) and its definitional extension \( F \), every kernel model of \( \Gamma \) can be extended to a kernel model of \( \Gamma \cup F \).

As a special case, convention (T) \( \forall \phi (T \phi \leftrightarrow \phi) \) satisfies trivially (5.3), so we obtain

**Corollary 5.5** Each kernel model of any \( \Gamma \subseteq \mathcal{L}^+ \) can be extended to a kernel model of \( \Gamma \cup \{ (T) \} \).

Holding for FOL and higher order classical logics, this does not contradict Tarski’s undefinability theorem. On the one hand, \( T \) above is just the identity operator, not any truth-predicate ‘decoding’ numbers (or names) as formulas. More significantly, substitution has not been assumed but this, requiring extension of LSO to open formulas, must be left for further work. A promising prospect for that work is the possibility – arising from the separation of metalevel and object-level – to distinguish restrictions on substitution of formulas from those on substitution of terms.

**6 Summary**

Paradoxes appearing only in metalanguage distinguish LSO from all main-stream approaches, where metalanguage is internalized in the object-language. Although the distinction can be recovered from such a coding, the view of paradoxes and their sources changes drastically. With internalized syntax, only convention (T) is responsible for all paradoxes, and its restrictions decide which coded statements become interpreted as truth-bearers, and which do not. Applied in classical context, a paradox becomes a contradiction causing explosion and search for further restrictions on convention (T). In LSO, paradoxes affect only metalevel, leaving all interpretations of object-language available.

Avoiding paradox in internalized metalanguage requires a holistic treatment in form of a single theory of truth-predicate, specifying the limits of applicability of convention (T). In LSO, on the other hand, the problem can be handled piecewise by avoiding paradoxes each time a new \( s \)-predicate or their group is introduced. This certainly invites to a search for uniform conditions excluding paradoxes, but allows also individual treatment of distinct predicates.

Another feature, besides paradoxes, distinguishing metalanguage from object-language are statements like the truth-teller, that remain indeterminate even when all atoms are consistently evaluated. Uneasiness with this fact seems to arise from conflating object-language with metalanguage and extrapolating to the latter the unique interpretation of the former under every valuation of atoms.

Representation of intensional and semantic statements in language graphs makes clear the structural identity of the respective paradoxes. Intensional and modal logics can refine the abstract schema of LSO, but its generality displays semantic and intensional paradoxes as arising due to the same, vicious circularity represented by the unresolvable odd cycles in language graphs.
This claim requires, of course, a reservation. Yablo paradox appears noncircular, unless some rather sophisticated concept of circularity is introduced. In LSO it is just graph cycles and Yablo graph $Y = (\omega, \{(i, j) \mid i < j\})$ has none. Theorem 3.16 ensures that language graphs are free from paradoxes, including Yablo’s, so they do not affect our metatheory. One can nevertheless capture the essential aspects of Yablo paradox, e.g., by the following theory $Y$ from [17]:

(a) a transitive binary relation $R$ on a nonempty subset of sentences,
(b) that has no endpoints, $\forall \alpha \exists \beta R(\alpha, \beta)$, and where
(B) $s$-predicate $P$ satisfies the formula: $\forall \alpha (P(\alpha) \leftrightarrow \forall \beta (R(\alpha, \beta) \rightarrow \neg P(\beta)))$.

A single sentence with a loop provides a model of $R$, but so does $\omega$ sentences with a strict total order. Semikernel containing $Y$ has no extension to a kernel, so this is just another instance of the general concept of paradox in LSO. Like all others, it arises due to the unfortunate definition (B) of $s$-predicate $P$, though not due to any (semantic) circularity. This raises the question about general structure of paradox, which is answered partially by graph representation and kernel theory. For finitary graphs (having either no infinite outgoing simple paths or no vertices with infinite outdegree), absence of odd cycles guarantees kernel existence by Richardson’s theorem [30].

In general, one has to exclude also some form of Yablo’s pattern. Such a pattern was proposed in [5] and, in a simpler form, in [34], where its exclusion is shown sufficient for a large class, but not all, of graphs. It is only conjecture to suffice in general.

Language graphs form only a very special class of graphs, and proofs of their solvability, Theorems 3.11 and 3.16, rely heavily on the features implied by the structure of the language. This also moves the question from general solvability of graphs to the conditions on valuations of $s$-predicates ensuring existence of solutions. These conditions can not be specified exhaustively in any effective way, but further investigation of their formulations, in terms of graphs or logic, is an interesting challenge.

Concerning theory $Y$, quoted above, the author observes that its “inconsistency [...] has nothing to do with truth, for it [...] arises irrespective of what $P$ means: other than the Yablo scheme itself (B) and the auxiliary axioms (a), (b), no specific axioms for $P$ are used in the deduction of the inconsistency.” Indeed, no other specific axioms for $P$ are used, but (B) is exactly such an axiom – causing the paradox and conforming to our model of paradoxes arising from definitions of metapredicates. It has nothing to do with truth predicate which only recovers paradoxical effects when metalanguage is internalized in object-language by Gödelization or other means.

Removing now $P$ from (B) yields a plain contradiction (just as does the unchanged formulation of $Y$ with variables ranging over objects rather than sentences). Indeed, paradox appears only by implying a contradiction, while with internalized syntax is only its special case. Both have the same patterns but, at least informally, very different roles, which LSO tries to preserve. Contradictions make modeling the world simply impossible. In that LSO remains classical. Paradoxes, on the other hand, are only problems of our way of speaking about the world and about the language, so contradictions they imply are not as devastating. Although they too make it impossible to form a coherent view of the whole world, they can often be maintained for a long time, until their contradictory consequences become insufferable. Importantly, they allow one to retain common sense, an adequate view of the world at the object-level which, so to say, can function as an independent semikernel in dissociation from the metalevel confusions.

These differences notwithstanding, our model complies fully with the diagnosis from [7], according to which paradoxes arise from taking for granted some assumptions that, on a closer analysis, display a contradiction. Often, such assumptions are hidden behind the name “definition”. Restricting $Y$ above to (a) and (b), one might claim (B) to be merely a definition of $P$. Such a view has permeated much of the discussion, e.g., around revision theory. LSO embodies this idea, since paradoxes arise here exactly due to the way $s$-predicates are defined by their valuations. So understood definitions are, however, just axioms or valuations – far from logically innocent. Specific definitions – valuations – of $s$-predicates amount to specific claims, just like do valuations of object-level predicates. At the metalevel, where self-reference is possible, they can give phenomena not occurring at the object-level, like paradoxes and undetermined sentences.

Granting that paradoxes yield inconsistency due to bad assumptions or ‘definitions’ does not extend to the extreme cases of the view, originating with Tarski’s diagnosis and voiced occasionally,
if only informally, in recent years, according to which natural language simply is inconsistent [2,4,11,22]. This view, arising from perceiving predicates on arithmetized syntax as an adequate model of (natural) metalanguage, need not be maintained if we replace them by operators. Much work remains but the inconsistency of LSO with the needed extensions remains to be seen – or, perhaps, countered.

LSO reflects the informal intuition that paradoxes do not reside in the language as such, but in the way we talk about it. S-predicates and their valuations/definitions represent just various ways of talking about language in the language. Each paradox makes a claim, represented by the valuation of the involved s-predicates, and can be avoided by avoiding the unfortunate claims. If nobody claims to be (always) lying, no liar paradox results and the language remains consistent.

To this, however, one wants to object! Statements like the liar L “This sentence is false” do not have to be claimed, they cause trouble by simply being there. Well, by simply being, the liar does claim its falsity. In the usual representation, it becomes $L \leftrightarrow \neg \lambda(L)$ or simply $L \leftrightarrow \neg L$, expressing the pretence to a truth-value satisfying its semantic claim, this unsatisfiable equivalence. The paradox amounts to the nonexistence of a boolean value satisfying this claim. This case, represented as a mere contradiction, is avoided as contradictions are in general, by withdrawing the unsatisfiable claim, here, $L \leftrightarrow \neg L$ or $L \leftrightarrow \neg \lambda(L)$. (The latter can be rejected, as LSO does not suffer from diagonalization lemma.) More elaborate cases, like somebody claiming only to be always lying, are represented explicitly as potential paradoxes in LSO. They make it more transparent that the only causes of troubles are implausible claims.

7 Appendix: Language graphs and (semi)kernels

7.1 Some facts about (semi)kernels

The following equivalent semikernel condition is used in some proofs.

**Fact 7.1** For any $L \subseteq V : E(L) \subseteq E^-(L) \subseteq V \setminus L \iff E(L) \subseteq E^-(L) \cap V \setminus L$.

**Proof.** If $E(L) \subseteq E^-(L) \subseteq V \setminus L$ then $E(L) \subseteq E^-(L) \cap V \setminus L$. If $E(L) \subseteq E^-(L) \cap V \setminus L$ then $E^-(L) \subseteq V \setminus L$, for if some $x \in E^-(L) \cap L$ then $E(x) \cap L \neq \emptyset$, i.e., $E(L) \not\subseteq V \setminus L$. □

The two facts below imply equisolvability of graphs, showing actually that the two have essentially the same solutions: each solution of one can be expanded to a solution of the other, and each solution of the other, restricted to the first, is its solution. These facts, applied implicitly on the drawings, justify also duplication of vertices $S_M$ as Aux, without affecting solutions.

A path $a_0 \ldots a_k$ is isolated if $E_G(a_i) = \{a_{i+1}\}$ for $0 \leq i < k$ and $E_G(a_i) = \{a_{i-1}\}$ for $0 < i < k$. A double edge, introduced earlier, is an isolated path of length 2. **Contraction** of such an isolated path amounts to identifying the first and the last vertex, joining their neighbourhoods and removing the intermediate vertices, i.e., obtaining graph $G'$ where $V_{G'} = V_G \setminus \{a_1 \ldots a_k\}$, $E_{G'}(a_0) = E_G(a_k)$ and $E_{G'}(a_k) = E_G(a_0) \cup E_G(a_k) \setminus \{a_{k-1}\}$. The first fact is a trivial observation.

**Fact 7.2** If $G'$ results from $G$ by contracting an isolated path of even length, then

$\forall K' \in \text{sol}(G') \exists ! K \in \text{sol}(G) : K' \subseteq K$, and $\forall K \in \text{sol}(G) : K \cap V_{G'} \in \text{sol}(G')$.

The same holds if $G'$ results from a transfinite number of such contractions, provided that no ray, i.e., an infinite outgoing path with no repeated vertex, is contracted to a finite path.

The second fact shows that identifying vertices with identical out-neighbourhoods preserves and reflects (semi)kernels. To define this operation, let $R_G \subseteq V_G \times V_G$ relate two vertices in $G$ with identical out-neighbourhoods, i.e., $R_G(a,b) \Leftrightarrow E_G(a) = E_G(b)$. It is an equivalence, so let $G^+$ denote the quotient graph over equivalence classes, $[v] = \{u \in V_G \mid R_G(u,v)\}$, with edges $E_G([v],[u]) \Leftrightarrow \exists w \in [v], u \in [u] : E_G(v,u)$. The operation can be iterated any number of times, denoted by $G^{n+1}$ and defined by: $G^{1} = G^+$ and $G^{n+1} = (G^{n})^+$. Vertices of $G^\alpha$ are taken as subsets of $V_G$, $[u] = \{v \in V_G \mid \exists i \leq \alpha : R_G([v],[u]^i)\}$. For limit ordinals $\lambda$, $G^{\lambda}$ is given by $V_{G^{\lambda}} = \{[u] \mid u \in V_G\}$ where $[u] = \bigcup_{i < \lambda} [u]^i = \{v \in V_G \mid \exists i < \lambda : R_G([v],[u]^i)\}$ and $E_G^{\lambda}([v],[u]) \Leftrightarrow \exists n \in \lambda : E_G^{n+1}([v]^{n+1},[u]^{n+1})$. 
Fact 7.3. For every ordinal \( n \), and \( SKr \) denoting either kernels or semikernels (sol or SK):

(a) \( K \in SKr(G) \Rightarrow ([v]^n \mid v \in K) \subseteq SKr(G^{in}) \), and

(b) \( K^{in} \in SKr(G^{in}) \Rightarrow \bigcup K^{in} \in SKr(G) \).

Proof. (1) The proof for \( n = 1 \) shows the claim also for every successor \( n \).
(a) \( K^k = \{ v \mid v \in K \} \) is independent, for if \( E_{G^k}([v],[w]) \) for some \( v, w \in K^k \), then \( E_G(v, w) \) for some \( v, w \in K \), contradicting independence of \( K \) if \( x \in K \) then \( [x] \subseteq K \), since \( \forall x, y \in [v] : E_G(x) = E_G(y) \), so \( E_G(x) \cap K = \emptyset \Leftrightarrow E_G(y) \cap K = \emptyset \).

If \( v \in V_{G_k} \setminus K^k \), then \( [v] \subseteq V_{G_k} \setminus K \subseteq E_G(K) \), so \( \forall v \in [v] \exists w \in K : E_G(v, w) \). Then \( [w] \in K^k \) and \( [v] \in \mathcal{E}_{G_k}(\{[v]\}) \subseteq \mathcal{E}_{G_k}(K^k) \). Thus \( V_{G_k} \setminus K^k \subseteq \mathcal{E}_{G_k}(K^k) \), so \( K^k \subseteq sol(G^k) \).

If \( K \in SK(G) \) and \( v \in \mathcal{E}_{G_k}(K^k) \), i.e., for some \( v \in [v] \in K \), then \( [v] \subseteq \mathcal{E}_{G_k}(w) \) and \( [w] \in K^k \), so \( v \in \mathcal{E}_{G_k}(K^k) \), i.e., \( \mathcal{E}_{G_k}(K^k) \subseteq \mathcal{E}_{G_k}(K^k) \), so \( K \subseteq SK(G^k) \).

(b) \( K = \bigcup K^k = \{ v \in V_G \mid [v]^k \subseteq K \} \). If \( v \in V_G \setminus K^k \) for some \( x \in K \), then \( v \notin K \) for if \( v \in K \), i.e., \( [v]^k \subseteq K \), then \( [v]^k \in \mathcal{E}_{G_k}(K^k) \subseteq \mathcal{E}_{G_k}(K^k) \) contradicting independence of \( K \). If \( v \in V_G \setminus K \), i.e., \( [v]^k \notin K \), then there is some \( u \in \mathcal{E}_{G_k}(K^k) \) for \( K = \bigcup K^k = \bigcup \mathcal{E}_{G_k}(K^k) \), independence of \( K \) follows as above. If \( v \in \mathcal{E}_{G_K}(K^k) \), then \( [v]^k \in \mathcal{E}_{G_k}(K^k) \subseteq \mathcal{E}_{G_k}(K^k) \), i.e., for some \( n \in \lambda : [v]^n \in \mathcal{E}_{G_k}(K^k) \). By IH, \( [v]^n \subseteq \mathcal{E}_{G_k}(K^k) \subseteq \mathcal{E}_{G_k}(K) \). Hence \( \mathcal{E}_{G_K}(K) \subseteq \mathcal{E}_{G_K}(K) \).

\( \square \)

7.2 Logical and graph equivalences

We formulate logical and some other notions of equivalence in terms of graphs. Two \( L^+ \) sentences are equivalent, in \( G_M(L^+) \), if they belong to the same kernels. \( L^+ \) sentences are (logically) equivalent if they are so in every language graph:

\[
\begin{align*}
&\text{for a graph } G \text{ and } A, B \in V_G : \quad A \overset{G}{\leftrightarrow} B \iff \forall K \in sol(G) : A \in K \Leftrightarrow B \in K \\
&\text{for } A, B \in S^+_M : \quad A \overset{M}{\leftrightarrow} B \iff A \overset{G^+_M}{\leftrightarrow} B \\
&\text{for } A, B \in S^+ : \quad A \overset{\omega}{\leftrightarrow} B \iff \forall M : A \overset{\omega}{\leftrightarrow} B.
\end{align*}
\]

A more specific equivalence will be used, corresponding to prenex operations. Each sentence can be written in PDNF, that is, prenex normal form with matrix in DNF. Two \( L^+_M \) sentences are PDNF equivalent, denoted by \( A \overset{P}{\leftrightarrow} B \), if they have (also) identical PDNFs. To show that PDNF equivalence implies \( L^+ \) equivalence, we use a more structural notion of equivalence in a graph.
By $E^*_G$, we denote the reflexive and transitive closure of $E_G$ and by $E^*_G(S)$, for $S \in V_G$, the subgraph of $G$ induced by all vertices reachable from $S$. A common cut of $A, B \in V_G$ is a set of vertices $C \subseteq E^*_G(A) \cap E^*_G(B)$, such that every path leaving $A$ and prolonged sufficiently far crosses $C$ and so does every path leaving $B$. ($C$ may intersect $A$ and $B$ and contain vertices on various cycles intersecting $A$ and $B$.) We say that $A$ and $B$ are cut equivalent, $A \lessdot B$, if there is a common cut $C$ such that for every correct (not falsifying (3.2)) valuation of $C$, every correct extension to $\{A, B\}$ forces identical value of $A$ and $B$. Obviously, if $A \lessdot B$ in a graph $G$, then also $A \lessdot B$, as each $K \in \text{sol}(G)$ determines a correct valuation of every common cut of $A$ and $B$.

**Fact 7.5** For $A, B \in S^+_M$ in $G_M(L^+)$, if $A \lessdot B$ then $A \lessdot B$, hence $A \lessdot B$.

**Proof.** Letting $G = G_M(L^+)$ and assuming $\text{sol}(G) \neq \emptyset$, we verify standard prenex transformations, considering only $s$-quantifiers, as object-quantifiers can be treated in the same way.

1. The claim holds trivially for $B$ obtained by renaming bound $s$-variables (avoiding name clashes) in $A$, as the two have the same subgraph. This is also the case for the subgraphs of $A = \neg \exists \phi D[\phi]$ and $B = \exists \phi \neg D[\phi]$.

2. $A = (\forall \phi D[\phi]) \land C \lessdot \forall \phi (D[\phi] \land C) = B$, with no free occurrences of $\phi$ in $C$. On the schematic subgraph below, $X_i, X_j, \ldots$ stand for all $S^+_M$ and common cut is marked by the waved line.

   ![Graph](image)

   Inspecting the graph, we see that, for any kernel $K$:

   $B \in K \iff (\forall D[X_i] \land C) \in K$ for all $X_i \iff (C \in K \land (D[X_i] \in K$ for all $X_i)) \iff A \in K$.

3. For $A = \neg \exists \phi D[\phi] \lessdot \forall \phi \neg D[\phi] = B$ the schematic subgraph is as follows:

   ![Graph](image)

   Obviously, for any kernel $K : A \in K \iff \Diamond \in K \iff (D[X_i] \notin K$ for all $X_i) \iff B \in K$.

Thus, every sentence in $L^+$ has an $\lessdot \vdash$-equivalent PDNF sentence. A useful consequence is that, considering below solvability of $G_M(L^\Phi)$ or $G_M(L^\Phi)$, we can limit attention to sentences in PDNF.

7.3 No paradoxes in $L^\Phi$ – solvability of $G(L^\Phi)$

Extending any classical language $L$ with $s$-quantifiers to $L^\Phi$ does not introduce any paradoxes. The following theorem shows a stronger claim that, in a domain $M$, all $L^\Phi_M$ sentences obtain unique values under every valuation of $L^\Phi_M$ sentences, which is determined by a valuation of atoms $A_M$.

**Theorem 3.11** In any $G_M(L^\Phi)$, each $\rho \in 2^{S^+_M} \text{ has a unique extension } \hat{\rho} \in \text{sol}(G_M(L^\Phi))$ with $\hat{\rho}|_{S_M} = \rho$. 
PROOF. Graph $G_M(L^F)$ consists of two subgraphs, the strong component with all $s$-quantified sentences, $G_M(L^F \setminus L) = \bigcup_{A \in S^F_M \setminus S_M} G_M(A)$, and the forest $G_M(L) = \bigcup_{B \in S_M} T_M(B)$ of trees for object-language sentences, with no edges from the latter to the former. For each $A \in S^F_M \setminus S_M$ in the former, there are (single or double) edges from external atoms $V \in \text{ext}(G_M(A))$, to the roots of $G_M(V)$, that are trees $T_M(V)$ when $V \in S_M$. By Lemma 3.12 below, valuation $\rho$ of $S_M = V_{G_M(L)}$, determines a solution $\rho_A$ of each $G_M^-(A)$ (subgraph of $G_M(A)$ without its DNF-feet), compatible with every valuation of $\text{ext}(G_M(A))$. Hence, these can be combined into $\rho \cup \bigcup_{A \in S^F_M \setminus S_M} \rho_A$ forcing value $\rho_V(V)$ at each $V \in \text{ext}(G_M(A))$, and thus determining solutions of all DNF-feet. Each $G_M(A)$ obtains thus a solution $\rho_A \supset \rho_A$, yielding a unique $\hat{\rho} = (\rho \cup \bigcup_{A \in S^F_M \setminus S_M} \rho_A) \in \text{sol}(G_M(L))$, extending $\rho$.

The missing lemma shows that for each sentence $A \in S^F_M \setminus S_M$, solution of the subgraph of $G_M(A)$ without its DNF-feet, denoted by $G_M^-(A)$, depends on the valuation of $S_M$, but not of external atoms $\text{ext}(G_M(A))$, as the second part of the lemma states. Valuation of $\text{ext}(G_M(A))$ affects, of course, values in DNF-feet in which they occur.

**Lemma 3.12** For every graph $G_M(L^F)$ and sentence $A \in S^F_M$, each valuation $\rho$ of $S_M$ and of external atoms of $G_M(A)$, $\rho \models [S_M \cup \text{ext}(G_M(A))]$, has a unique extension to $\rho_A \models \text{sol}(G_M(A))$. Restriction $\rho_{S_M}$, determines restriction of $\rho_A$ to $G_M^-(A)$: if $\rho_{S_M} = \sigma|_{S_M}$ then $\rho_A|_{G_M^-(A)} = \sigma_A|_{G_M^-(A)}$.

**Proof.** By Fact 7.5, we can limit attention to sentences in PDNF.

For $A \in S^F_M$, with the number $q(A) = n + 1 \geq 1$ of $s$-quantifiers and $s$-variables, and for $n$-sequence of sentences $\pi \in (S^F_M)^n$ substituted for the $n$ $s$-variables of $A$ bound by its first $n$ quantifiers, the roots of all feet, $A(\pi S) = D[\pi S]$, $S \in S^F_M$, are grandchildren of vertex $A(\pi) = \mathfrak{Y} \phi D[\pi \phi]$. (On the drawing, $\mathfrak{Y} = \exists$ and all feet have the common parent $\bullet$; when $\mathfrak{Y} = \forall$, their distinct parents are children of $A(\pi)$.) Each foot $A(\pi S)$ represents an application of the same boolean function $d^F(\phi) = D[\pi \phi]$, evaluating $D[\pi \phi]$ given valuation of its parameters $\pi, \phi$ and, possibly, some atoms $L_A \subset S_M$ occurring in the original matrix $D[\ldots]$. For any $\rho \in 2^{S_M}$, $L_A$ obtain fixed values so, considering $d^F$, we assume the effects of $\rho(L_A)$ taken into account.

1. The *internal* vertices of $\pi$, $\text{int}(\pi)$ are sentences occurring on the path after substitutions, and *external* ones are those which do not $A = S^F_M \setminus \text{int}(\pi)$. Some ‘sinks’ of the feet have single or double edges to vertices from $\pi$, which are $\text{int}(\pi)$, including $\tau_0 = A$ and $\mathfrak{Y} \phi D[\pi \phi]$ (when this is substituted for $\phi$ in $D[\pi \phi]$.) As branches from $\bullet$ instantiate $\phi$ with every sentence $S \in S^F_M$, all sentences from $\text{int}(\pi)$ do occur in some feet.

2. Depending on whether $\mathfrak{Y}$ is $\forall$ or $\exists$, the value at vertex $\mathfrak{Y} \phi D[\pi \phi]$, as a function of values of its grandchildren, is either

   \[ (*) \exists \phi D[\pi \phi] = \bigvee_{S \in S_M} d^F(S) \text{ or } \forall \phi D[\pi \phi] = \bigwedge_{S \in S_M} d^F(S). \]

We consider first the case when $|A| = q(A) - 1$, i.e., $A(\pi) = \mathfrak{Y} \phi D[\pi \phi]$ is the grandparent of the completely substituted (roots of) DNF-feet ($D[\pi A], D[\pi B]$, etc., on the drawing).

Every valuation of sentences from $\pi$, abbreviated as $\alpha \in 2^\pi$, specializes function $d^F(\phi)$ to a unary boolean function $d^F(\alpha)(\phi) = D[\alpha(\pi) \phi]$, and (*) to

\[ (***) \exists \phi D[\alpha(\pi) \phi] = \bigvee_{S \in S_M} d^F(\alpha(S)) \text{ or } \forall \phi D[\alpha(\pi) \phi] = \bigwedge_{S \in S_M} d^F(\alpha(S)). \]
iii. As a boolean function of one variable, $d^{\alpha}(\pi)(\phi)$ is either constant or not. If it is constant, i.e., $d^{\alpha}(\pi)(\phi) = d^{\alpha}(\pi)(\neg\phi)$, then $\exists \mathcal{D}[\phi] \alpha(\pi)\phi$ obtains the same value in either case of (**). Otherwise, $d^{\alpha}(\pi)(\neg\phi) = d^{\alpha}(\pi)(\phi)$ and, since for each $S \in S_{M}^{\phi}$ both $d^{\alpha}(\pi)(S)$ and $d^{\alpha}(\pi)(\neg S)$ enter evaluation of (**), this yields constant 0 at their least common predecessor (• when $\mathcal{F} = \exists$ and $\mathcal{A}(\pi)$ when $\mathcal{F} = \forall$). In this way, for every $\alpha \in 2^{\pi}$, $\mathcal{A}(\pi)$ obtains a unique value $\alpha'(\mathcal{A}(\pi))$, induced from all $D[\alpha(\pi)S]$ by (**), but determined already by $d^{\alpha}(\pi)(\phi)$, independently from valuation $\alpha(\mathcal{A}(\pi))$, i.e., if $\alpha_{0}, \alpha_{1} \in 2^{\pi}$ differ only at $\mathcal{A}(\pi)$, then $\alpha_{0}(\mathcal{A}(\pi)) = \alpha_{1}(\mathcal{A}(\pi))$, and (ii) independently from valuation of $\mathcal{A}(\pi)$, since each external vertex $S$ enters both evaluation of $d^{\alpha}(\pi)(S)$ and of $d^{\alpha}(\pi)(\neg S)$, with jointly constant contribution to (**). As just explained.

Point (i) means that cycles from the feet to $\mathcal{A}(\pi)$ admit a unique solution $\rho_{A(\pi),\alpha}$ to the subgraph $G_{M}(\mathcal{A}(\pi))$ of $G_{M}(\mathcal{A})$, given any $\rho \in 2^{S_{M} \cup \text{ext}(\pi)}$ and $\alpha \in 2^{\pi}$, where $\pi$ is $\pi$ without its last element $\mathcal{A}(\pi)$. By point (ii), $\rho|_{\text{ext}(\pi)}$ is inessential, so if $\rho|_{S_{M}} = \rho|_{S_{M}^{\pi}}$, then $\rho_{A(\pi),\alpha}(\mathcal{A}) = \rho_{A(\pi),\alpha}(\mathcal{A})$.

iv. This is the basis for the proof that for each $\alpha$ with $q(\mathcal{A}) \geq 1$ and each path $\pi$ from the root $\mathcal{A}$ with $|\pi| < q(\mathcal{A})$, each valuation of $\pi^{-}$ and $S_{M}$ determines a unique value of $\mathcal{A}(\pi)$. We use its formulation above, i.e., for each $\rho \in 2^{S_{M} \cup \text{ext}(\pi)}$ and each $\alpha \in 2^{\pi}$, vertex $V = \mathcal{A}(\pi)$ (above the roots of the feet) obtains a unique value $\alpha'(V)$, which depends at most on valuation of vertices on $\pi^{-}$ (above $V$), but neither on the value (i) of $\alpha(V)$ nor (ii) of $\rho(X)$, for any $X \in \text{ext}(\pi)$.

The claim is shown by induction on $h - l$, where $h \geq 1$ is the distance of the root $\mathcal{A}$ from the roots of the feet and $l$ is the distance of $\mathcal{V}$ from the root $\mathcal{A}$, $h > l \geq 0$. The basis $h - l = 1$ is iii.

v. The argument from iii works also in the induction step. For $0 \leq |\pi| = l < h - 1$, we have the following counterpart of the drawing from iii, with $\mathcal{A}(\pi) = \exists \mathcal{D}[\mathcal{A}(\pi)\phi]$, where $\mathcal{D}[\phi]$ is the sequence of remaining quantifiers, and $\psi_{1}, \psi_{2}$ at the bottom signal various substitutions for $\phi$.

\[
\begin{array}{c}
\mathcal{A} \\
\pi \\
\exists \mathcal{D}[\mathcal{A}(\pi)\phi] \\
\exists \mathcal{D}[\mathcal{A}(\pi)\psi_{1}] \\
\exists \mathcal{D}[\mathcal{A}(\pi)\psi_{2}] \\
\exists \mathcal{D}[\mathcal{A}(\pi)\psi_{3}] \\
A(\pi S) \\
\end{array}
\]

Given $\alpha \in 2^{\pi}$, IH applied to the lowest triangles on the drawing, i.e., subgraphs $G_{M}(\mathcal{A}(\pi S))$ with roots $\mathcal{A}(\pi S)$ for $S \in S_{M}^{\phi}$, gives to each $\mathcal{A}(\pi S)$ a unique value, independent of valuation of $\text{ext}(\pi S)$. Consequently $\mathcal{A}(\pi S)$ is a function of only $\pi$ and $\phi$, so that for any $\alpha \in 2^{\pi}$, it represents a function $d^{\alpha}(\pi)$ of $\phi$. The same argument and cases for $d^{\alpha}(\pi)$ as in iii show that the value $\alpha'(\mathcal{A}(\pi))$, induced to the common grandparent of all $\mathcal{A}(\pi S)$ under valuation $\alpha \in 2^{\pi}$, is equal whether $\alpha(\mathcal{A}(\pi)) = 1$ or $\alpha(\mathcal{A}(\pi)) = 0$, giving point (i) of induction. As for each $\mathcal{A}(\pi S)$ its value under $\alpha'$ is independent from valuation of $\text{ext}(\pi S)$ by IH, the induced value $\alpha'(\mathcal{A}(\pi))$ is independent from $\text{ext}(\pi S)$, giving point (ii) of induction. Consequently, $\alpha'(\mathcal{A}(\pi))$ is unique and independent of valuations of $\text{ext}(\pi)$ and of $\mathcal{A}(\pi)$, which establishes the induction step.

vi. Thus, the value of the root $\mathcal{A}$ is determined, for each $\rho \in 2^{S_{M}}$, independently from valuation of $\text{ext}(G_{M}(\mathcal{A}))$. Starting now from $\mathcal{A}$ and using claim iv downwards, the value of $\mathcal{A}(S)$, for each $S \in S_{M}^{\phi}$, is determined by $\rho$ and value of $\mathcal{A}$ (independently from valuation of $\text{ext}(G_{M}(\mathcal{A}))$). Since $\mathcal{A}$ is determined by $\rho$, so is the value of $\mathcal{A}(S)$. Proceeding inductively down the tree $T_{M}(\mathcal{A})(\mathcal{V})$, valuation $\rho_{A}$ of $T_{M}(\mathcal{A})^{-}$ is seen determined by $\rho$, independently from valuation of $\text{ext}(G_{M}(\mathcal{A}))$. The latter determines then values in all feet of $G_{M}(\mathcal{A})$, yielding a unique solution $\rho_{A}$ of $G_{M}(\mathcal{A})$, with $\rho_{A} \subset \rho_{A}$ and $\rho_{A}|_{S_{M} \cup \text{ext}(G_{M}(\mathcal{A}))} = \rho$. \qed
7.4 Expressive power of $\mathcal{L}^\phi$

By Theorem 3.14 below, extending $\mathcal{L}$ to $\mathcal{L}^\phi$ does not increase the expressive power, as the introduced s-quantification amounts to a complex form of quantification over boolean values. In models of $A = \forall \phi F[\phi]$, $F$ is true for all sentences $\phi$, including $A$ itself. Guaranteeing a well-defined boolean value for each sentence (in each structure), the theorem makes this "including itself" harmless, reducing $\forall \phi$ to propositional quantifier. To verify $A$ it suffices to verify $F[\phi]$ for $\phi = 1$ and $\phi = 0$. This follows provided that every sentential context $F[\phi]$ (having only $\phi$ free), is a congruence-preserving equivalence of sentences, i.e., such that for each pair of $\mathcal{L}^\phi$ sentences $A, B$,

$$A \Longleftrightarrow B \text{ implies } F[A] \Longleftrightarrow F[B]. \tag{7.6}$$

Given an internal equivalence $A \leftrightarrow B \Longleftrightarrow (A \wedge B) \vee (\neg A \wedge \neg B)$, it suffices that for every structure $M$ (abbreviating $(M, \rho)$), if $M \models A \leftrightarrow B$ then $M \models F[A] \leftrightarrow [B]$. These assumptions are satisfied for classical logic. Let $\top / \bot$ stand for an arbitrary tautology/contradiction in $\mathcal{L}$.

**Fact 7.7** For every $\mathcal{L}^\phi$ formula $F[\phi]$ with only $\phi$ free and for every $\mathcal{L}$-structure $M$:

$$M \models \forall \phi F[\phi] \text{ iff } M \models F[\top] \wedge F[\bot], \text{ and } M \models \exists \phi F[\phi] \text{ iff } M \models F[\top] \vee F[\bot].$$

**Proof.** If $M \models \forall \phi F[\phi]$ then, in particular, $M \models F[\top]$ and $M \models F[\bot]$, so $M \models F[\top] \wedge F[\bot]$. Conversely, assuming $M \models F[\top] \wedge F[\bot]$, let $S$ be an arbitrary $\mathcal{L}$-sentence. If $M \models S$ then also $M \models S \leftrightarrow \top$, hence $M \models F[S]\text{ by (7.6), since } M \models F[\top]$. If $M \nvdash S$ then also $M \models S \leftrightarrow \bot$, hence $M \models F[S], \text{ since } M \models F[\bot]$. In either case $M \models F[S]$, and, since $S$ was arbitrary, $M \models \forall \phi F[\phi]$.

If $M \models \exists \phi F[\phi]$, set $S$ to be a sentence for which $M \models F[S]$. Either $M \models S$ or $M \models \neg S$, i.e., $M \models \neg S$. In the first case $M \models S \leftrightarrow \top$ and in the latter $M \models S \leftrightarrow \bot$. Thus either $M \models F[\top]$ or $M \models F[\bot]$, hence $M \models F[\top] \vee F[\bot]$. Conversely, if $M \models F[\top] \vee F[\bot]$ then either $M \models F[\top]$ or $M \models F[\bot]$. In either case $M \models \exists \phi F[\phi]$. \hfill $\square$

In particular, the unique solution of $G(\emptyset^\phi)$ contains exactly true quantified boolean sentences, QBS. The right sides of the equivalences in Fact 7.7 give their standard semantics.

By Theorem 3.11, values of $\mathcal{L}$ sentences determine values of all $\mathcal{L}^\phi$ sentences. Consequently, if structures $M, N$ are elementarily equivalent in $\mathcal{L}$, $M \equiv N$, they are so also in $\mathcal{L}^\phi$, $M \equiv N$.

**Fact 7.8** For any $\mathcal{L}$ structures $M$ and $N$, $M \equiv N$ iff $M \equiv N$.

**Proof.** The non-obvious implication to the right follows by induction on the number of s-quantifiers. Let $M \equiv N$ denote that $M$ and $N$ model the same $\mathcal{L}^\phi$ sentences with up to $k$ s-quantifiers, so that $M \equiv N$ corresponds to $M \equiv N$, giving the induction basis. Consider first PDNF sentence $A = \forall \phi \exists \theta \psi_1 \ldots \psi_n[D[\theta, \psi_1, \ldots, \psi_n]]$, where $|\psi_n| = k \geq 0$. Suppose that

(m) $M \models A$, i.e., for every $F \in S^\phi$ : $M \models \exists \theta \psi_1 [D[F, \psi_1]]$, while

(n) $N \models A$, i.e., for some $F_0 \in S^\phi$ : $N \nvdash \exists \theta \psi_1 [D[F_0, \psi_1]]$.

$F_0$ has some s-quantifiers, as otherwise (m), (n) contradict IH, $M \equiv N$. Taking any $\mathcal{L}$-sentence $P_0$ such that $N \nvdash P_0 \leftrightarrow N \models P_0$, yields $N \nvdash \exists \theta \psi_1 [D[P_0, \psi_1]]$ by (7.6). This last sentence has $k$ sentential quantifiers so, by IH, $M \nvdash \exists \theta \psi_1 [D[P_0, \psi_1]]$, which contradicts (m). An analogical argument shows the induction step for $A = \exists \theta \psi_1 [D[\phi, \psi]]$.

For any theory in $\mathcal{L}^\phi$, Fact 7.7 makes it straightforward to construct a theory in $\mathcal{L}$ with the same model class. For any $\mathcal{L}^\phi$ sentence $A$ in PDNF, an $\mathcal{L}$ sentence $A^\phi$, with $\text{Mod}(A) = \text{Mod}(A^\phi)$, is obtained replacing $\forall \phi F[\phi]$ by $F[\top] \wedge F[\bot]$ and $\exists \phi F[\phi]$ by $F[\top] \vee F[\bot]$. E.g., starting with $A = \forall \phi \exists \psi [(C \wedge \phi) \vee (D \wedge \psi)]$, with $C, D \in S^-$, one application of Fact 7.7 yields

$$\exists \psi_1 [(C \wedge \top) \vee (D \wedge \psi) \vee (\top \wedge \phi)] \wedge \exists \psi_2 [(C \wedge \bot) \vee (D \wedge \psi) \vee (\bot \wedge \psi)],$$

which can be simplified to

$$\exists \psi_1 (C \vee (D \wedge \psi) \vee \psi) \wedge \exists \psi_2 (D \wedge \psi) \iff \exists \psi_1 (D \wedge \psi) \wedge \exists \psi_2 (D \wedge \psi),$$

Fact 7.7 applied to the last sentence yields the first sentence below

$$(D \wedge \top) \vee (D \wedge \bot) \iff D,$$

so $\text{Mod}(A) = \text{Mod}(D)$. Proceeding thus by induction on the number of s-quantifiers (in PDNF $\mathcal{L}^\phi$ sentences), Fact 7.7 yields $\forall A \exists A^\phi \in S^- : \text{Mod}(A) = \text{Mod}(A^\phi)$, establishing
Theorem 3.14 For every $\Gamma \subseteq \mathcal{L}^P$ there is a $\Gamma^- \subseteq \mathcal{L}$ with $\text{Mod}(\Gamma) = \text{Mod}(\Gamma^-)$. 

7.5 Solvability of $\mathcal{G}(\mathcal{L}^+)$ and of definitional extensions

Proof of Lemma 3.12 relies on each DNF-foot being a boolean function, ii-iii. It can be repeated, ensuring absence of paradoxes in $\mathcal{L}^+$, if for each s-predicate $P$, all $1 \leq i \leq n$ and $\{\phi_1, ..., \phi_n\} \setminus \{\phi_i\}$:

either $\forall \phi_i \in S_M^P : P(\phi_1, ..., \neg \phi_i, ..., \phi_n) = P(\phi_1, ..., \phi_i, ..., \phi_n)$

or $\forall \phi_i \in S_M^P : P(\phi_1, ..., \neg \phi_i, ..., \phi_n) = \neg P(\phi_1, ..., \phi_i, ..., \phi_n).$ (3.17)

Each constant s-predicate satisfies this condition, so each language graph $\mathcal{G}_M(\mathcal{L}^P)$ is solvable, although such a constant interpretation is hardly satisfactory. Verification of solvability for specific definitions of s-predicates can be hampered by the semantic character of condition (3.17). We show non-paradoxicality of a definitional extension $\mathcal{L}^P$ of a non-paradoxical language $\mathcal{L}^+$, namely, extension with a fresh predicate $P$ defined by a sentence

$$\forall \phi(P(\phi) \leftrightarrow \exists \psi F[\phi, \psi]),$$ (5.3)

where $F$ is an $\mathcal{L}^+$-formula (with free variables $\phi$ among those of the left side $P(\phi)$). As noted at (5.3) in Section 5.6, graph $\mathcal{G}_M(\mathcal{L}^P)$ can be obtained from $\mathcal{G}_M(\mathcal{L}^P)$ by stretching a double edge from $P(S)$ to its defining sentence $\exists \psi F[S, \psi]$, for every $S \in S_M^P$. Lemma 7.13 below, giving immediately Theorem 5.4, shows that for any language graph $G$ for $\mathcal{L}^+$, such an extension $G^P$, for $P$ axiomatized by (5.3), preserves solutions of $G$. Its proof amounts to elimination of symbol $P$, replacing each $P(S)$ by its definitions $\exists \psi F[S, \psi]$. Such a replacement, trivial in FOL, must proceed recursively on a cyclic graph (e.g., $P(P(S))$ needs repeated replacements) and involves some technicalities. These end with the paragraph before Lemma 7.13.

The proof assumes a language graph $G$ over some domain $M$, in which no two vertices have equal out-neighbourhoods. (If $G$ contains such vertices, as language graphs typically do, their identification preserves essentially the solutions by Fact 7.3, and we apply the construction and fact below to the so quotiented $(G)$.) The graph $G^P = \mathcal{G}_M(\mathcal{L}^P)$ contains $G$ as an induced subgraph.

As the first step, we quotient atoms of $G^P$ containing $P$. Let $\simeq$ be congruence on $\mathcal{L}_M^P$-sentences induced by the basic reflexive relation $P(S) \simeq_0 \exists \psi F[S, \psi]$, for every $\mathcal{L}_M^P$-sentence $S$. For every s-predicate $Q$ distinct from $P$, we identify every two atoms $Q(A_1, ..., A_n) \simeq Q(B_1, ..., B_n)$ when $A_i \simeq B_i$ for $1 \leq i \leq n$. Each equivalence class contains an atom $Q(S_1, ..., S_n)$ for some $S_i \in S_M^P$, not containing any $P$, so in the following we can assume only such atoms present. It is a simple observation that quotient $q : G^P \to H$, where $E_H(q(x)) = \{q(y) \mid y \in E_G(x)\}$ in the resulting graph $H$, reflects kernels, so the preimage of any kernel of $H$ is a kernel of $G^P$.

We now map $\gamma : H \to G$, performing a sequence of identifications $\gamma_i : H_{i-1} \to H_i$, for $0 < i \in \omega$ and $H_0 = H$. Each $\gamma_i$ is identity on the subgraph $G$ of $H_i$, identifying some vertices from $V_i \setminus V_G$ with some in $V_G$. First, we identify $\gamma_1(P(S)) = \exists \psi F[S, \psi]$, removing the double edge and the intermediate vertex $P(S)$ between $P(S)$ and its definitions $\exists \psi F[S, \psi]$, for $S \in S_M^P$.

Then $\gamma_{i+1}(v) = w$ when vertices $v \in V_i \setminus V_G$ and $w \in V_G$ have the same out-neighbourhood. More precisely, let $V_0 = V_H$, $E_0 = E_H$ and:

$$i = 1,$$ letting $R_{e_0} = \{\{P(S), \bullet P(S)\} \mid S \in S_M^P, \{\bullet P(S)\} = E_0(P(S))\}$ define:

$$\gamma_1(v) = \left\{ \begin{array}{ll} \exists \psi F[S, \psi], & \text{if } v = P(S) \text{ for any } S \in S_M^P \\ v, & \text{if } v \notin R_{e_0} \end{array} \right.$$

The resulting graph $H_1$ is given by:

$$V_1 = V_0 \setminus R_{e_0}, \text{ and } E_1(v) = \{\gamma_1(w) \mid w \in E_0(v)\} \setminus R_{e_0}$$

$$i + 1,$$ letting $R_{e_i} = \{v \in V_i \setminus V_G \mid 3w \in V_G : E_i(v) = E_i(w)\}$ define:

$$\gamma_{i+1}(v) = \left\{ \begin{array}{ll} w, & \text{if } v \notin R_{e_i} \text{ and } v \in V_G \text{ such that } E_i(v) = E_i(w) \text{ if } v \notin R_{e_i} \\ v, & \text{if } v \notin R_{e_i} \end{array} \right.$$ (7.9)

The resulting graph $H_{i+1}$ is given by:

$$V_{i+1} = V_i \setminus R_{e_i}, \text{ and } E_{i+1}(v) = E_i(\gamma_{i+1}(v)) \setminus R_{e_i}$$

$$\gamma(v) = \gamma_n(v), \text{ for } v \in V_H, \text{ where } n \in \omega \text{ is the least such that } \forall m > n : \gamma_m(v) = \gamma_n(v).$$
Function $\gamma$ is well-defined by the assumption that $G$ has no pair of vertices with identical out-neighbourhoods. For $A, B \in V_H$ and $n \in \omega$, we denote by $A \sim_n B$ that $\gamma_n(A) = \gamma_n(B)$, and by $A \sim B$ that $\gamma(A) = \gamma(B)$, i.e., $\exists n \in \omega : A \sim_n B$.

**Example 7.10** Let $P(\phi) \leftrightarrow \exists \psi(\phi \land \psi)$ and, for some $S \in S_M$, consider vertex $P(P(S)) \in V_H$. The relevant parts of the graph $H$ are sketched below, with $A/X, B/X, \ldots$ denoting vertices with $X$ substituted for the $\exists$-quantified $\psi$. The subscripts $L, R$ mark these instantiations in the respective subgraphs, e.g., $A/L = P(S) \land A$ and $A/R = \exists \phi(S \land \phi) \land A$. Sentences $A, B, \ldots$ (and $\overline{A}, \overline{B}, \ldots$) are duplicated in both subgraphs to increase readability, but they are actually the same vertices.

![Diagram of example 7.10](attachment:image.png)

1. $P(P(S)) \sim_1 \exists \psi(P(S) \land \psi)$ and $P(S) \sim_1 \exists \psi(S \land \psi)$, hence $E_1(P(P(S)) = \{ \gamma_1(P(S)) \} = \{ \exists \psi(S \land \psi) \} = E_1(\exists \psi(S \land \psi))$ and, consequently,
2. $P(S) \sim_2 \exists \psi(S \land \psi)$. Then, for each $A \in S_M^P$, $E_2(A/L) = \{ \exists \psi(S \land \psi), \overline{A} \} = E_2(A/R)$, so $A/L \sim A/R$, for every $A \in S_M^P$.
3. Consequently, $\bullet_L \sim_4 \bullet_R$ and then
4. $\exists \psi(\exists \phi(S \land \phi) \land \psi)) \sim_5 \exists \psi(P(S) \land \psi) \sim_1 P(P(S))$, leaving only $G$'s subgraph to the right. □

The equivalence $\sim$ is a congruence on $V_H$ in the sense that if all out-neighbours of $A$ and $B$ are $\sim$-equivalent then also $A \sim B$, i.e., for $E_H(A) = \{ A_i | i \in I \}$ and $E_H(B) = \{ B_i | i \in I \}$:

$$\text{if } (\forall i \in I : A_i \sim B_i) \text{ then } A \sim B.$$  \hspace{1cm} (7.11)

This holds since each sentence subgraph $G_M(A)$ (tree $T_M(A)$) has finite height $h(A)$, in particular distance from the root $A$ to atoms $P(S)$ of $G_M(A)$ is at most $h(A)$. Hence, if $\forall i \in I : A_i \sim B_i$ then $\exists n \leq \max\{h(A), h(B)\} \forall i \in I : A_i \sim B_i$. The equality $\gamma_n(A_i) = \gamma_n(B_i)$ implies, in turn, that $E_n(A) = \{ \gamma_n(A_i) | i \in I \} = \{ \gamma_n(B_i) | i \in I \} = E_n(B)$, which yields $A \sim_{n+1} B$.

**Fact 7.12** (a) $\forall S \in S_M^P \setminus S_M \exists Q \in S_M : Q \sim S$, hence $\gamma(H) = G$,
(b) $H$ and $G$ have essentially the same solutions,
(c) Every solution of $G$ extends to a unique solution of $G^P$.

**Proof.** Point (a) is shown by induction on the number $p$ of $P$s in a sentence $S \in S_M^P \setminus S_M$.

1. If $p = 1$ and $S$ is atomic, then $S = P(R)$ for some $R \in S_M$, so $S \sim_1 \exists \psi F[R, \psi] \in S_M$.
2. If $p = 1$ and $S$ is not atomic, we proceed by structural induction on $S$, with point 1 providing the basis and induction hypothesis $IH_2$:
   i. $\land_{i \in I} S_i$, for finite $I$. By $IH_2$, for each $S_i$ there is $Q_i \in S_M$ with $S_i \sim Q_i$, so $\land_{i \in I} S_i \sim \land_{i \in I} Q_i$ by (7.11), and $\land_{i \in I} Q_i \in S_M$.
   ii. $\neg A$. By $IH_2$, $A \sim Q$ for some $Q \in S_M$, so $\neg A \sim \neg Q$ by (7.11), while $\neg Q \in S_M$.

\[7\] This implication fails in general graphs for $\sim$ defined by (7.9) from some basis $\sim_1$, when $I$ is infinite and distance from $A_i, B_i, i \in I$, to relevant pairs $X \sim Y$ is unbounded.
iii. $S = \exists \phi A[\phi]$, where $\phi$ does not occur under $P$, so that $S = \exists \phi A[\phi, P(R)]$, for some $R \in S_M$ and context $A[\phi, \_]$ without no $P$. Since $P(R) \sim_1 \exists \psi F[R] \in S_M$, taking $Q = \exists \phi A[\phi, \exists \psi F[R]] \in S_M$, we obtain \[ A[T, P(R)] \sim_1 A[T, \exists \psi F[R]] \] for every $T \in S_M^P$ by (7.11), i.e., for all grandchildren of $S$ and $Q$. By (7.11), this yields $S \sim Q$.

iv. $S = \exists \phi A[P(C[\phi])]$, i.e., $S$ contains quantification into $P$, for some contexts $A[\_], C[\_]$ without any $P$, as $p = 1$. For grandchildren of $S$, namely, $A[P(C[T])]$ for all $T \in S_M^P$, the equivalence $P(C[T]) \sim_1 \exists \phi F[C[T], \phi]$ gives $A[P(C[T])] \sim_1 A[\exists \phi F[C[T], \phi]]$ by (7.11). Sentences on the left, for all $T \in S_M^P$, comprise all grandchildren of $S$, and those on the right all grandchildren of $Q = \exists \phi A[\exists \phi F[C[\phi], \psi]] \in S_M$, so $S \sim Q$ by (7.11).

3. For the induction step for $p > 1$, the two cases depend on whether $P$ is nested or not.

i. If the number of $P$s not nested under others is $n > 1$, consider all these highest $P$s in $T_M(S)$, i.e., $S = C[P(A_1), \ldots, P(A_n)]$, where $C[\_]$ contains no $P$s. For $R = \exists \phi F[A_1, \psi], \ldots, \exists \phi F[A_n, \psi]$, $S \sim R$ by (7.11). $R$ has $p - n < p$ $P$s so, by IH, $R \sim Q$ for some $Q \in S_M$. Hence $S \sim Q$.

ii. If all $P$s are nested under each other, then $S = C[P(A)]$ for some context $C[\_]$ without any $P$s, and with $p - 1$ occurrences of $P$ in $A$. $P(A) \sim_1 \exists \phi F[A, \psi]$ and, by IH, $\exists \phi F[A, \psi] \sim R$ for some $R \in S_M$, so that also $P(A) \sim R$. Then $C[P(A)] \sim_1 C[R]$, by (7.11) and $C[R] \in S_M$, as required.

The equality $\gamma(H) = G$ follows since each $S \in \mathcal{V}_H \backslash \mathcal{V}_G$ represents a sentence in $S_M^P \backslash S_M$.

(b) For $i \geq 0$, $H_i$ is the quotient of $H$ by $\sim_1, \ldots, \sim_i$. By Fact 7.2, $H_1$ has essentially the same solutions as $H$. (No ray is contracted to a finite path, because the case $P(S) \sim_1 \exists \phi F[S, \psi]$ is applied at most finitely many times along each path under each sentence $Q$, since $Q$ contains at most finitely many nested $P$s.) By Fact 7.3, the same holds for $H_i$ and every $H_i$, $i > 1$, including limits $H_\lambda$. Thus, $H$ and $\gamma(H) = G$ have essentially the same solutions.

(c) By the observation before this fact, quotient $G^P \rightarrow H$ reflects solutions, so that the preimage of every solution of $H$ is a solution of $G^P$. Using (b), each solution of $G$ extends to one for $G^P$. □

Let definitional extension refer to any well-ordered chain starting with any theory $\Gamma_0 \subseteq \mathcal{L}_0 \subseteq \mathcal{L}^+$ and adding, at step $i + 1$, axiom (5.3) with a fresh predicate $P \notin \mathcal{L}_i$ and $F[\phi, \psi] \in \mathcal{L}_i$, for language $\mathcal{L}_i$ of theory $\Gamma_i$ obtained at step $i$. In the limits, the language and theory are unions of all steps.

The following counterpart of model theoretic conservativity of usual definitional extensions holds.

**Lemma 7.13** Each solution of a language graph $G_0 = G_M(\mathcal{L}_0)$ extends to a solution of the graph of its definitional extension.

**Proof.** Fact 7.12.(c) gives the claim for an extension with a single predicate. By IH, definitional extension $G_i$ of $G_0$ with $P_1, \ldots, P_i$, preserves all solutions of $G$. Graph $G_{i+1}$, obtained now by adding $P_{i+1}$, whose definics $F_{i+1}$ can utilize $P_j, j \leq i$, preserves by Fact 7.12 solutions of $G_i$, and hence of $G$. This establishes successor step.

For any limit, the language $\mathcal{L}_M^\omega = \bigcup_{i \in \omega} \mathcal{L}_i^M$ extends the initial language $\mathcal{L}_M^0$ with all $\omega$ predicates $P_1, P_2, \ldots$ introduced on the way. Its graph $G_\omega = \bigcup_{i \in \omega} G_i$, with unions taken on vertices and on edges, contains all double edges from the new predicate’s instances to their definics. We repeat the proof with the unions of all equivalences used along the way. As the first step, let $\equiv_\omega$ be a congruence on $\mathcal{L}_M^\omega$-sentences induced from the relation $A \equiv_\omega B \iff \exists n \in \omega : A \equiv_\omega^n B$, where $\equiv_\omega$ is the congruence $\equiv_\omega$ on $\mathcal{L}_M^\omega$-sentences from step $n$. Identification of all atoms $Q(A_1 \ldots A_k) \equiv_\omega Q(B_1 \ldots B_k)$ when $A_i \equiv_\omega B_i$ for $1 \leq i \leq k$ gives a quotient $H$ reflecting kernels as before. Each equivalence class contains an atom from $\mathcal{L}_M^\omega$. Let $H$ denote the resulting graph, and $H_i$ its restriction to the subgraph induced by vertices of $G_i$ (with the atoms identified as just described), so that $H = \bigcup_{i \in \omega} H_i$.

In the chain $G_0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$, for each pair of subsequent $H_{i-1} \subseteq H_i$, the construction (7.9) yields $\gamma^i : H_i \rightarrow H_{i-1}$ satisfying Fact 7.12. Composing $\gamma^i(\gamma^j(\ldots(\gamma^i(H_0)) \ldots))$ gives surjective $\gamma_\omega : H \rightarrow G_0$, where $\gamma_\omega(H_i) = \gamma^i(H_i)$ for any $j \geq i$. Hence, the union $\gamma_\omega = \bigcup_{i \in \omega} \gamma^i$ gives a surjective quotient $\gamma_\omega : H \rightarrow G_0$, reflecting solutions. □

A non-paradoxical language $\mathcal{L}^+$ is one having a solvable graph $G_M(\mathcal{L}^+)$ so, by this lemma, its definitional extension remains non-paradoxical.
Theorem 5.4 For every $\Gamma \subseteq \mathcal{L}^+$ and its definitional extension $F$, every kernel model of $\Gamma$ can be extended to a kernel model of $\Gamma \cup F$.

7.6 (Cut) preserves consistency.

An important theorem from [21] states that if every induced proper subgraph has a semikernel then the graph has a kernel. For language graphs we would rather ask more specifically about a kernel containing a given theory, depending on the existence of semikernels extending the theory to some (finite) parts of the language. The following fact gives such a compactness-like claim for any language graph. (Recall the notation $S \subseteq X$ for $S$ being a finite subset of $X$.)

Fact 7.14 For $\Gamma \subseteq \mathcal{L}^+$ and any $G = G_M(\mathcal{L}^+)$, if for each $S \subseteq S_M^+$ there is a semikernel of $G$ containing $\Gamma$ and covering $S$, then $G$ has a kernel containing $\Gamma$.

Proof. Let $SK_F$ denote all semikernels of $G$ containing $\Gamma$. For a finite set $X \subseteq S_M^+$, denote by

$$SK_X = \{L \in SK_F \mid X \subseteq L\}$$

semikernels of $G$ containing $\Gamma$ and $X$, and

$$SK_X^+ = \{L \in SK_F \mid X \subseteq E_G[L]\}$$

semikernels of $G$ containing $\Gamma$ and covering $X$.

The set $F = \{SK_X \mid X \subseteq S^+\}$ has finite intersection property by the main assumption. We show that, for an ultrafilter $U \supseteq F$ on $\mathcal{P}(SK_F)$, existing by the ultrafilter lemma, a kernel of $G$ can be given by $K = \{S \in S_M^+ \mid SK_S \subseteq U\}$ (subscript $_{\subseteq}$ abbreviates now $_{\subseteq}(S)$ for a single sentence $S$).

$K$ covers $G$ because if $S \in S_M^+ \setminus K$, then $SK_S \not\subseteq U$ so $SK_S = \{L \in SK_F \mid S \not\subseteq L\} \not\subseteq U$. If also $SK_{\subseteq} \not\subseteq U$, then $SK_{\subseteq} = \{L \in SK_F \mid \neg S \not\subseteq L\} \not\subseteq U$. Hence, if both $S \not\in K$ and $\neg S \not\in K$, then $nS = SK_S \cap SK_{\subseteq} = \{L \in SK_F \mid S \not\subseteq L \land \neg S \not\subseteq L\} \not\subseteq U$. As $SK_S \cup SK_{\subseteq} = SK_S \subseteq F \subseteq U$, so $nS \cap SK_{\subseteq} = \emptyset \subseteq U$ contradicts $U$ being an ultrafilter. Hence $S \in K$ or $\neg S \in K$ for each $S \in S_M^+$.

Independence of $K$ is shown for each kind of vertex in $V_G = S_M^+$.

(i) For each sentence $S$, including atoms, $SK_S \cap SK_{\subseteq} = \emptyset$, so both cannot belong to $U$, hence either $S \not\in K$ or $\neg S \not\in K$. If $S \not\in K$, then $E_G(\neg S) = \{S\} \subseteq V_G \setminus K$, while for atomic $S$, $S \in K \Rightarrow E_G(S) = \{\neg S\} \subseteq V_G \setminus K$. For nonatomic $S$ with main connective other than $\neg$, one of the following cases applies.

(ii) For each conjunction $E_G(S_1 \land S_2) = \{\neg S_1, \neg S_2\}$, hence $SK_{S_1 \land S_2} \cap SK_{\neg S_i} = \emptyset$, for $i \in \{1, 2\}$, so either both $\neg S_i \not\in K$ or else $S_1 \land S_2 \not\in K$, i.e., $S_1 \land S_2 \not\in K \Rightarrow E_G(S_1 \land S_2) \subseteq V_G \setminus K$.

(iii) For each $\forall$-quantified sentence $E_G(\forall x(F(x))) = \{\neg F(m) \mid m \in M \lor \in S^+\}$, so $SK_{\forall x(F(x))} \cap SK_{\neg F(m)} = \emptyset$ for each $m \in M$ (or $\in S^+$), hence if $\forall x(F(x)) \in K$ then $E_G(\forall x(F(x))) \subset V_G \setminus K$. □

From a syntactic perspective, a shortcoming of this fact is that it concerns a single graph $G_M(\mathcal{L}^+)$, rather than a theory, and involves all $S_M^+$ sentences, not only $S^+$. The idea of a theory having a consistent extension to the whole language, provided that it has such extensions to its finite parts, is better captured by the next lemma, leading to the theorem that LS-unsolvability of any (specific) contradiction implies existence of a graph with a kernel, and hence unprovability of any contradiction also when using (cut). For a finite set $Q \subseteq S^+$, we let $\bot_Q = \bigvee_{S \in Q}(S \land \neg S)$.

Lemma 7.15 $(\forall Q \subseteq S^+: \Gamma \not\supseteq \bot_Q) \Rightarrow \exists G, K \in \text{sol}(G): \Gamma \subset K$.

Proof. To use Fact 7.14, we construct a graph $G$ over domain $M$ with semikernels containing $\Gamma$ and covering every finite subset of $S_M^+$. Letting $I$ index finite subsets of $S^+$, the assumption gives a semikernel $L_i$ of a graph $G$, containing $\Gamma$ and covering $\perp_i$, for every $i \in I$. Let $G$ be the language graph over $M = \prod M_i$, where $f^M(\prod m_i) = \prod f^{M_i}(m_i)$, constants $c^M = \prod c^{M_i}$, and define inductively the operation $L_i^+$, lifting terms $T_i^+ \rightarrow \mathcal{P}(T_i^+)$ and formulas $F_i^+ \rightarrow \mathcal{P}(F_i^+)$:

$$m_i^{+M} = (m_i)^{\perp M} = \{n \in M \mid m_i = n\} = \{m_i\} \times \prod_{j \neq i} M_j, \text{ for } m_i \in M_i, \text{ and likewise }$$

$$f^{+M}(m_i^{+M}) = \{f(n) \in T_i^+ \mid n \in m_i^{+M}\} \text{ and }$$

$$F(m_i^{+M}) = F(m_i)^{+M} = \{F(n) \in S_M^+ \mid n \in m_i^{+M}\}.$$ 

Notation $F(m_i^{+M})$ implies that the only $M_i$ elements are among $m_i$. In general, $S_i$ denotes an $L_i^+$ sentence with possibly some elements from $M_i$, that equals $S \subseteq S^+$ if no such elements occur.
Then $S^1_M$ is the set of sentences obtained by replacing each $m_i \in M_i$ by all $m_i^{vM}$. (S_i denotes also an $L^+ / L_M$ sentence S with all terms interpreted in/projected onto $M_i$.) Some observations:

i. $\bigcup \{m_i^{vM} | m_i \in M_i\} = M$ and $\bigcup \{S^1_M | S_i \in S^+_M\} = S^+_M$, and $\bigcup \{F(S)^vM | S_i \in S^+_M\} = \{F(S) | S \in S^+_M\}$.

ii. $\forall S_i, R_i \in S^+_M : S_i \neq R_i \Rightarrow S^1_M \cap R^1_M = \emptyset$ (= modulo renaming of bound variables)

iii. For $S \in S^+$, $S \in (S^1_M)$, and $\{S\} = (S^1_M)$ if $S$ has no non-variable terms, e.g., $(\forall xP(x))^vM = \forall xP(x)$. For $S$ with non-variable terms, e.g., a constant $c_i = c^{vM}$, $P(c_i)^vM = P(c_i)$ ($\forall e_i \in \{\{c_i \times \prod_j M_j\}\} \ni P(c_i)$). Inductively, $t^vM$ implies $f^vM(t_i)^vM = \{m \in M | m_i = f(t_i)^vM\}$, and then $S \in (S^1_M)$, for every $S \in S^+$.

iv. $S \notin L_i \Leftrightarrow \forall X \in L_i : S \neq X \Leftrightarrow \forall X \in L_i : S^1_M \cap X^vM = \emptyset \Rightarrow S^1_M \cap L^1_M = \emptyset$. In particular, $L^1_M$ is absorbing, $E_G(L^1_M) \subseteq E_G(L^1_M)$, follows by considering cases of its vertices:

1. If $P(m) \in L^1_M$, then $E_G(P(m)) = \{P(m)\} \subseteq E_G(L^1_M)$.

2. For a negated $\neg S \in L^1_M$, we have $\neg S \in L^1_M \Rightarrow \neg S \in L_i \Rightarrow S \in E_{\neg G}(L_i)$, and show $S \in E_G(L^1_M)$ for $E_G(\neg S) = \{S\}$ by cases of $S$. Where relevant, we mark possible $m \in M$ occurring in the considered sentences as extra parameters.

3. $\forall S \in L^1_M \Rightarrow \exists S \in L_i \Rightarrow \exists m_i \in M_i : \neg F(m_i, n_i) \in L_i \Rightarrow \forall xF(x, n) \in E_{\neg G}(L_i)$.

4. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

5. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in E_{\neg G}(L_i)$.

6. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

7. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

8. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

9. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

10. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

11. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

12. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

13. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

14. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

15. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

16. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

17. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

18. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

19. $\forall \phi F(\phi, n) \in L^1_M \Rightarrow \forall \phi F(\phi, n) \in L_i \Rightarrow \forall \phi F(\phi, n) \in E_G(L_i)$.

20. We show that $L^1_M$ is independent, i.e., $E_G(L^1_M) \subseteq V_G \setminus L^1_M$, considering cases of its vertices:

21. $P(m) \in L^1_M \Rightarrow P(m) \in L_i \Leftrightarrow P(m) \notin L^1_M$.

22. $\neg S \in L^1_M \Rightarrow \neg S \in L_i \Rightarrow S \in G(L_i)$.

23. $S^1(m) \cup S^2(n) \in L^1_M \Rightarrow S^1(m) \cup S^2(n) \in L_i \Rightarrow \neg S^1(m), \neg S^2(n) \in E_G(L_i)$.

24. $S^1(m) \cup S^2(n) \in L^1_M \Rightarrow S^1(m) \cup S^2(n) \in L_i \Rightarrow \neg S^1(m), \neg S^2(n) \in E_G(L_i)$.
Thus, if $\Gamma$ and existing by the assumption that

$$
\Gamma \not\vdash \bot
$$

then so does $L_{i}I^{M}$ covers by some $L_{i}I^{M}$. 

3. For an arbitrary $S \in S_{M}$, $\bot S = S \land \neg S$ is covered by $L_{(S)}$ which does not contain it. Hence $(S \land \neg S) \in E_{G_{(S)}}(L_{(S)})$, so either $\neg S \in L_{(S)}$ or $\neg \neg S \in L_{(S)}$. In the former case $S \in

$$
E_{G_{(S)}}(L_{(S)}) \subseteq E_{G_{(S)}}(L_{(S)})
$$

while in the latter $S \in L_{(S)}$. In each case, $L_{i}I^{M}$ covers $S = S^{M}$.

For an arbitrary $S \in S_{M} \setminus S^{+}$, i.e. $S = Fm$, where $m \in M$ are all $M$ elements occurring in $S$, contradiction $N(F) = \forall x(\neg (Fx \land \neg Fx) \land \neg \forall x(\neg (Fx \land \neg Fx))$ has, in some graph $G_{i}$, a countermodel $L_{i}$ covering it by $N(F) \in E_{G_{i}}(L_{i})$. Thus either $\forall x(\neg (Fx \land \neg Fx)) \in L_{i}$ or $\neg \forall x(\neg (Fx \land \neg Fx)) \in L_{i}$, but since the former sentence is a contradiction, the latter is the case. Then $\forall x(\neg (Fx \land \neg Fx)) \in L_{i}$, hence also $\neg (Fm_{1} \land \neg Fm_{1}) \in L_{i}$ for every $m_{1} \in M_{i}$. Since then $(Fm_{1} \land \neg Fm_{1}) \in E_{G_{i}}(L_{i})$, for every $m_{1} \in M_{i}$, either $\neg Fm_{1} \in L_{i}$ or $\neg \neg Fm_{1} \in L_{i}$. In the former case $Fm_{1} \in L_{i}$, while in the latter, $Fm_{1} \in E_{G_{i}}(L_{i})$. Thus, $L_{i}$ covers $Fm_{1}$, containing either $Fm_{1}$ or $\neg Fm_{1}$, for every $m_{1} \in M_{i}$.

By iii, $Fm \in L_{i}I^{M}$ or $\neg Fm \in L_{i}I^{M}$, which means that $L_{i}I^{M}$ covers $Fm$, for every $m \in M$.

These arguments for single sentences are extended to an arbitrary $Q = \{F_{1}m_{1}, ..., F_{k}m_{k}\} \in S_{M}$, by considering semikernel $L_{i}$ which covers, without containing, $\bot Q = N(F_{1}) \lor \cdots \lor N(F_{k})$, and existing by the assumption that $\Gamma \not\vdash \bot Q$. The subgraph of $\bot Q$ has the form

$$
\begin{array}{c}
\bot Q \\
\downarrow \\
N(F_{1}) \lor \cdots \lor N(F_{k})
\end{array}
$$

and hence $\bot Q \in E_{G_{i}}(L_{Q})$ implies $\bullet \in L_{Q}$, so $N(F_{j}) \in E_{G_{Q}}(L_{Q})$, for $1 \leq j \leq k$. By the argument for a single $N(F)$, this implies that $L_{Q}$ covers every $F_{j}m_{j}$.

If LSO does not derive from $\Gamma$ contradiction $\bot Q = \bigvee_{S \in Q}(S \land \neg S)$, for any $Q \in S^{+}$, there is thus a graph with a kernel containing $\Gamma$. Soundness, Fact 4.2, implies then $\Gamma \not\vdash \bot$ for each $\bot \in C$. Thus, if LSO with (cut) proves a contradiction from a theory $\Gamma$, then so does LSO without (cut).

**Theorem 4.3** For a countable $\Gamma \subset FOL^{+}$: $(\forall Q \in S^{+} : \Gamma \not\vdash \bot Q) \Rightarrow (\forall \bot \in C : \Gamma \not\vdash \bot)$.

### 8 Appendix: Soundness and completeness

Facts 8.1 and 8.3 below show soundness and completeness of LSO for semikernel semantics from (3.18), establishing Theorem 3.19. Fact 4.2 (4.2) shows these properties for LSO with (cut) for kernel semantics (3.10).

**Fact 8.1** The rules of LSO are sound and invertible for (3.18).
PROOF. Given an arbitrary language graph \(G^+\) (over an arbitrary domain \(M\)), soundness for each rule follows by showing that any semikernel \(L\) covering the conclusion satisfies it, assuming validity of the premise(s), while invertibility by showing that any semikernel \(L\) covering (each) premise satisfies it, assuming validity of the rule’s conclusion.

1. \((\land R)\). For soundness, assume \(\Gamma \models \Delta, A_1\) and \(\Gamma \models \Delta, A_2\), and let semikernel \(L\) cover the rule’s conclusion, under a given \(\alpha \in M^{\text{FOL}(\Gamma, \Delta)}\). Assume that \(\alpha(\Gamma) \subseteq L\), \(\alpha(\Delta) \subseteq E^-(L)\) and \(\alpha(A_1 \land A_2) \in E^-(L)\) if not, then \(L \models \alpha \Rightarrow \Delta, A_1 \land A_2\), as desired. Since \(\alpha(A_1 \land A_2) \in E^-(L)\) and \(E(\alpha(A_1 \land A_2)) = \{\neg \alpha(A_1), \neg \alpha(A_2)\}\) so, for some \(i \in \{1, 2\}, \neg \alpha(A_i) \subseteq L\), and then \(\alpha(A_i) \in E^-(L)\), contradicting the assumption \(\Gamma \models \Delta, A_i\).

For invertibility, let \(\Gamma \models \Delta, A_1 \land A_2\) and \(L\) cover \(A_1\) (or \(A_2\)) under \(\alpha\). If \((*)\) \(\alpha(\Gamma) \subseteq L\) and \(\alpha(\Delta \cup \{A_1\}) \subseteq E^-(L)\), then \(\Gamma' = L \cup \{\neg \alpha(A_1)\}\) is a semikernel, since \(E(\neg \alpha(A_1)) \subseteq \{\alpha(\Delta)\} \subseteq E^-(L) \subseteq E^-(\Gamma')\). Thus \(\Gamma'\) covers the conclusion, while \(\alpha(\Gamma) \cap E^-(L') = \emptyset\) and \(\alpha(\Delta \cup \{A_1 \land A_2\}) \cap \Gamma' = \emptyset\), so \(\Gamma' \not\models \Delta, A_1 \land A_2\), contrary to \(\Gamma \models \Delta, A_1 \land A_2\). Hence \((*)\) fails, so \(\alpha(\Gamma) \cap E^-(L) \neq \emptyset\) or \(\alpha(\Delta \cup \{A_1\}) \subseteq L \neq \emptyset\), yielding the claim. Assignments to free FOL-variables do not affect the argument, so covering by \(L\) below is to be taken relatively to a given \(\alpha\), which we do not mention, except for \((\forall \eta)\).

2. \((\land L)\). For soundness, assume \(\Gamma, A_1, A_2 \models \Delta\), let semikernel \(L\) cover the rule’s conclusion, \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(A_1 \land A_2 \subseteq L\), then \(E(A_1 \land A_2) = \{\neg A_1, \neg A_2\} \subseteq E^-(L)\), so \(E(\neg \alpha(A_1)) \subseteq \{\alpha(\Delta)\} \subseteq E^-(L)\). Thus \(A_1 \land A_2 \subseteq E^-(L)\) and \(L \models \Gamma, A_1 \land A_2 \Rightarrow \Delta\).

For invertibility, assume \(\Gamma, A_1 \land A_2 \models \Delta\), let semikernel \(L\) cover the rule’s premise, and assume \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(A_1, A_2 \subseteq L\), which is the only way \(L\) can contradict \(\Gamma, A_1, A_2 \models \Delta\), then \(\{\neg A_1, \neg A_2\} \subseteq E^-(L)\), and \(L' = L \cup \{A_1 \land A_2\}\) is also a semikernel:

\[
E(L') = E(L \cup \{A_1 \land A_2\}) = E(L) \cup E(\{A_1 \land A_2\}) \subseteq E^-(L) \cup \{\neg A_1, \neg A_2\} \subseteq E^-(L) \subseteq V(L \cup \{A_1 \land A_2\}).
\]

The last inclusion follows because \(E^-(L) \subseteq V \setminus L\) and \(A_1 \land A_2 \notin E^-[L]\), since \(A_1 \land A_2 \subseteq L\) contradicts \(\Gamma, A_1 \land A_2 \models \Delta\) (as \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\)), while \(A_1 \land A_2 \in E^-(L)\) contradicts independence of \(L\), implying \(\neg A_i \in L\) (for \(i = 1\) or \(i = 2\)), while \(\neg A_i \in E^-(L)\) since \(A_i \subseteq L\).

Since \(L' \not\models \Gamma, A_1 \land A_2 \Rightarrow \Delta\) contradicts the assumption, either \(A_1 \notin L\) or \(A_2 \notin L\), and \(L \models \Gamma, A_1 \land A_2 \Rightarrow \Delta\) as desired.

3. \((\land R)\). For soundness, assume \(\Gamma, A \models \Delta\), let semikernel \(L\) cover the rule’s conclusion, and assume \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(\neg A \in E^-(L)\), we are done, while if \(\neg A \in E^-(L)\) then \(A \subseteq L\), which contradicts the assumption, since now \(\Gamma \cup \{A\} \subseteq L\) and \(\Delta \subseteq E^-(L)\).

For invertibility, assuming \(\Gamma \models \Delta, \neg A\), let \(L\) cover the rule’s premise, \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(A \in L\) then \(\neg A \in E^-(L)\) and \(L \not\models \Gamma \Rightarrow \Delta, \neg A\), contradicting the assumption. Hence \(A \in E^-(L)\), as required for \(L \models \Gamma, A \Rightarrow \Delta\).

4. \((\neg L)\). For soundness, assume \(\Gamma \models \Delta, A\), let \(L\) cover the rule’s conclusion, \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(\neg A \in E^-(L)\), we are done, while if \(\neg A \in L\) then \(A \in E(\neg A) \subseteq E^-(L)\), contradicting the assumption, since now \(\Gamma \cup \{A\} \subseteq L\) and \(\Delta \cup \{A\} \subseteq E^-(L)\).

For invertibility, assume \(\Gamma, \neg A \models \Delta\), let \(L\) cover the rule’s premise, \(\Gamma \subseteq L\) and \(\Delta \subseteq E^-(L)\). If \(A \in E^-(L)\) then \(L' = L \cup \{\neg A\}\) is a semikernel, because \(L\) is and \(E(\neg A) = \{A\} \subseteq E^-(L)\). But \(L'\) contradicts the assumption, so \(A \notin L\), as required for \(L \models \Gamma \Rightarrow \Delta, A\).

5. \((\forall \eta)\). For soundness, assume \(F(t), \Gamma, \forall x F(x) \models \Delta\) and let \(L\) cover the rule’s conclusion. If \(\forall x F(x) \notin L\), i.e., \(\forall x F(x) \in E^-(L)\), then \(\Gamma \cup \{\forall x F(x)\} \models E^-(L) \neq \emptyset\), so \(L \models \Gamma, \forall x F(x) \Rightarrow \Delta\). If \(\forall x F(x) \in L\) then also \(F(t) \in L\), since \(\neg F(t) \in E(\forall x F(x)) \subseteq E^-(L)\) and \(E(\neg F(t)) = \{F(t)\}\). As \(L\) covers the premise, either \(\Gamma \cap E^-(L) \neq \emptyset\), since \(F(t) \notin E^-(L)\), or \(\Delta \cap L \neq \emptyset\). Either case yields the claim for \(L\), so \(L \models \Gamma, \forall x F(x) \Rightarrow \Delta\).

For invertibility, assuming \(\Gamma, \forall x F(x) \models \Delta\), any \(L\) covering the premise of the rule covers also its conclusion, yielding the claim.

6. \((\forall \eta)\). For soundness, let \((\ast)\) \(\Gamma \models \Delta, F(y)\), with eigenvariable \(y\), and \(L\) cover the rule’s conclusion, under a given assignment \(\alpha\) to \(V(\Gamma, \Delta, \forall x F(x)) \neq y\). Assume also \(\alpha(\Gamma) \subseteq L\) and \(\alpha(\Delta) \subseteq E^-(L)\).
If \( \alpha(\forall x F(x)) \notin L \) then \( \alpha(\forall x F(x)) \in E^-(L) \) and some \( \alpha(\neg F(m)) \in L \), since \( E(\alpha(\forall x F(x))) = \{\alpha(\neg F(m)) \mid m \in M\} \). Extending \( \alpha \) with \( \alpha(y) = m \), we obtain \( L \not\models_\alpha \Gamma \rightarrow \Delta, F(y) \), contrary to \((*)\). Thus, \( \alpha(\forall x F(x)) \in L \) and \( L \not\models_\alpha \Gamma \rightarrow \Delta, \forall x F(x) \).

For invertibility, if \( L \not\models_\alpha \Gamma \rightarrow \Delta, F(y) \), for \( \alpha(y) = m \), i.e., \( \alpha(\Gamma) \subseteq L \), \( \alpha(\Delta) \subseteq E^-(L) \) and \( \alpha(F(m)) \in E^-(L) \), then \( L' = L \cup \{\exists \neg F(m)\} \) is a semikernel, because \( L \) is and \( E(\alpha(\neg F(m))) = \{\alpha(F(m))\} \subseteq E^-(L) \subseteq E^-(L') \). \( L' \) covers the conclusion since \( \alpha(\forall x F(x)) \in E^-(\alpha(\neg F(m))) \), but \( L' \not\models_\alpha \Gamma \rightarrow \Delta, \forall x F(x) \).

7. \( (\forall \varphi^+) \). The argument repeats that for \( (\forall S) \). For soundness, assume \( \Gamma, \forall S \varphi(\phi) \models \Delta \) and let \( L \) cover the rule’s conclusion. If \( \forall S \varphi(\phi) \notin L \) then \( \forall S \varphi(\phi) \in E^-(L) \), yielding \( L \models \Gamma, \forall S \varphi(\phi) \rightarrow \Delta \). If \( \forall S \varphi(\phi) \in L \) then also \( F(S) \subseteq L \), since \( \neg F(S) \in E(\forall S \varphi(\phi)) \subseteq E^-(L) \) and \( E(\neg F(S)) = \{F(S)\} \). Thus \( L \) covers also the premise, hence, either \( \Gamma \cap E^-(L) \neq \emptyset \), since \( F(S) \notin E^-(L) \), or \( \Delta \cap L \neq \emptyset \). Either case yields the claim for \( L \), so \( \Gamma, \forall S \varphi(\phi) \models \Delta \).

For invertibility, assuming \( \Gamma, \forall S \varphi(\phi) \models \Delta \), any \( L \) covering the premise of the rule covers also its conclusion, yielding the claim.

8. \( (\forall \varphi^+) \). For soundness, let \( \Gamma \models \Delta, F(S) \) for every \( S \), and \( L \) cover the rule’s conclusion. If \( \forall S \varphi(\phi) \in L \) then \( L \) satisfies the conclusion. If \( \forall S \varphi(\phi) \notin L \) then \( \forall S \varphi(\phi) \in E^-(L) \) and some \( \neg F(S) \in L \), since \( E(\forall S \varphi(\phi)) = \{\neg F(S) \mid S \in S^+\} \). Now \( L \) covers also the premise \( \Gamma \rightarrow \Delta, F(S) \) and \( F(S) \notin L \), hence either \( \Gamma \cap E^-(L) \neq \emptyset \) or \( \Delta \cap L \neq \emptyset \). Each case yields the claim that \( L \) satisfies the conclusion.

For invertibility, assume \( \Gamma \models \Delta, \forall S \varphi(\phi) \), and let \( L \) cover a premise \( \Gamma \rightarrow \Delta, F(S) \). If \((*) \) \( \Gamma \subseteq L \) and \( \Delta \cup \{F(S)\} \subseteq E^-(L) \) then \( L' = L \cup \{\neg F(S)\} \) is a semikernel, since \( E(\neg F(S)) = \{F(S)\} \subseteq E^-(L) \subseteq E^-(L') \). \( L' \) covers also \( \forall S \varphi(\phi) \), since \( \forall S \varphi(\phi) \in E^-(\neg F(Q)) \), for every \( Q \in S^+ \), in particular, \( \forall S \varphi(\phi) \in E^-(\neg F(1)) \). Thus \( \Gamma \cup \Delta \cup \{F(S), \forall S \varphi(\phi)\} \subseteq E^-[L'] \), while \( \Gamma \cap E^-(L') = \emptyset \) and \( \forall S \varphi(\phi) \notin L' \), contrary to \( \Gamma \models \Delta, \forall S \varphi(\phi) \). Hence \((*) \) fails, so either \( \Gamma \cap E^-(L) \neq \emptyset \) or \( \Delta \cup \{F(S)\} \cap L \neq \emptyset \), yielding the claim.

9. S-equality rules are sound and invertible, because atoms occurring in premises but not conclusions are redundant (due to point 3 of Definition 3.5 of language graph). E.g., \( S = S \) in the premise of (ref) is always satisfied (being a sink), hence satisfaction of the premise implies satisfaction of the conclusion. Conversely, satisfaction of the conclusion by any semikernel allows its extension with any sink, in particular, with \( S = S \). Analogous argument works for (rep) and (neq).

The following simple consequence of Definition 3.5 is used in the completeness proof below.

**Fact 8.2** In any graph \( G_M^+ \), the following relations hold between the form of a nonatomic sentence \( X \in S_M^+ \) and forms of its out- and in-neighbours:

(a) \( E^-(X) = \{\neg X\} \) when \( X \) does not start with \( \neg \),

(b) \( E^-(\neg X) = \{\neg X\} \cup \{X \wedge S \mid S \in S_M^+\} \cup \{\forall \exists \varphi.D(\phi) \mid \exists S \in S_M^+: D(S) = X\} \cup ...

... \cup \{\forall x.D(x) \mid \exists t \in T_M : D(t) = X\}

(c) when \( X \) does not start with \( \neg \), then each out-neighbour of \( X \) does,

(d) \( E(\neg X) = \{X\} \).

For atomic \( X \), \( E^-(X) = \{\neg X\} = E(X) \) and \( E^-(\neg X) = \{X\} = E(\neg X) \).

The proof of completeness can apply the standard techniques because proofs in LSO, even if infinite, are well-founded trees with axioms as leaves. A few adjustments are needed for handling deviations from 

**Proof:**

1. The infinitary rule (\( \forall \varphi^+ \)), needed because substitution of fresh eigenvariables for s-variables, although sound, does not necessarily lead to a countermodel in an unsuccessful derivation, since s-variables are not sentences. (Replacing (\( \forall \varphi^+ \)) by a usual \( \forall \varphi \)-rule using eigen-variables would yield a complete system for a modified notion of \( \models \), admitting extensions of the language with new s-constants.) In the proof, we ensure not only that all formulas are processed and all terms are substituted by (\( \forall \varphi \)), but also that all sentences are substituted by (\( \forall \varphi^+ \)). Missing subformula property, due to substitution of all sentences for s-variables, is handled by retaining the principal formula from the conclusion in all its premises, in a bottom-up construction of a derivation tree. As a special case of violation of this property, a branch can be cyclic, with the same sequent appearing infinitely often. Any nonaxiomatic (e.g., cyclic) branch provides a countermodel.
Fact 8.3 For a countable $\Gamma \cup \Delta \subseteq \text{FOL}^+$ : $\Gamma \not\vdash \Delta \Rightarrow \exists \exists L \in SK(\text{FOL}^+) : \nu(L) \neq \Gamma \Rightarrow \Delta$.

Proof. We fix an enumeration $E^+ \subseteq \text{FOL}^+$ so that each occurs infinitely often, an enumeration $E_T = t_1, t_2, \ldots$ of terms $T_X$ so that each occurs infinitely often, and an enumeration $E_S = S_1, S_2, \ldots$ of FOL+ formulas without free s-variables and with s-predicates applied only to sentences $S^+$, so that each occurs infinitely often. (FOL variables, requiring special care, are treated in the standard way and ignored below, e.g., we keep also an enumeration of eigenvariables). We enumerate all triples $\langle S_i, t_j, S_k \rangle \in E_S \times E_T \times E^+$, with each $\langle S_i, t_j, \Delta \rangle$ and $\langle S_i, \ldots, S_k \rangle$ occurring infinitely often. This is interleaved with an enumeration of all pairs $E_S \times E_S$, with each pair occurring infinitely often.

1. We construct a derivation tree, starting with the root $\Gamma \vdash \Delta$, which is to be proven. An active sequent $\vdash$—initially, only the root—is a nonaxiomatic leaf of the tree constructed bottom-up so far. We proceed along the enumeration of the triples and pairs considering, for each $\langle S_i, t_j, S_k \rangle$, the cases of active occurrences (in the active sequents) of $S_i$. The pairs $\langle S_i, S_j \rangle$ are given at the end.

   i. If $S_i \in A^+$, or $S_i$ has no active occurrences, proceed to the next triple.

   ii. Otherwise, proceed retaining $S_i$ from the active sequent, which instantiates the conclusion of the relevant rule, in the new leaves obtained from the rule’s premisses. E.g., if $S_i = A \land B$ then every active sequent of the form $\Gamma', A \land B, \Gamma'' \vdash \Delta$ is replaced by $A, B, \Gamma', A \land B, \Gamma'' \vdash \Delta$ while every active sequent of the form $\Gamma \vdash \Delta', A \land B, \Delta''$ by $\Gamma \vdash \Delta', A \land B, \Delta''$.

   iii. If $S_i = \forall x D(x)$, every active sequent of the form $\Gamma', \forall x D(x), \Gamma'' \vdash \Delta$, is replaced by the derivation with a new leaf adding $D(t_j)$ to its antecedent

   $\Gamma', \forall x D(x), \Gamma'' \vdash \Delta$.

   iv. If $S_i = \forall \phi D(\phi)$ then replace every active sequent of the form $\Gamma', \forall \phi D(\phi), \Gamma'' \vdash \Delta$ by $\Gamma', \forall \phi D(\phi), \Gamma'' \vdash \Delta$.

   v. For a pair $\langle S_i, S_j \rangle$, we apply rules for $\bar{=}$. If $S_i \neq S_j$, we add atom $S_i \bar{=} S_j$ to the consequent of every active sequent. Otherwise, we add it to the antecedent. Finally, for each active sequent containing $S_i \bar{=} Q$ in its antecedent, along with any formula $A(S_i)$, we add to it $A(Q)$.

2. A branch gets closed when its leaf is an axiom, and the tree is obtained as the $\omega$-limit of this process. If all branches are closed (finite), the derivation yields a proof of the root.

   Otherwise, an infinite branch allows us to construct a countermodel of all sequents on this branch, including the root sequent. (Such an infinite branch can represent a finite process of derivation terminating with a nonaxiomatic sequent, which remains unchanged in an infinite tail of the branch. It can also be cyclic. These special cases are treated uniformly with an infinite branch without any repeated sequents.)
3. The claim is that if $\beta$ is an infinite branch, with $\beta'_L/\beta'_R$ all formulas occurring in $\beta$ on the left/right of $\top$, then there is a language graph $G$ with a semikernel $L$ such that $\beta'_L \subseteq L'$ and $\beta'_R \subseteq E^-(L')$. The rest of the proof establishes this claim.

Absence of any axiom in $\beta$ implies that $\beta'_L \cap \beta'_R = \emptyset$, which is often applied implicitly. $\beta'_L = \beta_L \cup EQL$, where $EQL$ are $\lnot$-atoms $S \subseteq S$ occurring on the left. $\beta'_R = \beta_R \cup EQR$, where $EQR$ are $\lnot$-atoms occurring on the right, with $EQR$ denoting the set of their negations.

If $\beta$ contains any FOL-atoms, construct first a FOL-structure $M$, giving a countermodel to $(\beta_L \cap S_M) \Rightarrow (\beta_R \cap S_M)$, in the standard way. Otherwise, set $M = \emptyset$. Let $G = G_M$ (when $M = \emptyset$, this is the graph for QBS). We show that (def) $L = \beta_L \cup (E(\beta_R) \cap E^-(\beta_R))$ is a semikernel of $G$, with $\beta_R \subseteq E^-(L)$ and $\beta_L \subseteq L$. Then $L' = EQL \cup EQR \cup L$ is a required semikernel of $G$.

4. First, $\lnot$-atoms can be treated separately. Since $\beta'_L \cap \beta'_R = \emptyset$, each $\lnot$-atom $A \in EQR$ has the form $S \subseteq T$ for syntactically distinct sentences, while each such atom in $EQL$ has the form $S \subseteq S$. Any semikernel of $G$, in particular $L$, can be extended to semikernel $L' = L \cup EQL \cup EQR$, as the added vertices are sinks of $G$, by Definition 3.5. Thus $E(EQL \cup EQR) = \emptyset$, while $EQR \subseteq E^-(L') \cap (V \setminus L)$.

5. To show $L \in S K(G)$, we show first $\beta_R \subseteq E^-(L)$, which follows from definitions of $L$ and $G$ by considering the cases for $A \in \beta_R$. Use of Fact 8.2/Definition 3.5 is marked by superscript $^\perp$.

i. If $A \in A^+$ then $E(A) \overset{S^2}{=} \{ \lnot A \} \overset{S^2}{=} E^-(A)$, so $\lnot A \in L$ by (def) and $A \overset{S^2}{=} E^-(L)$.

ii. If $A = \lnot C$ then $C \in \beta_L \subseteq L$, so $\beta \overset{S^2}{=} E^-(L)$.

iii. If $A = C \land D$ then $C \in \beta_R$ (or $D \in \beta_R$), so $\lnot C \overset{S^2}{=} E^-(C) \cap E(C \land D) \subseteq E^-(\beta_R) \cap E(\beta_R) \subseteq L$, and thus $A = C \land D \subseteq E^-(\lnot C) \cap E^-(L)$. (The case of $D \in \beta_R$ is analogous.)

iv. If $A = \forall x. D(x)$ then $D(c) \in \beta_R$, for some $c \in M$, so $\lnot(D(c) \overset{S^2}{=} E^-(D(c)) \cap E(\forall x. D(x)) \subseteq L$, and $A \overset{S^2}{=} E^-(\lnot D(c)) \subseteq E^-(L)$.

v. If $A = \forall \phi. D(\phi)$ then $D(S) \in \beta_R$ for some $S \in S^+$, so $\lnot D(S) \overset{S^2}{=} E^-(D(S)) \cap E(\forall \phi. D(\phi)) \subseteq L$, and $A \overset{S^2}{=} E^-(\lnot D(S)) \subseteq E^-(L)$.

6. We show $E(L) \subseteq E^-(L) \cap (V \setminus L)$, partitioning $L = \beta_L \cup Z$, where $Z = (E(\beta_R) \cap E^-(\beta_R)) \setminus \beta_L$, and establish first $E(\beta_L) \subseteq E^-(L) \cap (V \setminus L)$, considering cases of $A \in \beta_L$.

i. For atoms $A \in A^+$, $A \in \beta_L \subseteq L$ and $A \not\in \beta_R$ imply $\lnot A \not\in \beta_R$ and, since $E(\lnot A) \overset{S^2}{=} \{ A \}$, $\lnot A \not\in E^-(\beta_R)$. Thus $E(A) \overset{S^2}{=} \{ \lnot A \} \subseteq E^-(A) \cap V \setminus L \subseteq E^-(L) \cap V \setminus L$.

ii. $A = \lnot C \in \beta_L$ implies $C \in \beta_R$, so $E(A) \overset{S^2}{=} \{ C \} \subseteq \beta_R \subseteq E^-(L)$ by 5.

We show $E(A) \subseteq V \setminus L$. $C \not\in \beta_R$ since $\beta_L \cap \beta_R = \emptyset$. Suppose $C \in E(\beta_R) \cap E^-(\beta_R)$. If $C = \lnot D$ then $\lnot D \in E^-(\beta_R)$, i.e., $E(\lnot D) \overset{S^2}{=} \{ D \} \subseteq \beta_R$, while $A = \lnot C = \lnot \lnot D \in \beta_L$ implies also $\lnot D \in \beta_R$ and $D \in \beta_L$, contradicting $\beta_L \cap \beta_R = \emptyset$.

Otherwise, i.e., if $C$ does not start with $\lnot$, then for any $F \in \beta_R$ for which $C \in E(F)$, Fact 8.2.(c-d) forces $F = \lnot C = A$, contradicting $\beta_R \cap \beta_L = \emptyset$.

iii. $A = B \land C \in \beta_L$ implies $\{ B, C \} \subseteq \beta_L$ and $\{ \lnot B, \lnot C \} \subseteq \beta_L$, so $E(B \land C) \overset{S^2}{=} \{ \lnot B, \lnot C \} \subseteq V \setminus \beta_L$ and $E(B \land C) = \{ \lnot B, \lnot C \} \subseteq E^-(\beta_L)$. If, say, $\lnot B \in E^-(\beta_R)$, then $B \in \beta_R$ would contradict $\beta_L \cap \beta_R = \emptyset$. The same if $\lnot C \in E^-(\beta_R)$. Thus, $E(B \land C) \subseteq E^-(L) \cap V \setminus L$.

iv. $A = \forall \phi. D(\phi) \in \beta_L \Rightarrow \{ D(S) \mid S \in S^+ \} \subseteq \beta_L$, so $E(\forall \phi. D(\phi)) \overset{S^2}{=} \{ \lnot D(S) \mid S \in S^+ \} \subseteq E^-(\{ D(S) \mid S \in S^+ \}) \subseteq E^-(\beta_L) \subseteq E^-(L)$.

If any $\lnot D(S) \in L$ then either $\lnot D(S) \in \beta_L$, so $D(S) \in \beta_R$, or $\lnot D(S) \in E(\beta_R) \cap E^-(\beta_R)$, which implies $D(S) \in \beta_R$, since $E(\lnot D(S)) \overset{S^2}{=} \{ D(S) \}$. In either case, $D(S) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \emptyset$. Thus $E(\forall \phi. D(\phi)) \subseteq V \setminus L$.

v. For $A = \forall x. D(x)$, the argument is as in iv. $\forall x. D(x) \in \beta_L$ implies $\{ D(t) \mid t \in T_M \} \subseteq \beta_L$, so $E(\forall x. D(x)) \overset{S^2}{=} \{ \lnot D(t) \mid t \in T_M \} \subseteq E^-(\{ D(t) \mid t \in T_M \}) \subseteq E^-(\beta_L) \subseteq E^-(L)$.
If any \( \neg D(t) \in L \), then either \( \neg D(t) \in \beta_L \), so \( D(t) \in \beta_R \), or \( \neg D(t) \in E(\beta_R) \cap E^-(\beta_R) \), which implies \( D(t) \in \beta_R \), since \( E(\neg D(t)) \subseteq \{D(t)\} \). In either case, \( D(t) \in \beta_R \) contradicts \( \beta_L \cap \beta_R = \emptyset \).

Thus \( E(\forall x.D(x)) \subseteq V \setminus L \).

7. Also each sentence \( S \in Z = (E(\beta_R) \cap E^-(\beta_R)) \setminus \beta_L \) satisfies \( E(S) \subseteq E^-(L) \cap (V \setminus L) \)

i. If \( S \in Z \) does not start with \( \neg \), then \( E^-(S) \subseteq \{\neg S\} \), so \( \neg S \in \beta_R \), implying \( S \in \beta_L \), so \( S \notin Z \).

ii. If \( S = \neg A \in Z \subseteq E^-(\beta_R) \) then \( E(\neg A) \subseteq \beta_R \subseteq E^-(L) \). If \( A \in Z \), then it starts with \( \neg \) by 7.1, i.e., \( A \in B \) and \( E(\neg B) \subseteq \{B\} \subseteq E^-(\beta_R) \). Since also \( A \in \beta_R \) so \( B \in \beta_L \), contradicting \( \beta_L \cap \beta_R = \emptyset \).

Hence \( A \notin Z \) and \( A \notin \beta_L \) (since \( A \in \beta_R \)), i.e., \( A \notin L = Z \cup \beta_L \), so that \( E(\neg A) = \{A\} \subseteq V \setminus L \).

By 6 and 7, \( E(L) = E(\beta_L) \cup E(Z) \subseteq E^-(L) \cap (V \setminus L) \), so \( L \in SK(G) \) by Fact 7.1.

Unlike in variants of circular proof theory, an infinite branch gives always a rise to a countermodel.

A paradigmatic example of a cyclic proof, with the same sequent reappearing infinitely often in a branch, can be the attempted derivation of \( \forall \phi, \phi \):

\[
\frac{\frac{\therefore \vdash \forall \phi, \phi}{\therefore \vdash \neg \neg \forall \phi, \phi}}{\therefore \vdash \neg \forall \phi, \phi}
\]

Any sentence \( \neg \forall \phi \land \forall \phi \) gives a counterexample when any of its branches does not terminate with an axiom, providing a countermodel. Such is, in particular, the leftmost branch where \( \forall \phi, \phi \) is substituted for \( \phi \) in the root sentence and expanded further, giving a copy of the whole tree and, eventually, a special branch \( \beta \) with \( \beta_L = \emptyset \) and \( \beta_R = \{\forall \phi, \phi\} \). This infinite branch provides also a countermodel, with \( \forall \phi, \phi = \emptyset \). This looks strange, but is verified by inspecting graph \( G_M(AK) \) in Example 3.9, according to which \( \forall \phi, \phi \) does act as a witness to its own falsity.

A branch with a repeated sequent can be terminated, with the conclusion of unprovability, if one can verify that subsequent substitutions, higher up in the derivation, will also yield the same sequent. A single repetition is not enough, as it may be due to a specific substitution, while subsequent ones might yield new sequents.

The last fact to be shown is soundness and completeness with (cut) for the global semantics of kernels defined in (3.10).

**Fact 4.2** For a countable \( \Gamma \cup \Delta \subseteq FOL^+ \), \( \Gamma \models \Delta \) iff \( \Gamma \vdash \Delta \).

**Proof.** Soundness and invertibility follow by essentially the same argument as in Fact 8.1, with some simplifications due to each kernel \( K \in sol(G) \) covering the whole graph, \( E_G(K) = V_G \setminus K \).

We fix an arbitrary graph \( G \) and show each case for an arbitrary fixed \( K \in sol(G) \).

1. \((\wedge R)\). For soundness, let \( \Gamma \models \Delta, A_1, \Gamma \models \Delta, A_2, \) and \( a \in M^{\forall(\Gamma, \Delta)} \). Then \( (a(\Gamma) \cap V)(K) \neq \emptyset \) or \( a(\Delta) \cap K \neq \emptyset \), in which case also conclusion is satisfied under \( a \), or else \( \{a(A_1), a(A_2)\} \subseteq K \). Then \( \{\neg a(A_1), \neg a(A_2)\} \subseteq V \setminus K \), and hence \( a(A_1 \land A_2) \subseteq K \) since \( E(a(A_1 \land A_2)) = \{\neg a(A_1), \neg a(A_2)\} \).

For invertibility, let \( \Gamma \models \Delta, A_1 \land A_2 \). If \( a(\Gamma) \cap V \setminus K \neq \emptyset \) or \( a(\Delta) \cap K \neq \emptyset \), then \( K \) satisfies also both premises under \( a \). If neither is the case, then \( a(A_1 \land A_2) \in K \), hence \( E(a(A_1 \land A_2)) = \{\neg a(A_1), \neg a(A_2)\} \subseteq K \), for \( i \in \{1, 2\} \), hence \( K \models a(\Gamma) \Rightarrow A_i, A_i \).

Assignments to free FOL-variables do not affect the arguments below. They are relative to a given \( a \), which we do not mention, except for \( (\forall R) \). In each case, we assume that \( \Gamma \subseteq K \) and \( \Delta \subseteq V \setminus K \) focusing on the active/principal formulas.

2. \((\wedge L)\). For soundness, assuming \( \Gamma, A_1, A_2 \models \Delta \) (and \( \Gamma \subseteq K \) and \( \Delta \subseteq V \setminus K \), \( A_i \in V \setminus K \), for \( i = 1 \) or \( i = 2 \). Then \( \neg A_i \in K \), since \( E(\neg A_i) = A_i \), and \( A_1 \land A_2 \in E(\neg A_1) \subseteq E^-(K) \subseteq V \setminus K \).

Thus \( K \models \Gamma, A_1 \land A_2 \Rightarrow \Delta \).

For invertibility, assume \( \Gamma, A_1 \land A_2 \models \Delta \). If \( A_1, A_2 \in K \), which is the only way \( K \) can contradict \( \Gamma, A_1 \land A_2 \models \Delta \), then \( E(\Gamma, A_1 \land A_2) = \{\neg A_1, \neg A_2\} \subseteq E^-(\{A_1, A_2\}) \subseteq E^-(K) = V \setminus K \), and hence \( A_1 \land A_2 \in K \), contradicting \( K \models \Gamma, A_1 \land A_2 \Rightarrow \Delta \).

3. \((\neg R)\). For soundness, \( \Gamma, A \models \Delta \) implies \( A \in V \setminus K \), so \( \neg A \in K \), since \( E(\neg A) = A \).

For invertibility, \( \Gamma \models \Delta, \neg A \) implies \( \neg A \in K \), so \( E(\neg A) = \{A\} \subseteq V \setminus K \) and \( K \models \Gamma, A \Rightarrow \Delta \).

4. \((\neg L)\). For soundness, \( \Gamma \models \Delta, A \) implies \( A \in K \) hence \( \neg A \in V \setminus K \) and \( K \models \Gamma, \neg A \Rightarrow \Delta \).
For invertibility, \( \Gamma, \neg A \models_c \Delta \) implies \( \neg A \in V \setminus K \), hence \( E(\neg A) = A \in K \) and \( K \models \Gamma \Rightarrow \Delta, A \).

5. (\( \forall \beta \)). For soundness, assume \( F(t), \Gamma \cup xF(x) \models_c \Delta \). If \( \forall xF(x) \notin K \), i.e., \( \forall xF(x) \in V \setminus K \), then \( (\Gamma \cup \{\forall xF(x)\}) \cap E(\neg(\Gamma)) \neq \emptyset \), so \( K \models \Gamma \cup xF(x) \Rightarrow \Delta \). If \( \forall xF(x) \in K \) then also \( F(t) \in K \), since \( \neg F(t) \in E(\neg(\neg F(t))) \subseteq V \setminus K \), so \( E(\neg F(t)) \cap K \neq \emptyset \) while \( E(\neg F(t)) = \{F(t)\} \). Thus either \( \Gamma \cap (V \setminus K) \neq \emptyset \) or \( \Delta \cap K \neq \emptyset \), yielding \( K \models \Gamma \cup xF(x) \Rightarrow \Delta \).

Invertibility follows by weakening since \( \Gamma, \forall xF(x) \models_c \Delta \) implies \( F(t), \Gamma \cup xF(x) \models_c \Delta \).

6. (\( \forall \beta \)). For soundness, let \( (\forall \beta) \Gamma \models_c \Delta, F(y) \), with eigenvariable \( y \notin V(\Gamma, \Delta) \), and \( \alpha(\Gamma) \subseteq K \) and \( \alpha(\Delta) \subseteq V \setminus K \). If \( \alpha(\forall xF(x)) \notin K \) then \( \alpha(\forall xF(x)) \in E(\neg(K)) \) and some \( \alpha(\neg F(m)) \in K \), since \( \mathbf{E}(\alpha(\forall xF(x))) = \{\alpha(\neg F(m)) | m \in M\} \). Extending \( \alpha \) with \( \alpha(y) = m \) yields \( \alpha(\Gamma) \subseteq K \) and \( \alpha(\Delta, F(y)) \subseteq V \setminus K \), contrary to \((\forall \beta)\). Hence \( \alpha(\forall xF(x)) \in K \).

For invertibility, if \( \alpha(\Gamma) \subseteq K \) and \( \alpha(\Delta, F(y)) \subseteq V \setminus K \), for \( \alpha(y) = m \), then \( \neg \alpha(\neg F(m)) \in K \), since \( \mathbf{E}(\neg(\neg F(m))) = \alpha(\neg F(m)) \subseteq V \setminus K \). Then \( \alpha(\forall xF(x)) \in \mathbf{E}(\neg(\neg F(m))) \subseteq \mathbf{E}(\neg(K)) = V \setminus K \), giving \( \alpha(\Gamma) \subseteq K \) and \( \alpha(\Delta, 1.Fx) \subseteq V \setminus K \), which contradicts \( K \models \Gamma \Rightarrow \Delta, \forall \beta.Fx \).

7. (\( \forall \beta \)). The argument repeats that for (\( \forall \beta \)). For soundness, assume \( \Gamma, F(S), \forall \beta(\phi) \models_c \Delta \). If \( \forall \beta(\phi) \in K \) then also \( F(S) \in K \), since \( \neg F(S) \in \mathbf{E}(\forall \beta(\phi)) \subseteq \mathbf{E}(\neg(K)) \) and \( \mathbf{E}(\neg F(S)) = \{F(S)\} \). Hence, as \( F \subseteq K \) and \( \Delta \subseteq V \setminus K \), it holds \( \forall \beta(\phi) \in V \setminus K \) and \( K \models \Gamma, \forall \beta(\phi) \Rightarrow \Delta \).

Invertibility, assuming \( \Gamma, \forall \beta(\phi) \models_c \Delta \), weakening yields \( F(S), \Gamma, \forall \beta(\phi) \models_c \Delta \).

8. (\( \forall \beta \)). For soundness, let \( \Gamma \models_c \Delta, F(S) \) for every \( S \in S^+ \). If \( \forall \beta(\phi) \notin F \) then \( \forall \beta(\phi) \in \mathbf{E}(\neg(K)) \) and some \( \neg F(S) \in K \), since \( \mathbf{E}(\forall \beta(\phi)) = \{\neg F(S) | S \in S^+\} \). Since \( F(S) \notin K \), either \( \Gamma \Rightarrow \mathbf{E}(\neg(K)) \neq \emptyset \) or \( \Delta \cap K \neq \emptyset \), contradicting the assumption \( \Gamma \subseteq K \) and \( \Delta \subseteq V \setminus K \). Hence \( \forall \beta(\phi) \in K \) and \( K \) satisfies the rule’s conclusion.

For invertibility, assume \( \Gamma \models_c \Delta, \forall \beta(\phi) \), and \( \forall \beta(\phi) \in K \). If for some \( S \in S^+ \), \( F(S) \in V \setminus K \), then \( \neg F(S) \in K \) since \( \mathbf{E}(\neg F(S)) = \{F(S)\} \subseteq V \setminus K \). But \( \neg F(S) \in \mathbf{E}(\forall \beta(\phi)) \), contradicting independence of \( K \). Hence, \( F(S) \in K \) for all \( S \in S^+ \).

9. The rules for \( \models \) are sound and invertible by the same argument as in case of semikernels, point 1.v in proof of Fact 8.3. Every kernel contains all sinks, so \( S \models \Gamma \) for syntactically distinct sentences, and all \( S \models \) belong to every kernel.

This concludes the proof of soundness. For completeness, we modify the construction from the proof of Fact 8.3, by interleaving the enumeration of all triples \( E_\delta \times E_\gamma \times E^+ \) and pairs \( E_\delta \times E_\delta \) with enumeration \( E_M^\delta \) of all FOL+ formulas without free s-variables, where each such formula occurs only once. Following this interleaved enumeration yields now a new case, 1.5 of an \( A \in E_\delta^\prime \), in which we expand each active sequent \( \Gamma \Rightarrow \Delta \) with the premises of (cut) over \( A \), i.e., with \( \Gamma \Rightarrow \Delta, A \) and \( \Delta, A \Rightarrow \Delta \). A semikernel falsifying any one of them, falsifies the conclusion. Given an infinite nonaxiomatizable branch \( \beta \), a language graph \( G_\beta M \) is obtained as in the proof of Fact 8.3, over domain \( M \) consisting of free variables and ground terms used in the standard construction of a FOL countermodel for \( \beta \models S_M \). Point 3 of the proof of Fact 8.3 shows \( \beta \) to determine a semikernel \( K \) of \( G_\beta M \), falsifying each sequent on \( \beta \). Now, \( \beta \) contains one of the premises of an application of (cut) for each \( A \in E_\delta^\prime = S_M^+ \). As every sentence \( S_M^+ \) occurs thus in \( \beta_L \) or \( \beta_R \), semikernel \( K \) covers all \( S_M^+ \), so it is a kernel of \( G_\beta M \).

References