Extensions in Graph Normal Form

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Abstract

Graph Normal Form, introduced earlier for propositional logic, is shown to be a normal form also for first-order logic. It allows to view syntax of theories as digraphs, while their semantics as kernels of these digraphs. Graphs are particularly well suited for studying circularity, and we provide some general means for verifying that circular or apparently circular extensions are conservative. Traditional syntactic means of ensuring conservativity, like definitional extensions or positive occurrences guaranteeing exsitence of fixed points, emerge as special cases.

1 Graph Normal Form

Graph Normal Form, GNF, for propositional logic was introduced in [1] and applied in [5] to analysis of paradoxes. We begin by showing in this section that it is normal form also for first-order logic, FOL. Section 2 shows how to represent theories in GNF as graphs and their classical semantics as graph kernels. Logical circularity emerges as graph cycles, of which only odd ones are vicious, leading possibly to inconsistency. Using graph-theoretic terms, Section 3 formulates conditions ensuring conservativity of extensions, which generalize those utilizing usual syntactic restrictions. Definitional extensions provide a special case, while the conditions are applicable in many situations involving extensions with circular or apparently circular definitions, like fixed point definitions.

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Given a FOL language $\mathcal{L}$ and a set D, by $\mathbb{T}_D$ we denote the free term algebra over D, and by $\textsc{At}(D)$ the atomic formulas formed by application of any predicate symbol $P \in \mathcal{L}$ to elements of D. The usual atoms are thus $\textsc{At}(\mathbb{T}_X)$, for some set of variables X.

Definition 1.1 *A formula of a FOL language $\mathcal{L}$ is in GNF if it is an equivalence where*

- *the left side is an atom, $LS \in \textsc{At}(\mathbb{T}_X)$,*
- *the right side, RS, is a (universally quantified) conjunction of negated atoms,*
- *all free variables of RS occur in LS, $\mathcal{V}(RS) \subseteq \mathcal{V}(LS)$.*

A theory, i.e., a set of formulas, Γ is in GNF if each $F \in \Gamma$ is in GNF.

Atomic formulas are special cases of GNF, consisting of LS with empty RS. Predicate symbols not occurring in LS of any formula of a GNF theory Γ are referred to as "undefined" by Γ. (Occurring elsewhere, they are as defined as others, but this figure of speech will find some justification.)

Given any formula F in prenex normal form, PNF, its $GNF(F)$ is obtained by constructing first a restricted form of Morleyization, $GNF^{-}(F)$, introducing fresh predicates by a series of definitional extensions. In the resulting formulas, we write the universal quantifier $\forall x$ as $\bigwedge_x$.

Example 1.2 *In a PNF formula, say* $\phi = \forall x \exists y Pxy$,[1] *we first replace* $\exists$ *by* $\neg\forall\neg$, *obtaining* $\forall x \neg\forall y \neg Pxy$, *and then introduce fresh variable* z *and fresh predicates* A, S *with equivalences:*

$Az \leftrightarrow \bigwedge_x \neg Sx,$
$Sx \leftrightarrow \bigwedge_y \neg Pxy.$

For $\exists y \forall x Pxy$, *we first obtain* $\neg\forall y \neg\forall x Pxy$, *introduce a fresh variable* z *and then*

$Nz \leftrightarrow \neg Rz$
$Rz \leftrightarrow \bigwedge_y \neg Qy$
$Qy \leftrightarrow \bigwedge_x \neg\overline{P}xy$
$\overline{P}xy \leftrightarrow \neg Pxy$

Substituting back the introduced predicates shows the equivalence to the original formula:

$Az \leftrightarrow \bigwedge_x \neg Sx \leftrightarrow \bigwedge_x \neg \bigwedge_y \neg Pxy,$
$Nz \leftrightarrow \neg Rz \leftrightarrow \neg \bigwedge_y \neg Qy \leftrightarrow \neg \bigwedge_y \neg \bigwedge_x \neg\overline{P}xy \leftrightarrow \neg \bigwedge_y \neg \bigwedge_x \neg\neg Pxy \leftrightarrow \neg \bigwedge_y \neg \bigwedge_x Pxy.$

An open formula can be treated in the same way as its universal closure, or by keeping its free variables in the fresh predicate in the LS. Nothing significant would change if, for the sentence ϕ above, we replaced $Az \leftrightarrow \bigwedge_x \neg Sx$ by $Ax \leftrightarrow \neg Sx$. This, with the equivalence for Sx, can be taken as $GNF^-(\exists y Pxy)$.

Remaining propositional matrix, if nonatomic, is rewritten further, introducing new predicates, in the manner analogous to Example 1.2. If this example and description do not suffice, the reader may consult the appendix: Definition 4.1 details the transformation into GNF^-.

If ϕ is a formula in a language $\mathcal{L}$, then $GNF^-(\phi)$ denotes a GNF theory in an extended language $\mathcal{L}^-$, resulting from the procedure exemplified above. The choice of the predicate symbols is inessential as long as they are all different, so we speak also about the $GNF^-(\phi)$. The predicate Az in $GNF^-(\phi)$, not occurring in any RS and satisfying $GNF^-(\phi) \models Az \leftrightarrow \phi$, is referred to as the *top predicate* of $GNF^-(\phi)$. For a theory Γ, its $GNF^-(\Gamma)$ is obtained as the union of $GNF^-(\phi)$, for each $\phi \in \Gamma$, where distinct $GNF^-(\phi)$ introduce distinct predicate symbols.

The following fact states the basic feature of $GNF^-(\Gamma)$, illustrated in Example 1.2 and proven in the appendix. An *explicit definition*, in a FOL language $\mathcal{L}$, of a predicate $B \notin \mathcal{L}$ is $Bx \leftrightarrow \phi$, where ϕ is an $\mathcal{L}$-formula with free variables $\mathcal{V}(\phi) \subseteq x$. A *definitional extension* of a theory Γ is a well-founded chain of explicit definitions, starting with Γ.

Fact 1.3 *For every FOL theory* Γ, $GNF^-(\Gamma)$ *is a definitional extension of* Γ.

Theory $GNF^-(\phi)$ is satisfiable even if ϕ is not. To obtain equisatisfiability of ϕ and its GNF, we add one more formula.

Definition 1.4 $GNF(\phi)$ *is* $GNF^-(\phi)$ *with a fresh predicate symbol* A', *where* $ar(A') = ar(A)$ *for the top predicate* A *of* $GNF^-(\phi)$, *and the GNF formula*

$$A'x \leftrightarrow (\neg Ax \wedge \neg A'x). \tag{1.5}$$

For a theory Γ, $GNF(\Gamma) = \bigcup_{\phi \in \Gamma} GNF(\phi)$, *where distinct* $GNF(\phi)$ *introduce distinct symbols.*

Example 1.6 *For* $\phi = \forall x \exists y Pxy$ *from Example 1.2, its* $GNF^-(\phi)$

$Az \;\leftrightarrow\; \bigwedge_x \neg Sx$
$Sx \;\leftrightarrow\; \bigwedge_y \neg Pxy$

becomes $GNF(\phi)$ *when extended with:* $A'z \leftrightarrow (\neg Az \wedge \neg A'z)$.

Although $GNF(\Gamma)$ is no longer a definitional extension of Γ, due to formula (1.5), it shares its essential feature: every model of Γ has a unique expansion to a model of $GNF(\Gamma)$, because for

[1] We write Bx for $B(x)$, with x denoting a list of variables matching the arity of B, i.e., $x \in X^{ar(B)}$. Btx denotes an application of the predicate B to term(s) t, possibly with some variables x, where x may also comprise other arguments of B. $ar(Bt)$ denotes then the arity of the derived predicate resulting from such a substitution. Abbreviation $Bx \leftrightarrow \bigwedge_y Cxy$ assumes $x \cap y = \emptyset$ but admits all cases of $\mathcal{V}(Cxy) \cap x \subseteq x$. Substitution of t for x yields then Bt and Cty, where $\mathcal{V}(Cty) = (\mathcal{V}(Cxy) \setminus x) \cup \mathcal{V}(t)$. Such syntactic details would only clutter the main ideas, distinguishing cases which appear uniform from the level relevant to our considerations.

a structure M : $M \models A'z \leftrightarrow (\neg Az \wedge \neg A'z)$ if and only if $M \models Az$. On the other hand, reduct of every model of $GNF(\Gamma)$, forgetting A', is a model of Γ. This gives the following fact, where $X \leftrightarrows Y$ denotes a bijection. (We show existence of injections $X \to Y$ and $Y \to X$, also when X, Y are classes, assuming their theory with Schröder-Bernstein theorem, e.g., NGB.)

Fact 1.7 *$Mod(\Gamma) \leftrightarrows Mod(GNF(\Gamma))$, for each FOL theory Γ.*

Proof. If $M \models \Gamma$ then, as a consequence of Fact 1.3, M has a unique expansion M^- to the language $\mathcal{L}^-$, so that $M^- \models \Gamma \cup GNF^-(\Gamma)$. Since $M \models Ax$ for each $Ax \in \Gamma$, also $M^- \models Ax$. Interpreting $A' = \emptyset$ in M^-, yields an expansion M' satisfying (1.5), i.e., $M' \models GNF(\Gamma)$. No other interpretation of A' over M^- satisfies (1.5), so the expansion of M to $M' \in Mod(GNF(\Gamma))$ is unique.

Conversely, if $M' \models GNF(\Gamma)$ then, in particular, $M' \models Ax$, for each $Ax \in \Gamma$, since M' satisfies each equivalence (1.5). Hence $M'|_{\mathcal{L}} \in Mod(\Gamma)$, where $M'|_{\mathcal{L}}$ denotes the $\mathcal{L}$-reduct of M'. □

GNF is thus a normal form for FOL, and the rest of the paper studies only theories in GNF.

2 Graphs as syntax

The syntax of a GNF theory can be represented by a graph, while the semantics amounts to specific properties of this syntax graph. By a "graph" we mean directed graph, namely, a pair $G = (V_G, A_G)$ with vertices, V_G, and edges, $A_G \subseteq V_G \times V_G$. For $x \in V_G$, by $A_G(x)$ we denote the set of vertices with an edge from x, that is $A_G(x) = \{y \in V_G \mid \langle x, y\rangle \in A_G\}$ and, dually, $A^-_G(y) = \{x \in V_G \mid \langle x, y\rangle \in A_G\}$. $A^*_G, A^{\underline{*}}_G$ denote the reflexive transitive closure of A_G, A^-_G. This notation is extended to sets, e.g., $A^*_G(X) = \bigcup_{v\in X} A^*_G(v)$, etc.

One proviso is needed. Definition 1.1 admits in a GNF theory several equivalences with the same predicate in LSs, say, $Pa \leftrightarrow ...$ and $Pb \leftrightarrow ...$, with a, b being constants or even unifiable terms. Such general cases are treated in the appendix. Until then, definitions, results and examples are formulated for first-order logic without equality or function symbols, FOL$^-$, to convey the main ideas undisturbed by more detailed technicalities, which are presented in the appendix. Thus $\mathbb{T}_X = X$ and the only terms in LS of each GNF formula are variables, $LS \in \mathrm{A{\small T}}(X)$. We assume also that each predicate symbol P occurs at most once in LS of some formula of a theory, to which we refer as P's equivalence.

We let B_i range over predicate symbols from the language $\mathcal{L}$ of a GNF theory Γ. An arbitrary formula from Γ (with the restrictions just mentioned) is represented schematically by the pattern:

$$Bx \leftrightarrow \bigwedge_y \big(\neg B_1xy \wedge ... \wedge \neg B_nxy\big). \tag{2.1}$$

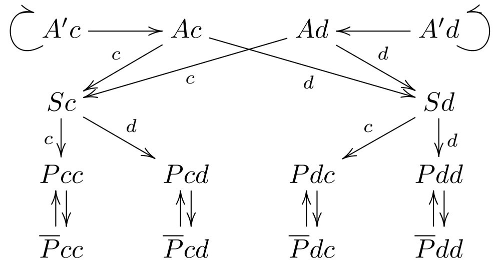
Definition 2.2 *For a GNF theory Γ in FOL$^-$ and set D, the graph $G = \mathcal{G}_D(\Gamma)$ is given by:*

1. *$V_G = \mathrm{A{\small T}}(D) \cup \overline{\mathrm{A{\small T}}}(D)$, with $\overline{\mathrm{A{\small T}}}(D)$ given in point 3.*
2. *For each axiom (2.1) of Γ, each vertex $Bd \in \mathrm{A{\small T}}(D)$ instantiating its LS Bx has the outgoing edges to $A_G(Bd) = \{B_idc \mid 1 \le i \le n, c \in D^{ar(B_id)}\}$.*
3. *For each undefined predicate symbol P, we add a fresh symbol $\overline{P}$ and the 2-cycle $Pd \leftrightarrows \overline{P}d$ for each $d \in D^{ar(P)}$. Vertices $\overline{P}d$ form $\overline{\mathrm{A{\small T}}}(D)$.*

A nonempty D is a domain of a FOL-structure interpreting the language $\mathcal{L}$ of the theory. We exclude the empty graph, $\mathcal{G}_\emptyset(\Gamma) = (\emptyset, \emptyset)$, from considerations, and denote by $Gr(\Gamma)$ the class of graphs $\mathcal{G}_D(\Gamma)$, for all nonempty sets D.

Example 2.3 *For $\phi = \forall x \exists y Pxy$ with $\Gamma = GNF(\phi)$ from Example 1.6, repeated to the left, and* *for the set $D = \{c, d\}$, Definition 2.2 yields the graph $\mathcal{G}_D(\Gamma)$ to the right:*

$$\begin{aligned} A'z &\leftrightarrow \neg Az \wedge \neg A'z \\ Az &\leftrightarrow \textstyle\bigwedge_x \neg Sx \\ Sx &\leftrightarrow \textstyle\bigwedge_y \neg Pxy \end{aligned}$$



The subgraph induced[2] by $\{Ac, Ad, Sc, Sd\}$ and all Pxy vertices corresponds to $GNF^{-}(\phi)$, with vertices $\overline{P}xy$ and their 2-cycles to Pxy added according to Definition 2.2.3. The top vertices $A'c, A'd$ with their edges arise from formula (1.5) in Definition 1.4.

Generally, in $\mathcal{G}_D(\Gamma)$, each vertex Ad, for $d \in D$, has edges to $|D|$ copies of a subgraph with a source Sx and edges to Pxy, for all $x, y \in D$. Each pair Pxy, $\overline{P}xy$ forms a 2-cycle.

For each instance Bd, $d \in D^{ar(B)}$, of an atomic axiom Bx, Definition 2.2.2 gives the empty set of neighbours, including thus Bd in $sinks(G) = \{v \in V_G \mid A_G(v) = \emptyset\}$. The copies $\overline{P}d$, added along with the 2-cycles for each undefined Pd in point 3, ensure that such atoms are not sinks and can obtain arbitrary values, restricted only by the rest of the theory, as explained below.

2.1 Kernels as models

Given a GNF theory Γ, we denote its usual models by $Mod(\Gamma)$, while its models over a given set D by $Mod_D(\Gamma)$. Graphs from Definition 2.2 provide an equivalent representation of these models.

Graph $\mathcal{G}_D(\Gamma)$ mixes syntax, using the predicate symbols from the language of Γ, with semantics, applying these symbols to the elements of the interpretation domain D. Typically, by "atoms" we refer to such mixed expressions. One uses such a notational abbreviation when, for a formula $\phi(x)$, one writes $\phi(d)$ for some $d \in D$. It may denote (i) the formula $\phi(x)$ with a new constant - naming the object d – substituted for x, or else, (ii) the interpretation of the formula $\phi(x)$ under a valuation of variables assigning the object d to x. Vertices of our graphs come closest to (i) – formulas with names of the objects from D substituted for variables. They obtain truth values, becoming (ii), relatively to the solutions of the graph, which we now define.

A *kernel* (or a *solution* [4]) of a graph G is a subset $K \subseteq V_G$ which
(a) is independent, i.e., $A_G^{-}(K) \subseteq V_G \setminus K$, and
(b) absorbs its complement, i.e., $A_G^{-}(K) \supseteq V_G \setminus K$,
in short, such that $A_G^{-}(K) = V_G \setminus K$. Equivalently, a kernel is an assignment $\alpha \in \mathbf{2}^{V_G}$ such that
$\forall x \in V_G : \alpha(x) = \mathbf{1} \Leftrightarrow (\forall y \in A_G(x) : \alpha(y) = \mathbf{0})$.
A graph G is *solvable* when $Ker(G) \neq \emptyset$, where $Ker(G)$ denotes the set of all kernels of G.

Vertices of a graph $\mathcal{G}_D(\Gamma)$ contain all D instances of all atomic formulas. An assignment $v \in D^{\mathcal{V}(\phi)}$ to free variables $\mathcal{V}(\phi)$ of a formula ϕ, along with a kernel K, determine a valuation of all atoms over $\mathcal{V}(\phi)$, with atoms in K assigned $\mathbf{1}$, as given in line 1 below. Satisfaction of arbitrary formulas is defined from this basis in the usual way, with the needed adjustments in the last two lines. (For a structure M (or kernel of a graph) over a set D, a formula ϕx and $d \in D$, we write $M \models \phi d$ if M satisfies ϕx with x valuated to d.)

$$\begin{array}{rlcl} 1. & (G,K) \models_v Px & \Leftrightarrow & P(v(x)) \in K, \text{for atomic } Px \\ 2. & (G,K) \models_v \neg\phi & \Leftrightarrow & (G,K) \not\models_v \phi \\ 3. & (G,K) \models_v \phi_1 \wedge \phi_2 & \Leftrightarrow & (G,K) \models_v \phi_1 \text{ and } K \models_v \phi_2 \\ 4. & (G,K) \models_v \forall x \phi x & \Leftrightarrow & \forall d \in D : (G,K) \models_v \phi d \\ \textit{and} & (G,K) \models \phi & \Leftrightarrow & \forall v \in D^{\mathcal{V}(\phi)} : (G,K) \models_v \phi \\ & G \models \phi & \Leftrightarrow & \forall K \in Ker(G) : (G,K) \models \phi \\ & \Gamma \models \phi & \Leftrightarrow & \forall G \in Gr(\Gamma) : G \models \phi \end{array} \tag{2.4}$$

[2]Given $X \subseteq V_G$, the subgraph of G *induced by* X is $(X, A_G \cap (X \times X))$.

Repeating the standard definition of satisfaction, this gives $\Gamma \models \phi$ coinciding with the standard notion $Mod(\Gamma) \models \phi$, provided that kernels of graphs in $Gr(\Gamma)$ correspond to $Mod(\Gamma)$. We establish this now, showing first that kernels of $\mathcal{G}_D(\Gamma)$ correspond to $Mod_D(\Gamma)$. (Given a graph G, $K \models \phi$ abbreviates $(G, K) \models \phi$.)

Fact 2.5 *For a GNF theory Γ in FOL$^-$ and any nonempty set D, there are injections*
$kr : Mod_D(\Gamma) \to Ker(\mathcal{G}_D(\Gamma))$ *and* $md : Ker(\mathcal{G}_D(\Gamma)) \to Mod_D(\Gamma)$,
such that $M \models \phi \Leftrightarrow kr(M) \models \phi$ *and* $K \models \phi \Leftrightarrow md(K) \models \phi$ *for* $M \in Mod_D(\Gamma), K \in Ker(\mathcal{G}_D(\Gamma))$ *and formula ϕ of the language of Γ.*

Proof. An injection $kr : Mod_D(\Gamma) \to Ker(\mathcal{G}_D(\Gamma))$ is obtained as follows. Let D^+ be a model of Γ over a set D, $G = \mathcal{G}_D(\Gamma)$ be as in Definition 2.2, and
$kr(D^+) = K = \{Bd \in \mathrm{AT}(D) \mid D^+ \models Bd\} \cup \{\overline{P}d \in \overline{\mathrm{AT}}(D) \mid D^+ \not\models Pd\}$,
where $\overline{\mathrm{AT}}(D)$ is as in Definition 2.2.3. If $Bd \notin K$, for Bd instantiating LS Bx of some axiom (2.1), say F, then $D^+ \not\models Bd$ and $D^+ \models \neg Bd$. Since D^+ satisfies F, for some $B_i xy$ in its RS and some $c \in D^{ar(B_i d)}$, also $D^+ \models B_i dc$, i.e., $B_i dc \in K$. Since $B_i dc \in A_G(Bd)$ by Definition 2.2, so $Bd \in A^-_G(K)$. For any instance Pd of an undefined Px, if $Pd \notin K$, i.e., $D^+ \not\models Pd$, then $\overline{P}d \in K$ so $Pd \in A^-_G(K)$. If $\overline{P}d \notin K$, then $D^+ \models Pd$, so $Pd \in K$ and $\overline{P}d \in A^-_G(K)$. Hence $V_G \setminus K \subseteq A^-_G(K)$.

If $Bd \in A^-_G(K)$, with Bd instantiating LS Bx of some axiom F, then $B_i dc \in K$ for some $B_i dc \in A_G(Bd)$, that is, $D^+ \models B_i dc$ and $D^+ \models \neg Bd$, since $D^+ \models F$. Hence $Bd \notin K$. For Pd instantiating undefined Px, if $Pd \in A^-_G(K)$, then $\overline{P}d \in K$, i.e., $D^+ \not\models Pd$, so $Pd \notin K$. If $\overline{P}d \in A^-_G(K)$, then $Pd \in K$, i.e., $D^+ \models Pd$ so $\overline{P}d \notin K$. Hence $A^-_G(K) \subseteq V_G \setminus K$. The two inclusions give $A^-_G(K) = V_G \setminus K$, so $K \in Ker(G)$.

Since for each atom Bd, $D^+ \models Bd \Leftrightarrow K \models Bd$, this equivalence extends to arbitrary formula ϕ. Obviously, if $D^+ \neq E^+$ for two models of Γ, then $kr(D^+) \neq kr(E^+)$.

An injection $md : Ker(G) \to Mod_D(\Gamma)$ is obtained as follows. Given a $K \in Ker(G)$, we define $md(K) = M$ over D by $M \models Bd$ iff $Bd \in K$, for each $Bd \in \mathrm{AT}(D)$. We verify that M satisfies each $F \in \Gamma$, having the form (2.1). If $M \models Bd$ then $Bd \in K$ so that $B_i dc \notin K$ for all B_i in the RS of B's equivalence (2.1), and all $c \in D^{ar(B_i d)}$. Hence $M \not\models B_i dc$, i.e., $M \models \neg B_i dc$ for all such B_i and c. Thus M satisfies the implication from left to right of B's equivalence (2.1). If $M \not\models Bd$ then $Bd \notin K$ and $Bd \in A^-_G(K)$, since K is a kernel of G. This means that $A_G(Bd) \cap K \neq \emptyset$, that is, $B_i dc \in K$ for some $B_i xy$ in the RS of B's equivalence (2.1) and some $c \in D^{ar(B_i d)}$. Thus M satisfies also the right to left implication of B's (2.1), and we conclude that $M \models \Gamma$.

Since for each atom Bd (also undefined by Γ), $md(K) \models Bd \Leftrightarrow K \models Bd$, this equivalence extends to all formulas. Obviously, if two kernels K_1, K_2 of G are different, then so are $md(K_1)$ and $md(K_2)$, giving different values to at least one atom Bd. Thus md is injective. $\square$

One sees easily that kr and md are inverses of each other.

As sinks belong to every kernel of a graph, Definition 2.2.3 adds 2-cycle $Pd \leftrightarrows \overline{P}d$ at all instances of undefined Px, to admit both boolean assignments to such Pd. The subgraph $A'z \to Az$..., for each axiom's top predicate Az, forces kernels to include all instances of Az.

Example 2.6 *A model over $\{c, d\}$ of the formula $\forall x \exists y Pxy$ must satisfy either Pcc or Pcd, and either Pdd or Pdc. These determine exactly the kernels of $\mathcal{G}_{\{c,d\}}(\Gamma)$ from Example 2.3. For every kernel $K \cap \{A'c, A'd\} = \emptyset$, hence $\{Ac, Ad\} \subseteq K$, forcing $\{Sc, Sd\} \cap K = \emptyset$. $Sc \notin K$ requires $\{Pcc, Pcd\} \cap K \neq \emptyset$, while $Sd \notin K$ requires $\{Pdc, Pdd\} \cap K \neq \emptyset$.*

Fact 2.5, augmented by the extension from $GNF^-(\Gamma)$ to $GNF(\Gamma)$, Definition 1.4, and Fact 1.7, yield the following correspondence between models of any FOL$^-$ theory Γ and kernels of the graphs from $Gr(\Gamma)$. More precisely, a graph model consists of a pair (G, K) with $G \in Gr(\Gamma)$ and $K \in Ker(G)$, and the class $Mod(\Gamma)$ corresponds to $GMod(\Gamma) = \bigcup\{\{G\} \times Ker(G) \mid G \in Gr(\Gamma)\}$.

Fact 2.7 *In FOL$^-$:*

1. *For an arbitrary GNF theory $\Gamma : Mod(\Gamma) \leftrightarrows GMod(\Gamma)$.*
2. *For an arbitrary theory $T : Mod(T) \leftrightarrows GMod(GNF(T))$.*

3. For an arbitrary GNF theory $\Gamma : Mod(\Gamma) \neq \emptyset \Leftrightarrow Ker(\mathcal{G}_\omega(\Gamma)) \neq \emptyset$.

PROOF. 1. By Fact 2.5, we have injections $Mod_D(\Gamma) \leftrightarrows Ker(\mathcal{G}_D(\Gamma))$ for each set D. This gives the obvious injections $Mod(\Gamma) = \bigcup_{D\in Set} Mod_D(\Gamma) \leftrightarrows \bigcup_{D\in Set} \left(\{\mathcal{G}_D(\Gamma)\} \times Ker(\mathcal{G}_D(\Gamma))\right) = GMod(\Gamma)$.
2. By Fact 1.7, $Mod(T) \leftrightarrows Mod(GNF(T))$, so the claim follows by point 1.
3. A Γ with an uncountable model has a countable one, by Skolem-Löwenheim, so $Ker(\mathcal{G}_\omega(\Gamma)) \neq \emptyset$ by Fact 2.5. If Γ has a finite model, it also has an infinite one by a standard argument for FOL$^-$. Conversely, if $Ker(\mathcal{G}_\omega(\Gamma)) \neq \emptyset$, then Fact 2.5 gives a countable model of Γ. $\square$

2.2 Some facts about kernels

We gather some facts about kernels to be used later on. Since existence of a kernel for some graph in $Gr(\Gamma)$ is equivalent to consistency of Γ, we start by quoting a couple of results on kernel existence. A graph is *kernel perfect* if each induced subgraph has a kernel. Often, establishing solvability, one shows actually kernel perfectness and we, too, will use this stronger notion. The central result in kernel theory is the following theorem of Richardson.

Theorem 2.8 ([3]) *A graph G without odd cycles is kernel perfect if (a) for each $x \in V_G : A_G(x)$ is finite or (b) there are no rays (infinite, simple, outgoing paths).*

In particular, a finite graph without odd cycles is kernel perfect. We will also encounter the following notion and fact. A digraph G is *bipartite* if so is its underlying undirected graph (forgetting directions of edges), that is, if V_G can be partitioned into two independent subsets, so that each A_G-edge connects a vertex in one subset to a vertex in the other.

Fact 2.9 ([6]) *A bipartite graph is kernel perfect.*

Kernels can be transferred from homomorphic images to preimages. A *graph homomorphism* from G to H is a function $h : V_G \to V_H$ such that

$$\forall x \in V_G : h(A_G(x)) = A_H(h(x)), \tag{2.10}$$

where h is extended pointwise to subsets, i.e., for $X \subseteq V_G : h(X) = \{h(x) \mid x \in X\}$.

Fact 2.11 *For a homomorphism $h : G \to H$, if $K \in Ker(H)$ then $K^- \in Ker(G)$, where $K^- = h^-(K) = \{y \in V_G \mid h(y) \in K\}$.*

PROOF. When K is independent then so is K^-, because $h(A_G(x)) \subseteq A_H(h(x))$ by (2.10).
For any $y \in V_G \setminus K^-$, i.e., $h(y) \notin K$, there is an $h \in A_H(h(y)) \cap K$, since $V_H \setminus K \subseteq A_H^-(K)$. By (2.10) $h(A_G(y)) \supseteq A_H(h(y))$, so $h \in h(A_G(y)) \cap K$, hence for some $g \in A_G(y) : h(g) = h$. Since $h \in K$, so $g \in K^-$ and $y \in A_G^-(K^-)$. Thus $V_G \setminus K^- \subseteq A_G^-(K^-)$ and $K^- \in Ker(G)$. $\square$

An isomorphism of graphs G and H, denoted by $G \simeq H$, is a bijective homomorphism either way.

Fact 2.12 *For every GNF theory Γ in FOL$^-$ and two sets D, E, if there is a surjection (bijection) $D \to E$, then there is a surjective (bijective) homomorphism $\mathcal{G}_D(\Gamma) \to \mathcal{G}_E(\Gamma)$.*

PROOF. A surjection (bijection) $\beta' : D \to E$ gives a surjection (bijection) on atoms $\beta : \mathrm{AT}(D) \to \mathrm{AT}(E)$ by $\beta(B(d)) = B(\beta'(d))$. We verify that it is a homomorphism. For $X \in \{D, E\}$, replace indices $\mathcal{G}_X(\Gamma)$ by X, e.g., $V_X = V_{\mathcal{G}_X(\Gamma)}$, etc. For each $B(d) \in V_D$
$A_E(\beta(B(d))) = A_E(B(\beta'(d))) = \{B_i(\beta'(d), e) \mid 1 \leq i \leq n, e \in E^{k_i}\}$, for $k_i = ar(B_i(\beta'(d)))$
On the other hand, $A_D(B(d)) = \{B_i(d, c) \mid 1 \leq i \leq n, c \in D^{k_i}\}$, so that
$\beta(A_D(B(d))) = \{\beta(B_i(d, c)) \mid 1 \leq i \leq n, c \in D^{k_i}\} = \{B_i(\beta'(d), \beta'(c)) \mid 1 \leq i \leq n, c \in D^{k_i}\}$.
Since $\beta' : D \to E$ is surjective (bijective), it gives a surjection (bijection) $D^{k_i} \to E^{k_i}$ for every $k_i \in \omega$, so that $\beta(D^{k_i}) = E^{k_i}$ and $\beta(A_D(B(d))) = A_E(\beta(B(d)))$. $\square$

Hence, a bijection $D \leftrightarrows E$ gives an isomorphism $\mathcal{G}_D(\Gamma) \simeq \mathcal{G}_E(\Gamma)$. We can therefore identify graphs in $Gr(\Gamma)$ by their domain's cardinality and set $Gr(\Gamma) = \{\mathcal{G}_\kappa(\Gamma) \mid \kappa > 0\}$, excluding the empty graph $\mathcal{G}_0(\Gamma)$. The graph in Example 2.3 is thus $\mathcal{G}_2(A)$. By Fact 2.5, $\mathcal{G}_\kappa(\Gamma)$ captures up to isomorphism all models of Γ over domains with cardinality κ. Incidentally, a simple version of upward Skolem-Löwenheim theorem follows: if Γ in FOL$^-$ has a model of cardinality κ, then for any $\lambda > \kappa$, surjection $\lambda \to \kappa$ gives by Fact 2.12 a homomorphism $\mathcal{G}_\lambda(\Gamma) \to \mathcal{G}_\kappa(\Gamma)$, which reflects kernels of the latter by Fact 2.11, yielding a model over λ by Fact 2.5.

Homomorphisms reflect also bipartitions, according to the (proof of the) following fact.

Fact 2.13 *If $h : G \to H$ is a homomorphism and H is bipartite, then so is G.*

Proof. We show that if $\langle A_1, A_2\rangle$ is a bipartition of H, then $\langle A_1^-, A_2^-\rangle$ is a bipartition of G, where $A_i^- = \{x \in V_G \mid h(x) \in A_i\}$. Let $\{i, j\} = \{1, 2\}$. If $h(x) = y \in A_i$ then $A_H(y) \cap A_i = \emptyset$, so there is no $z \in A_G(x) \cap A_i^-$, since otherwise $h(z) \in h(A_G(x) \cap A_i^-) \subseteq h(A_G(x)) \cap h(A_i^-) = A_H(y) \cap A_i$. So $A_G(A_i^-) \subseteq A_j^-$, and dually, $A_G(A_j^-) \subseteq A_i^-$, i.e., $\langle A_i^-, A_j^-\rangle$ is a bipartition of G. □

Finally, we sometimes start with a partial assignment of boolean values (select a part of a kernel) and propagate its consequences, that is, induce values to some other vertices. Briefly, a vertex must be assigned **0** if it has an edge to a vertex assigned **1**, while if all outneighbours are assigned **0**, the vertex itself must be **1**. More formally, given a partial assignment σ with domain $X \subseteq V_G$, we start with $V_0 = V_G$, set

$\sigma_0^{\mathbf{1}} = \{x \in X \mid \sigma(x) = \mathbf{1}\} \cup (sinks(G) \setminus X)$ and $\sigma_0^{\mathbf{0}} = \{x \in X \mid \sigma(x) = \mathbf{0}\} \cup (A_G^-(\sigma_0^{\mathbf{1}}) \setminus X)$,

and iterate the following:

$V_{i+1} = V_i \setminus (\sigma_i^{\mathbf{1}} \cup \sigma_i^{\mathbf{0}})$, and in limits $\lambda : V_\lambda = \bigcap_{i<\lambda} V_i$,

G_j is the subgraph of G induced by V_j

$\sigma_{i+1}^{\mathbf{1}} = sinks(G_{i+1})$ and $\sigma_{i+1}^{\mathbf{0}} = A_G^-(\sigma_{i+1}^{\mathbf{1}}) \cap V_{i+1}$.

We stop when $V_{\kappa+1} = V_\kappa$, obtaining the induced $(\bigcup_{i\leq\kappa} \sigma_i^{\mathbf{0}} \times \{\mathbf{0}\}) \cup (\bigcup_{i\leq\kappa} \sigma_i^{\mathbf{1}} \times \{\mathbf{1}\})$. Depending on σ, this may not be a function, but we induce only when it is. If needed, more details can be found in [6].

3 Extensions

Transforming a given theory into GNF may be a cumbersome task worthwhile only in special cases. However, GNF is often encountered directly. Typically, extensions introduce definitions of new predicates in GNF like, for instance, definitional extensions or fixed point definitions. A crucial property of extensions is often conservativity or even (unique) expandability of every model of the theory under extesion and to ensure it, various syntactic restrictions are utilized. Graph representation gives a new perspective on such situations, in particular, when circularity is involved.

We therefore view now extensions as the primary objects. An extension is simply a GNF theory Δ, with undefined predicates (if any) marking connections to any actually extended theory Γ. This is how extensions are often used: as a generic possibility of augmenting a wide range of theories. Just like transitive closure can be applied to any binary relation, we can think of an extension as potentially applicable to any theory possessing predicates with the arities of the undefined predicates of the extension.

Given Δ as an independent object, its undefined predicates may need renaming to match the appropriate predicates of Γ, avoiding such an identification of predicates defined in Δ. An application of Δ to Γ amounts to ensuring this, which is straightforward, so we assume the naming details are always resolved and write the result of such an extension as $\Delta(\Gamma)$. Assuming such a renaming, $\Delta(\Gamma)$ can be thought of as $\Delta \cup \Gamma$. For every (set of) cardinality κ, the graph equality $\mathcal{G}_\kappa(\Delta(\Gamma)) = \mathcal{G}_\kappa(\Delta) \cup \mathcal{G}_\kappa(\Gamma)$ holds because of this assumption, so that a sink of $\mathcal{G}_\kappa(\Delta)$, say Pd, is identified with the vertex Pd of $\mathcal{G}_\kappa(\Gamma)$. Whenever the choice of κ is inessential, we speak about the graph $\mathcal{G}(\Delta)$ of the extension and the graph $\mathcal{G}(\Gamma)$ of the extended theory.

A significant consequence of the above restrictions is that while sinks of $\mathcal{G}(\Delta)$ end up among vertices of $\mathcal{G}(\Gamma)$, there are no edges from $\mathcal{G}(\Gamma)$ to $\mathcal{G}(\Delta)$ – the extended theory does not refer to any predicates defined by the extension. In principle, $\mathcal{G}(\Delta)$ may have more sinks, with only some intended as vertices of the extended theory. But since sinks must belong to every kernel, they affect $\mathcal{G}(\Delta)$ in a unique way. Assuming their effects (by inducing) taken into account, we can simplify the presentation by postulating that all sinks of the extension's graph are mapped onto some vertices of the graph of the actually extended theory Γ.

Definitional extensions are conservative, but we distinguish also two other intermediary notions. Given languages $\mathcal{L} \subseteq \mathcal{L}^+$, an $\mathcal{L}^+$ structure M^+ is an *expansion of* $\mathcal{L}$-structure M, if the $\mathcal{L}$-reduct of M^+ equals M, i.e., $M^+|_{\mathcal{L}} = M$. Recall also that an *explicit definition*, of a predicate $B \notin \mathcal{L}$ in a language $\mathcal{L}$, is $Bx \leftrightarrow \phi$, where ϕ is an $\mathcal{L}$-formula with free variables $\mathcal{V}(\phi) \subseteq x$.

Definition 3.1 *Let $\Gamma \subseteq \Gamma^+$ be theories over languages $\mathcal{L} \subseteq \mathcal{L}^+$. $\Delta = \Gamma^+ \setminus \Gamma$ is a*

- definitional extension *of Γ if it is a well-founded chain of explicit definitions, starting with Γ,*
- model unique extension *of Γ if each model of Γ has a unique expansion to a model of Γ^+,*
- model extension *of Γ if each model of Γ has an expansion to a model of Γ^+,*
- conservative extension *of Γ if for every $\mathcal{L}$-sentence $\phi : \Gamma \vdash \phi \Leftrightarrow \Gamma^+ \vdash \phi$.*

The notions are listed with decreasing strength: every definitional extension is a model unique extension, which is a model extension, and every model extension is conservative. In general, none of these inclusions can be reversed.

We begin with a few simple examples. There is no requirement that predicate(s) introduced in an extension be true or satisfiable so, when not formulated in GNF, we convert extension to GNF^- and not to the full GNF.

Example 3.2 *Definitional extension Δ given by $Dx \leftrightarrow Fx \vee \neg Hx$ has $GNF^-(\Delta)$ shown to the left and graph $\mathcal{G}(\Delta)$ to the right:*

$$\Delta: \quad \begin{aligned} &Dx \leftrightarrow \neg Bx \\ &Bx \leftrightarrow \neg\overline{H}x \wedge \neg Fx \\ &\overline{H}x \leftrightarrow \neg Hx \end{aligned}$$

$$\begin{tikzcd} Dx \arrow[r] & Bx \arrow[r] \arrow[d] & Fx \\ & \overline{H}x \arrow[r] & Hx \end{tikzcd}$$

An application of Δ to any actual theory Γ amounts to matching F and H to some unary predicates of Γ. For any κ, the graph $\mathcal{G}_\kappa(\Delta)$ has a copy of the above $\mathcal{G}(\Delta)$ for each $x \in \kappa$. The sinks Fx and Hx of $\mathcal{G}_\kappa(\Delta)$ obtain in $\mathcal{G}_\kappa(\Delta(\Gamma))$ the edges which Fx, Hx have in $\mathcal{G}_\kappa(\Gamma)$.

Example 3.3 *The extension $Nx \leftrightarrow Fx \vee (Hx \wedge \neg Nx)$, that is, $Nx \leftrightarrow \neg(\neg Fx \wedge \neg(Hx \wedge \neg Nx))$, gives $GNF^-(N)$ to the left and graph $\mathcal{G}(N)$ to the right:*

$$(N) \quad \begin{aligned} &Nx \leftrightarrow \neg Bx \\ &Bx \leftrightarrow \neg Fx \wedge \neg Cx \\ &Cx \leftrightarrow \neg\overline{H}x \wedge \neg Nx \\ &\overline{H}x \leftrightarrow \neg Hx \end{aligned}$$

$$\begin{tikzcd} Nx \arrow[r] & Bx \arrow[r] \arrow[dl] & Fx \\ Cx \arrow[u] \arrow[r] & \overline{H}x \arrow[r] & Hx \end{tikzcd}$$

The graph has no solution for $Fx = \mathbf{0}, Hx = \mathbf{1}$, so (N) does not model extend any Γ consistent with $\exists x(\neg Fx \wedge Hx)$. From (N) we can actually prove $Fx \vee \neg Hx$, so this extension is not even conservative for any such Γ.[3]

Example 3.4 *Let (N') be as (N) in Example 3.3, but with Bx replaced by the following $B'x$:*

$$(N') \qquad B'x \leftrightarrow \neg Fx \wedge \neg Cx \wedge \neg Nx$$

$$\begin{tikzcd} Nx \arrow[r, shift left] & B'x \arrow[l, shift left] \arrow[r] \arrow[dl] & Fx \\ Cx \arrow[u] \arrow[r] & \overline{H}x \arrow[r] & Hx \end{tikzcd} .$$

Since $(N) \models Bx \leftrightarrow \neg Nx$, this might seem insignificant but the change is dramatic. The new edge from $B'x$ to Nx makes the induced subgraph $\{B'x, Cx, Nx\}$ kernel perfect. Consequently, the graph is solvable for every assignment to its sinks, so (N') is a model extension of every theory.

The following more complex example illustrates also the effects of quantifiers.

[3] In chapter 4.I of [2], viewing (N) as a 'definition', the authors remark that it "allows us to prove *a priori* that all H's are F's.". Well, all H's are F's *assuming* (N), but while a definition may seem *a priori*, there is nothing *a priori* about an extension with nonlogical axioms. Since every theory can be written in the GNF format, typical if not required for definitions, definitions might perhaps be viewed as special kinds of axioms.

3.1 Transitive closure

Given a binary relation E, a natural attempt to define its transitive closure is by adding the axiom

(TC) $TCxy \leftrightarrow Exy \vee \exists z : Exz \wedge TCzy$.

We ask first about the general relation between possible models of E and models of E extended with (TC). In GNF, the definition of TC becomes the four equivalences in (3.5). The dotted edges, marked with $\forall z$ on the sketch of $\mathcal{G}(TC)$ in Figure 3.6, signal branching to all instances of the target formula, with some marked explicitly on the dotted edges leading to them. The edges $\overline{E}xy \to Exy$ to the sinks Exy of $\mathcal{G}(TC)$ are left implicit.

$$\begin{aligned} TCxy &\leftrightarrow \neg Bxy \\ Bxy &\leftrightarrow \neg Exy \wedge \textstyle\bigwedge_z \neg Cxzy \end{aligned} \qquad \begin{aligned} Cxzy &\leftrightarrow \neg \overline{E}xz \wedge \neg \overline{TC}zy \\ \overline{E}xy &\leftrightarrow \neg Exy \;\text{ and }\; \overline{TC}zy \leftrightarrow \neg TCzy \end{aligned} \tag{3.5}$$

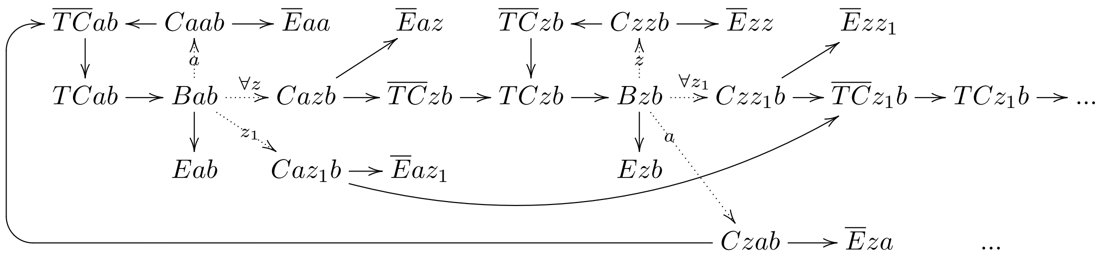


Figure 3.6: (TC) extension and (part of) graph $\mathcal{G}(TC)$.

i. First, if $Eaa = \mathbf{1}$, i.e., $\overline{E}aa = \mathbf{0}$, then we can obtain a model with $TCab = \mathbf{1}$ by simply choosing $\overline{TC}ab = \mathbf{0}$, which yields $Caab = \mathbf{1}$ and hence $Bab = \mathbf{0}$, independently of the choice of assignments further down the graph. This corresponds to the fact that, when $Eaa = \mathbf{1}$, the instance $TCab \leftrightarrow Eab \vee (Eaa \wedge TCab)$ of (TC) with $z = a$, becomes trivially satisfiable by merely choosing $TCab = \mathbf{1}$.

ii. Considering irreflexive E, (TC) still does not capture transitive closure, though this is less obvious. The case $\mathcal{G}_\omega(TC)$ of $\mathcal{G}(TC)$ has namely a kernel including $TCab$ when there is an infinite chain $R = \{z_0, z_1, z_2, ...\}$ with $z_0 = a$, and $Ez_ib = \mathbf{0} = \overline{E}z_iz_{i+1}$ for each $z_i \in R$. The last condition means $Ez_iz_{i+1} = \mathbf{1}$, i.e., R is a ray (or enters a cycle), which gives a very specific and unintended meaning to any b being E-reachable from a, as if an infinite walk reached every vertex (then TCz_ib for each $z_i \in R$). Still, in this situation there is also another model in which $TCab = \mathbf{0}$.

The main aim of this example is not to reexamine undefinability of transitive closure in FOL but to note that the graph above has a kernel for every assignment to the E-vertices: for an arbitrary theory of E, (TC) is its model extension. This follows from the observation that $\mathcal{G}(TC)$ is bipartite. We justify it by the following general argument.

For a GNF theory Γ, a simpler schematic graph $\mathcal{S}_\Gamma$ conveys often much information. Its vertices are (labeled by) the predicate symbols alone, ignoring the arguments. Each equivalence gives edges from the predicate in its LS, to each predicate occurring in its RS. Each of the graphs in Examples 3.2, 3.3 and 3.4 is isomorphic to such a schematic graph of its theory. For the extension (TC) from (3.5), the schematic graph $\mathcal{S}_{TC}$ is

$$\begin{tikzcd} TC \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \overline{TC}. \arrow[lll, bend right] \\ \mathcal{S}_{TC} & E & \overline{E} \arrow[l] & \end{tikzcd} \tag{3.7}$$

Every path in the graph $\mathcal{G}_\kappa(\Gamma)$, for any κ, results from unfolding some path in such a schematic graph $\mathcal{S}_\Gamma$. The latter is also the homomorphic image of the former under the *canonical* homomorphism, identifying all vertices with the same predicate symbol.

The graph $\mathcal{S}_{TC}$ is trivially bipartite (since so is its underlying undirected graph, having no odd undirected cycles). Using the just described canonical homomorphism and Fact 2.13, we conclude that $\mathcal{G}(TC)$ is bipartite and, by Fact 2.9, kernel perfect. This holds generally. Whenever the schematic graph $\mathcal{S}_\Gamma$ is bipartite, then each graph $\mathcal{G}_\kappa(\Gamma)$ is kernel perfect.

The simplified graph $\mathcal{S}_{TC}$ allows thus to conclude that every $\mathcal{G}(TC)$ is solvable for every assignment to its sinks Exy. Consequently, (TC) is a model extension of any theory of E. This triviality about (TC) becomes a useful fact, when formulated generally.

Fact 3.8 *An extension Δ is a model (unique) extension if for every cardinality κ, the graph $\mathcal{G}_\kappa(\Delta)$ is (uniquely) solvable for every assignment to its sinks.*

This follows because, given a theory Γ (extendable by Δ), any kernel M of $\mathcal{G}_\kappa(\Gamma)$ determines values of $sinks(\mathcal{G}_\kappa(\Delta))$; since no edges go from $\mathcal{G}_\kappa(\Gamma)$ to $\mathcal{G}_\kappa(\Delta)$, while the latter is solvable for every assignment to its sinks, the kernel M of $\mathcal{G}_\kappa(\Gamma)$ can be extended to a kernel of $\mathcal{G}_\kappa(\Delta(\Gamma))$.

In Example 3.4, (N') is a model extension. As observed there, this follows because its graph is kernel perfect so, in particular, has a solution for every assignment to its sinks.

In Example 3.2, (D) is a definitional extension, while its graph $\mathcal{G}(D)$ is a dag without any rays. By the first result in kernel theory from [4], such a graph has a unique kernel and, in fact, is kernel perfect. In our graph terms, the effect of the syntactic restrictions on a definitional extension is that its graph becomes a rayless dag. The model uniqueness, implied by definitional extension, follows then because values at all vertices of a rayless dag are determined by those assigned to its sinks. It is a special case of Fact 3.8 which implies, more generally, that Δ is a model extension whenever $\mathcal{G}(\Delta)$ is kernel perfect. This general statement is used below.

3.2 Fixed points and positive occurrences

Restrictions on fixed point definitions provide another example of syntactic means ensuring condition of Fact 3.8. One defines a predicate by (*) $Bx \leftrightarrow \phi x$, with B occurring in the formula ϕ (with $\mathcal{V}(B) = x = \mathcal{V}(\phi)$). If M is a structure interpreting the symbols from ϕ, let (M, X) denote its expansion with the interpretation of B as $X \subseteq M$. A model of (*) over a given M is then a fixed point of the operator defined by $\mathbb{B}^M(X) = \{m \in M \mid (M, X) \models \phi m\}$.[4] Often, one chooses only least or greatest fixed points, but we address only the consistency conditions, that is, the mere existence of fixed points.

A simple restriction, ensuring monotonicity of $\mathbb{B}$ and existence of fixed points, forbids negative occurrences of B in ϕ. In terms of GNF, this amounts to forbidding any occurrence of B under an odd number of negations, when replacing predicates in the RS of B's equivalence, by the RSs of their equivalences. Such a substitution, performed in Example 1.2 and below, provides a procedure for identifying negative occurrences in GNF.

Example 3.9 *Recall definitions from Example 3.3:*

$$\begin{array}{ll} (N) & Nx \leftrightarrow \neg Bx \\ & Bx \leftrightarrow \neg Fx \wedge \neg Cx \\ & Cx \leftrightarrow \neg \overline{H}x \wedge \neg Nx \\ & \overline{H}x \leftrightarrow \neg Hx \end{array}$$

$$\begin{tikzcd} Nx \arrow[r] & Bx \arrow[r] \arrow[dl] & Fx \\ Cx \arrow[u] \arrow[r] & \overline{H}x \arrow[r] & Hx \end{tikzcd}$$

Each equivalence below marks one step of the successive substitution:

$Nx \leftrightarrow \neg Bx \leftrightarrow \neg(\neg Fx \wedge \neg Cx) \leftrightarrow \neg(\neg Fx \wedge \neg(\neg \overline{H}x \wedge \neg Nx))$

In the last formula, N occurs under 3 negations, displying thus its negative occurrence.

Each $\neg$ in GNF amounts to an edge in the corresponding graph, so a negative occurrence of Nx, in the RS of (some such substitution instance of) the equivalence for Nx, amounts to an odd cycle in the corresponding graph. Each odd cycle signals negative occurrence of all predicates in its vertices. In (N) above, N, B and C all have such occurrences. Forbidding negative occurrences amounts to excluding odd cycles.

Strictly speaking, what must not occur negatively is the same atom, say Td, for some d in the domain, and not merely the predicate symbol T. Negative occurrences of T in $T1 \leftrightarrow \neg T2$ or

[4] GNF, conforming to the format of (*), illustrates also that for each FOL theory, over a language with n predicates, models over a set M are fixed points of such an operator over subsets of M^n.

$Tx \leftrightarrow \neg Tsx$ may appear circular, but they do not create any cycles in the graph, as long as $1 \neq 2$ and $x \neq sx$. An (odd) cycle emerges only from a (negative) occurrence in RS of an atom, like Td, that is an instance of Tx in LS, from which Td is reached in the substitution process above.

The traditional restriction ensures the economy of applications, forbidding negative occurrences of the mere predicate symbol. This excludes odd cycles from the theory's simplified graph $\mathcal{S}_\Gamma$, enabling an argument similar to that lifting bipartition of $\mathcal{S}_{TC}$ in (2.11) to any graph $\mathcal{G}(TC)$. Having no odd cycles, the graph $\mathcal{S}_\Gamma$ is kernel perfect if it is finite, by Theorem 2.8. Since it is a homomorphic image of each $\mathcal{G}(\Gamma)$, every such graph is solvable by Fact 2.11. Consequently, a definition without negative occurrences of the defined predicate, and with a finite simplified graph, has fixed points over interpretation domain of each cardinality. Usefulness of such syntactic criteria is easily associated with their limitations. Kernel perfectness or, more generally, the condition of Fact 3.8, provides means for establishing existence of fixed points also in cases with negative occurrences, as illustrated by the following example.

Example 3.10 *The extension (A) to the left has the simplified graph $\mathcal{S}_{(A)}$ to the right*

$$(A) \quad \begin{array}{l} Ax \leftrightarrow \neg Bx \\ Bx \leftrightarrow \neg Cx \wedge \neg Hx \\ Cx \leftrightarrow \neg Ax \wedge \neg Bx \end{array}$$

$$\begin{tikzcd} A \arrow[rr, bend left] & C \arrow[l] \arrow[r, shift left] & B \arrow[l, shift left] \arrow[r] & H \end{tikzcd}$$

Substituting as described above, yields $Ax \leftrightarrow \neg(\neg(\neg Ax \wedge \neg Bx) \wedge \neg Hx)$, so the predicate A, and even the same atom Ax, occurs negatively, which is reflected by the odd cycle in the graph. All predicates from the odd cycle have negative occurrences but notwithstanding this, the graph is kernel perfect. Thus fixed points of (A) exist and, moreover, (A) is a model unique extension: $Hx = \mathbf{1}$ forces $Bx = \mathbf{0} = Cx$ and $Ax = \mathbf{1}$, while $Hx = \mathbf{0}$ forces $Bx = \mathbf{1}$ and $Ax = \mathbf{0} = Cx$.

3.3 Universal extensions

The examples suggest a generalization of Definition 3.1.

Definition 3.11 *An extension Δ is* universal *model unique/model/conservative if it is such for every theory Γ.*

Fact 3.8, which does not mention any extended theory Γ and actually states that Δ is a universal model (unique) extension, can be strengthened to this generalized context.

Fact 3.12 *An extension Δ is a universal model (unique) extension iff, for every cardinality κ, the graph $\mathcal{G}_\kappa(\Delta)$ is (uniquely) solvable for every assignment to its sinks.*

The if direction follows as in Fact 3.8, while for the opposite, any assignment to the sinks Γ of $\mathcal{G}_\kappa(\Delta)$, taken as the extended theory, which makes $\mathcal{G}_\kappa(\Delta)$ unsolvable, provides also a model of the theory Γ having no expansion to a model of $\Delta(\Gamma)$. Also, any assignment to the sinks Γ for which $\mathcal{G}_\kappa(\Delta)$ has two kernels, provides a model for Γ with two different expansions.

The extension (TC) is a universal model extension, because its graph $\mathcal{G}_\kappa(TC)$ is kernel perfect for every κ, having a homomorphisms onto the bipartite $\mathcal{S}_{TC}$ in (3.7). Hence, every model of E can be extended to a model of $E \cup (TC)$. It is not, however, a universal model unique extension, as observed in point **i** under Figure 3.6.

A universal model unique extension occurs, as for instance definitional extension, when its graph is a rayless dags. Every assignment to sinks induces then a unique valuation of all vertices. Such model unique extensions occur also in many other situations.

Example 3.13 *The extension Θ to the left has the simplified graph $\mathcal{S}_\Theta$ to the right (for every κ, the graph $\mathcal{G}_\kappa(\Theta)$ consists of κ copies of this graph):*

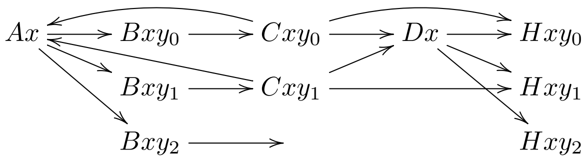
$$\Theta: \quad \begin{array}{l} Ax \leftrightarrow \neg Bx \\ Bx \leftrightarrow \neg Cx \\ Cx \leftrightarrow \neg Ax \wedge \neg Dx \wedge \neg Hx \\ Dx \leftrightarrow \neg Hx \end{array}$$

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \arrow[ll, bend right] \arrow[r] \arrow[rr, bend left] & D \arrow[r] & H \end{tikzcd}$$

The graph is not kernel perfect, as witnessed by the odd cycle $\{A, B, C\}$, showing also negative *occurrences of its atoms. Still, $\mathcal{S}_\Theta$ is uniquely solvable for every assignment to its sink H: $H = \mathbf{1}$ gives $C = D = A = \mathbf{0}$ and $B = \mathbf{1}$, while $H = \mathbf{0}$ yields $D = \mathbf{1} = B$ and $C = A = \mathbf{0}$. In spite of the negative occurrences, Θ is a universal model unique extension.*

Example 3.14 *Below, in a more complicated version Δ of Θ from Example 3.13, some predicates have different arities:*

$$\Delta: \quad \begin{aligned} &Ax \leftrightarrow \textstyle\bigwedge_y \neg Bxy \\ &Bxy \leftrightarrow \neg Cxy \\ &Cxy \leftrightarrow \neg Ax \wedge \neg Dx \wedge \neg Hxy \\ &Dx \leftrightarrow \textstyle\bigwedge_y \neg Hxy \end{aligned}$$



The graph $\mathcal{G}_\kappa(\Delta)$ has κ copies of the above graph, one for each $x \in \kappa$, and in each such copy, Ax (Dx) has edges to κ vertices Bxy_i (Hxy_i), for each $y_i \in \kappa$. Each vertex Bxy_i starts a copy of the subgraph following Bxy_0 (and Bxy_1), with an edge from each Cxy_i to the same Ax and Dx.

Now, $\mathcal{S}_\Delta = \mathcal{S}_\Theta$. The canonical homomorphism from any $\mathcal{G}_\kappa(\Delta)$ onto $\mathcal{S}_\Delta$ reflects kernels of $\mathcal{S}_\Delta$ by Fact 2.11. So $\mathcal{G}_\kappa(\Delta)$ is solvable whenever, for each $x \in \kappa$, all $Hxy_i = \mathbf{0}$ or all $Hxy_i = \mathbf{1}$. To conclude that Δ is a universal model extension, we have to consider also the case of $Hxy_i = \mathbf{1}$ only for some y_i. Then $Dx = Cxy_i = \mathbf{0}$, making $Bxy_i = \mathbf{1}$ and $Ax = \mathbf{0}$. All other Cxy_j, Bxy_j are then determined by the respective Hxy_j. Thus Δ is a universal model extension (in fact, unique), all negative occurrences and odd cycles notwithstanding.

4 Appendix

Section 4.1 details construction of $GNF^-(\phi)$ for a formula ϕ, while 4.2 extends the results to first-order logic with function symbols and equality.

4.1 GNF

Function $\langle \_ \rangle_\_$ in Definition 4.1 transforms quantifier prefix of a PNF formula into GNF, sending the quantifier-free matrix for further processing by function $pGNF^-$ from Definition 4.2. The numerical argument i in $\langle \_ \rangle_i$, counting the number of generated symbols, ensures that all introduced predicate symbols are distinct. All $\exists x$ in the initial formula are assumed replaced by $\neg\forall x \neg$. A block of quantifiers $\forall x_1 ... \forall x_n$ not separated by $\neg$ is abbreviated below as a single quantifier $\forall x$, with x abbreviating the sequence $x_1 ... x_n$. In the generated formulas, $\bigwedge_x$ can be expanded to $\bigwedge_{x_1} \cdots \bigwedge_{x_n}$.

The two main cases in Definition 4.1 correspond to ϕ starting with $\neg\forall$ (1 and 3) or $\forall$ (2 and 4), each having two subcases, when ϕ is open (o) or closed (c). One starts typically with case 1 or 2, then performs step 2o until reaching the last quantifier, to which equation 2o, 3o or 4o is applied. Point 5 sends the remaining quantifier-free matrix ρ to $pGNF^-\langle \rho \rangle_j$ given in Definition 4.2. In case of only one quantifier, there is only one step, which may be any of 1 through 4.

Definition 4.1 *For a formula $\phi(X)$ in PNF, with $X = \mathcal{V}(\phi)$, we define $GNF^-(\phi) = \langle \phi(X) \rangle_1$,* *with the recursive function* $\langle\_\rangle_{\_}$ *given by (z is a fresh free variable):*

$$
\begin{array}{llcll}
1o. & \langle\neg\forall x\neg\psi(X,x)\rangle_i & = & \{\ B_i(X)\leftrightarrow\neg B_{i+1}(X)\ \} & \cup\ \ \langle\forall x\neg\psi(X,x)\rangle_{i+1}\\
1c. & \langle\neg\forall x\neg\psi(x)\rangle_i & = & \{\ B_i(z)\leftrightarrow\neg B_{i+1}(z)\ \} & \cup\ \ \langle\forall\neg\psi(x)\rangle_{i+1}\\
2o. & \langle\forall x\neg\psi(X,x)\rangle_i & = & \{\ B_i(X)\leftrightarrow\bigwedge_x\neg B_{i+1}(X,x)\ \} & \cup\ \ \langle\psi(X,x)\rangle_{i+1}\\
2c. & \langle\forall x\neg\psi(x)\rangle_i & = & \{\ B_i(z)\leftrightarrow\bigwedge_x\neg B_{i+1}(x)\ \} & \cup\ \ \langle\psi(x)\rangle_{i+1}\\
\multicolumn{5}{l}{\textit{when quantifier-free } \rho \textit{ does not start with } \neg:}\\
3o. & \langle\neg\forall x\rho(X,x)\rangle_i & = & \{\ B_i(X)\leftrightarrow\neg B_{i+1}(X)\ \} & \cup\ \ \langle\forall x\rho(X,x)\rangle_{i+1}\\
3c. & \langle\neg\forall x\rho(x)\rangle_i & = & \{\ B_i(z)\leftrightarrow\neg B_{i+1}(z)\ \} & \cup\ \ \langle\forall x\rho(x)\rangle_{i+1}\\
4o. & \langle\forall x\rho(X,x)\rangle_i & = & \{\ B_i(X)\leftrightarrow\bigwedge_x\neg B_{i+1}(X,x), & \\
 & & & \ \ B_{i+1}(X,x)\leftrightarrow\neg B_{i+2}(X,x)\ \} & \cup\ \ \langle\rho(X,x)\rangle_{i+2}\\
4c. & \langle\forall x\rho(x)\rangle_i & = & \{\ B_i(z)\leftrightarrow\bigwedge_x\neg B_{i+1}(x), & \\
 & & & \ \ B_{i+1}(x)\leftrightarrow\neg B_{i+2}(x)\ \} & \cup\ \ \langle\rho(x)\rangle_{i+2}\\
\multicolumn{5}{l}{\textit{when } \rho \textit{ is quantifier-free, continue with Definition 4.2:}}\\
5. & \langle\rho\rangle_j & = & pGNF^-\langle\rho\rangle_j. &
\end{array}
$$

For instance:

$$
\begin{array}{lrcll}
i) & \langle\neg\forall x\neg\rho(x)\rangle_1 & \overset{1c}{=} & \{B_1(z)\leftrightarrow\neg B_2(z)\} & \cup\ \ \langle\forall x\neg\rho(x)\rangle_2\\
 & \langle\forall x\neg\rho(x)\rangle_2 & \overset{2c}{=} & \{B_2(z)\leftrightarrow\bigwedge_x\neg B_3(x)\} & \cup\ \ \langle\rho(x)\rangle_3...\\
ii) & \langle\forall x\neg\forall y\neg\rho(x,y)\rangle_1 & \overset{2c}{=} & \{B_1(z)\leftrightarrow\bigwedge_x\neg B_2(x)\} & \cup\ \ \langle\forall y\neg\rho(x,y)\rangle_2\\
 & \langle\forall y\neg\rho(x,y)\rangle_2 & \overset{2o}{=} & \{B_2(x)\leftrightarrow\bigwedge_y\neg B_3(x,y)\} & \cup\ \ \langle\rho(x,y)\rangle_3...\\
iii) & \langle\neg\forall x\neg\forall y\rho(x,y)\rangle_1 & \overset{1c}{=} & \{B_1(z)\leftrightarrow\neg B_2(z)\} & \cup\ \ \langle\forall x\neg\forall y\rho(x,y)\rangle_2\\
 & \langle\forall x\neg\forall y\rho(x,y)\rangle_2 & \overset{2c}{=} & \{B_2(z)\leftrightarrow\bigwedge_x\neg B_3(x)\} & \cup\ \ \langle\forall y\rho(x,y)\rangle_3\\
 & \langle\forall y\rho(x,y)\rangle_3 & \overset{4o}{=} & \{B_3(x)\leftrightarrow\bigwedge_y\neg B_4(x,y), & \\
 & & & \ \ B_4(x,y)\leftrightarrow\neg B_5(x,y)\} & \cup\ \ \langle\rho(x,y)\rangle_5...
\end{array}
$$

Definition 4.2 *Given a quantifier-free* $\rho(x)$ *and index* j*, to form* $pGNF_2^-\langle\rho\rangle_j$*:*

1. *write* $\rho(x)$ *in CNF as:* $(L_{1_1}\vee...\vee L_{1_{z1}})\wedge...\wedge(L_{n_1}\vee...\vee L_{n_{zn}})$ *– each* L_{i_k} *is a literal;*
2. *using DeMorgan, transform it to:* $\neg(L'_{1_1}\wedge...\wedge L'_{1_{z1}})\wedge...\wedge\neg(L'_{n_1}\wedge...\wedge L'_{n_{zn}})$*,*
   – where L' *removes* $\neg$ *from negative literals* L *and adds* $\neg$ *to the positive ones;*
3. *add the formula* $B_j(x)\leftrightarrow\neg B_{j+1}(x_1)\wedge...\wedge\neg B_{j+n}(x_n)$*, where* $x=x_1\cup...\cup x_n$ *and each* B_i*, for* $1\le i\le n$*, has the arity* x_i *of the* i*-th conjunct* $L_{i_1}\vee...\vee L_{i_{zi}}$*;*
4. *for each* $1\le i\le n$*, add* $B_{j+i}(x_i)\leftrightarrow\overline{L}_{i_1}\wedge...\wedge\overline{L}_{i_{zi}}$*, where for* $1\le j\le zi$ *and the predicate symbol* R *in* L'_{i_j}*, if* $L'_{i_j}=\neg R(x_{i_j})$ *then* $\overline{L}_{i_j}=L'_{i_j}=\neg R(x_{i_j})$*, while if* $L'_{i_j}=R(x_{i_j})$*, then* $\overline{L}_{i_j}=\neg\overline{R}(x_{i_j})$ *for a fresh symbol* $\overline{R}$ *with* $ar(\overline{R})=ar(R)$*, and additional* $\overline{R}(x_{i_j})\leftrightarrow\neg R(x_{i_j})$*; free variables* $x_i=x_{i_1}\cup...\cup x_{i_{zi}}$*.*

$pGNF^-\langle\rho\rangle_j$ contains formulas from points 3 and 4, and $pGNF^-\langle\rho\rangle_j\models\rho(x)\leftrightarrow B_jx$. For instance:

$pGNF^-\langle(Cxy\to Dx)\wedge Ey\rangle_0$:

1. CNF $=(\neg Cxy\vee Dx)\wedge(Ey)$
2. $\neg(Cxy\wedge\neg Dx)\wedge\neg(\neg Ey)$
3. $B_0xy\leftrightarrow\neg B_1xy\wedge\neg B_2y$
4. $B_1xy\leftrightarrow\neg\overline{C}xy\wedge\neg Dx$,
   $\overline{C}xy\leftrightarrow\neg Cxy$, and $B_2y\leftrightarrow\neg Ey$.

$pGNF^-\langle\neg Px\rangle_0$:

1. CNF $=\neg Px$
2. $\neg(Px)$, with $L'x=Px$
3. $B_0x\leftrightarrow\neg B_1x$
4. $B_1x\leftrightarrow\neg\overline{P}x$
   $\overline{P}x\leftrightarrow\neg Px$

Fact 4.3 (1.3) *For every theory* Γ *in FOL,* $GNF^-(\Gamma)$ *is a definitional extension of* Γ*.*

Proof. We view the process from Definitions 4.1 and 4.2 bottom-up, so that for $j>i$, B_j is introduced before B_i. Step i adds the explicit definition $\bigwedge_x B_i(x)\leftrightarrow\bigwedge_y\neg B_{i+1}(x,y)$ of the fresh predicate B_i in the language of the previous stage $i+1$ (containing, in addition to the original symbols, the predicate symbols B_j for $j>i$). Thus $GNF^-(A)$ is a definitional extension of A, for every $A \in \Gamma$. This yields the claim for finite Γ. If Γ is infinite, then we well-order Γ using axiom of choice. Since distinct $GNF^{-}(A_i)$ introduce distinct predicates, the resulting well-founded chain of explicit definitions is a definitional extension of Γ. $\square$

4.2 FOL with function symbols and equality

The construction of $GNF^{-}(\Gamma)$ for a theory Γ in FOL with equality follows Definitions 4.1 and 4.2, with equality treated as a binary predicate. Definitions and facts from Section 1 remain unchanged.

However, introduction of terms complicates the straightforward Definition 2.2 of a theory's graph. For the first, axioms may now have a more specific form than schema (2.1), with terms instead of variables. This general form is abbreviated as

$$Btx \leftrightarrow \bigwedge_{y} \big(\neg B_1 xy \wedge ... \wedge \neg B_n xy\big), \tag{4.4}$$

where x in Btx may stand for variables occurring within the term t or as other arguments of B, while each $B_i xy$ can be a predicate symbol B_i applied to some terms with variables among x, y. A special case has Ba, with a constant a, in LS.

Consequently, the same predicate applied to different terms can be now defined by different formulas, while the interpretations of these terms may then coincide. For instance, given constants a, b, a unary predicate can be (partially) defined by

(1) $Pa \leftrightarrow \neg Qab$
$\quad Pb \leftrightarrow \bigwedge_{y,z} \neg Rbyz$.

As long as a and b are interpreted as different elements of the domain, the graph may have different edges going out of vertices Pa and Pb. But if $a = b$, the mere identification of Pa with Pb in the graph will not reflect the logic, according to which $\neg Qab \leftrightarrow \bigwedge_{y,z} \neg Rbyz$. Note the difference from the single GNF axiom

(2) $Px \leftrightarrow (x = a \wedge \neg Qab) \vee (x = b \wedge \bigwedge_{y,z} \neg Rbyz)$.

Its graph is constructed as in Definition 2.2, except that also equality is needed. Unlike the two formulas in (1), it forces $Px = \mathbf{0}$ for all x distinct from a and b. To handle situations like (1) we introduce, along with terms, $\mathbb{T}_X \neq X$, also the equality predicate $eq(s,t)$, often abbreviated by eq_{st}, with the standard axioms. Vertices of the graph $\mathcal{G}_D(\Gamma)$ contain all atoms $Btd \in \mathrm{At}(\mathbb{T}_D)$, which are partitioned into two sets. Ins contains all atoms Btd which result from substituting some $d \in D^{ar(Bt)}$ for all variables x in the LS Btx of some axiom (4.4). For the remaining atoms, we include dual vertices $\overline{\mathrm{At}}(\mathbb{T}_D) = \bigcup_{B \in \mathcal{L}} \overline{\mathrm{At}}(\mathbb{T}_D, B)$, where $\overline{\mathrm{At}}(\mathbb{T}_D, B) = \{\overline{B}s \mid s \in \mathbb{T}_D, Bs \notin Ins\}$.

A brief explanation of the definition follows underneath.

Definition 4.5 *For Γ in FOL language $\mathcal{L}$ and a set D, the graph $G = \mathcal{G}^{+}_{D}(\Gamma)$ is given by:*

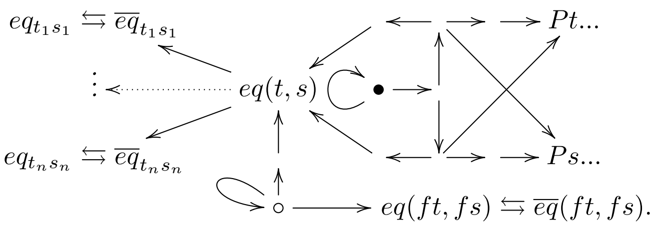
1. $V_G = \mathrm{At}(\mathbb{T}_D) \cup \overline{\mathrm{At}}(\mathbb{T}_D) \cup \{eq_{st} \mid s, t \in \mathbb{T}_D\} \cup \{\overline{eq}_{st} \mid s, t \in \mathbb{T}_D\} \cup \{\bullet_t \mid t \in \mathbb{T}_D \setminus D\} \cup Aux$, *where Aux are auxiliary vertices used below.*
2. *For each pair of distinct terms, $s, t \in \mathbb{T}_D$, we form 2-cycle $eq_{st} \leftrightarrows \overline{eq}_{st}$; for each pair of distinct $a, b \in D$, we add a vertex with a loop and the edge: $\circlearrowleft \bullet \rightarrow \overline{eq}(a,b) \leftrightarrows eq(a,b)$.*
3. *For each term $t \in \mathbb{T}_D \setminus D$ (including constants), we form first the complete digraph $C(t)$ over vertices $V_{C(t)} = \{eq(t,d) \mid d \in D\}$ to which we add vertex $\bullet_t$ with a loop and an edge $\bullet_t \rightarrow eq(t,d)$, for each $eq(t,d) \in V_{C(t)}$, schematically: $\circlearrowleft \bullet_t \dashrightarrow C(t)$.*
4. *We add the standard equality axioms for $eq(\_,\_)$, that is, for each $p, q, r \in \mathbb{T}_D$:*

   (r) vertex eq_{pp} is a sink – for reflexivity,

   (s) the subgraph
$$\begin{tikzcd} eq_{pq} \arrow[d, shift left] & \bullet \arrow[l] \arrow[r] \arrow[loop below] & \overline{eq}_{qp} \arrow[d, shift left] \\ \overline{eq}_{pq} \arrow[u, shift left] & \bullet \arrow[l] \arrow[r] \arrow[loop above] & eq_{qp} \arrow[u, shift left] \end{tikzcd}$$
– for symmetry,

(t) the subgraph
$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] \arrow[d] \arrow[drr] & \overline{eq}_{pq} \leftrightarrows eq_{pq} & \\ \overline{eq}_{qr} \leftrightarrows eq_{qr} & & eq_{pr} \leftrightarrows \overline{eq}_{pr} \end{tikzcd}$$
– for transitivity.

(c) For each function f/predicate P with arity n and pairs of terms $t_i, s_i \in \mathbb{T}_D$, $1 \leq i \leq n$, with $(t_1...t_n) = t \neq s = (s_1...s_n)$, we add the congruence subgraph with the sources $\circ$/$\bullet$:



If t, s are single terms, edges going out of $eq(t,s)$ can be replaced by the 2-cycle to $\overline{eq}(t,s)$.

5. *For atomic axioms $s = t$ or $s \neq t$ in Γ, with $s, t \in \mathbb{T}_X$, we augment each instance $sd, td \in \mathbb{T}_D$ of the 2-cycle from point 2 with a new vertex $\bullet$ with the loop and the edge:*
   (a) for each $s = t \in \Gamma, d \in D^{ar(s,t)}$: $\circlearrowright \bullet \rightarrow eq(sd,td) \leftrightarrows \overline{eq}(sd,td)$;
   (b) for each $s \neq t \in \Gamma, d \in D^{ar(s,t)}$: $eq(sd,td) \leftrightarrows \overline{eq}(sd,td) \leftarrow \bullet \circlearrowleft$.
6. *For each equivalence (4.4) with Btx in LS and for each $d \in D^{ar(Bt)}$, vertex Btd obtains the outgoing edges to $A_G(Btd) = \{B_i dc \mid 1 \leq i \leq n, c \in D^{ar(B_i d)}\}$.*
7. *For each $\overline{B}s \in \overline{\mathrm{AT}}(\mathbb{T}_D)$, vertex Bs obtains the 2-cycle $Bs \leftrightarrows \overline{B}s$.*

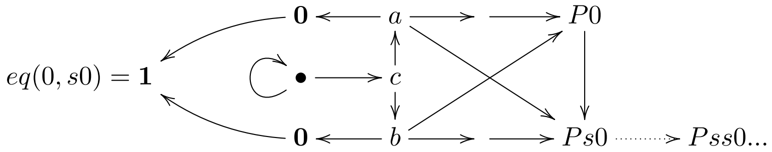
Auxiliary vertices Aux are all $\bullet$ and anonymous vertices in the indicated subgraphs. For each kernel $K \in Ker(G)$, the subgraphs in respective points ensure the following properties:

2. For distinct $a, b \in D$, $eq(a,b) \notin K$, representing inequality.
3. Unique interpretation in D of every function application. With eq representing equality, these subgraphs ensure that each application of a function to arguments from D returns a unique element of D, in particular, that each constant is interpreted as some unique $d \in D$. This follows because, in a complete graph, the kernels are exactly individual vertices, so that each kernel of $C(t)$ is exactly one $eq(t,d)$.
4. Satisfaction of the standard equality axioms by $eq(\_,\_)$. Equivalence is ensured by (r), (s) and (t), while in the subgraphs (c), vertex $\circ$ captures the congruence axiom $t = s \rightarrow ft = fs$, and vertex $\bullet$ its predicate version $t = s \rightarrow (Pt \leftrightarrow Ps)$. Vertices Pt, Ps initiate the subgraphs according to point 6.
5. Satisfaction of the atomic nonlogical with (in)equality axioms.
6. Satisfaction of other nonlogical axioms (4.4).
7. If a predicate B is only partially defined (like in (1)), then each $Bd \notin Ins$ can be interpreted arbitrarily, provided $eq(d,tc) \notin K$ for each $Btc \in Ins$ (and the chosen interpretation does not collide with other restrictions).

Saying below that something follows "by subgraphs..." refers to the points above, applied to any kernel restricted to these subgraphs. For instance, if $K \in Ker(\mathcal{G}^+_D(\Gamma))$ then, by subgraphs 5.(a), $eq(sd,td) \in K$ for $s = t \in \Gamma$ and each $d \in D^{ar(t,s)}$, because kernel requires exclusion of vertex $\bullet$ with the loop, which forces $eq(sd,td) \in K$.

Example 4.6 *For a predicate P, if there is no equivalence with Px in LS, then 7.(b) yields the 2-cycle $Pd \leftrightarrows \overline{P}d$, for each $d \in D^{ar(P)}$. For example (1) from the begining of this subsection, point 6 yields the edges*
– *$Pa \rightarrow Qab...$*
– *$Pb \rightarrow Rbd_1d_2...$ for all $d_1, d_2 \in D^2$.*
By subgraphs 4.5.4.(c) with Pa, Pb and Ps, $s \in D^{ar(P)}$, if $K \in Ker(G)$ and $eq(a,s) \in K$ then $K \models Pa \leftrightarrow Ps$, while if $eq(a,b) \in K$ then also $K \models \neg Qaa \leftrightarrow \bigwedge_{y,z} \neg Rayz$. But if $\{\overline{eq}(d,a), \overline{eq}(d,b)\} \subseteq K$, then by subgraphs 7.(b), either $Pd \in K$ or $\overline{P}d \in K$.

Example 4.7 *For the axiom (P) $Px \leftrightarrow \neg Psx$, and the set with one element 0, the terms $\mathbb{T}_{\{0\}}$ are (isomorphic to) natural numbers and the graph becomes a ray $P0 \to Ps0 \to Pss0 \to ...$, with subgraphs 4.5.4.(c) for each pair s^n0, s^m0, $n \neq m$. Since all these terms are interpreted identically over $\{0\}$, for any kernel $K \in Ker(\mathcal{G}^+_{\{0\}}(P))$, the subgraph 4.5.4.(c) with the source $\bullet$ for $P0, Ps0$ is such that $eq(0, s0) \in K$, while the edge $P0 \to Ps0$ yields $P0 \in K \Leftrightarrow Ps0 \notin K$:*



Consequently, this subgraph, and hence the whole graph, has no kernel, reflecting the nonexistence of models of (P) over one element domain.

Taking as the underlying set the natural numbers $\mathbb{N}$, with the standard interpretation of s as $+1$, the graph again becomes the ray $P0 \to P1 \to P2 \to ....$ But now no equality $eq(p, q)$ holds except for $eq(p, p)$. The instance of the subgraph above swaps $\mathbf{0}$ and $\mathbf{1}$, obtaining two kernels, with Ps^n0 for all even numbers $n \geq 0$, or for all odd $n > 0$.

Earlier facts for FOL$^-$ (except 2.7.3) remain valid for FOL with Definition 4.5 replacing 2.2. According to the following fact, $\mathcal{G}^+_D(\Gamma)$ captures all possible models of Γ over a set D, as $\mathcal{G}_D(\Gamma)$ did for FOL$^-$ according to Fact 2.5.

Fact 4.8 (cf.2.5) *For a GNF theory Γ in FOL and any nonempty set D, there are injections*

$$kr : Mod_D(\Gamma) \to Ker(\mathcal{G}^+_D(\Gamma)) \text{ and } md : Ker(\mathcal{G}^+_D(\Gamma)) \to Mod_D(\Gamma),$$

such that $M \models \phi \Leftrightarrow kr(M) \models \phi$ and $K \models \phi \Leftrightarrow md(K) \models \phi$, for each $M \in Mod_D(\Gamma), K \in Ker(\mathcal{G}^+_D(\Gamma))$ and formula ϕ of the language of Γ.

Proof. The proof has the structure of the proof of Fact 2.5, with the additional treatment of equality, and the more general form (4.4) of axioms.

(kr) Letting $G = \mathcal{G}^+_D(\Gamma)$, an injection $kr : Mod_D(\Gamma) \to Ker(G)$ is obtained by mapping a model $D^+ \models \Gamma$, over a set D, on $kr(D^+) \in Ker(G)$ given by all values induced from:

$$K = \{Bd \in \mathrm{At}(\mathbb{T}_D) \mid D^+ \models Bd\} \cup \{\overline{B}d \in \overline{\mathrm{At}}(\mathbb{T}_D) \mid D^+ \not\models Bd\}$$
$$\cup \{\overline{eq}(s, t) \mid D^+ \not\models s = t,\ s, t \in \mathbb{T}_D\} \cup \{eq(s, t) \mid D^+ \models s = t,\ s, t \in \mathbb{T}_D\}.$$

The second summand is empty in case $\overline{\mathrm{At}}(D) = \emptyset$ and each Bd obtains a value relatively to its outneighbours $A_G(Bd)$ determined by the axioms.

Vertices included into $kr(D^+)$ by inducing from K, but not mentioned in the definition of K above, are among the auxiliary vertices in graphs 4.5.4.(c). They do not affect the argument below, so we identify $kr(D^+) = K$.

i. Equality in D^+ is reflected by eq in G. Since each term applied to elements of D yields a unique element of D, K determines a unique solution to each subgraph from point 3 of Definition 4.5 with $eq(t, d) = \mathbf{1}$ for $d \in D$ interpreting the term $t \in \mathbb{T}_D$, i.e., $D^+ \models t = d$, which induces $\mathbf{0}$ to $\bullet_t$. The last two summands of K determine unique solutions to all subgraphs from Definition 4.5.4:

(r), since $eq(p, p)$ is a sink,

(s), since $eq(p, q) \Leftrightarrow eq(q, p)$ both $\bullet$ vertices in subgraph (s) obtain induced value $\mathbf{0}$ and

(t), since $eq(p, g)$ and $eq(q, r)$ imply $eq(p, r)$, so $\bullet$ in subgraph (t) obtains induced value $\mathbf{0}$.

For (c), if $eq(t, s) \in K$, i.e., $D^+ \models t = s$, then $D^+ \models ft = fs$ for each function f, and $D^+ \models Bte \leftrightarrow Bse$ for each predicate B and $e \in D^{ar(Bt)} = D^{ar(Bs)}$. Thus, if $eq(t, s) \in K$ then $eq(ft, fs) \in K$, while $Bte \in K \Leftrightarrow Bse \in K$, and the subgraph from 4.(c) obtains a solution since inducing from these values ensures $A_G(\circ) \cap K \neq \emptyset$ and $A_G(\bullet) \cap K \neq \emptyset$.

The last two summands of K give also unique solutions to the subgraphs from 4.5.5.

ii. Instances $Bd \in Ins$ defined by (4.4) are treated as in the proof of Fact 2.5, with a small proviso. It may happen that $Bd \notin Ins$, while for some axiom with Btx in LS, $D^+ \models d = tc$, so $D^+ \models Bd \Leftrightarrow D^+ \models Btc$. Then $eq(d, tc) \in K$ and $Bd \in K \Leftrightarrow Btc \in K$, so K solves the subgraph 4.5.4.(c) with Bd, Btc in place of Ps, Pt.

We show first $V_G \setminus K \subseteq A^-_G(K)$. This inclusion follows for vertices $eq(\_,\_)$ and $\overline{eq}(\_,\_)$ by **i** (and for Aux by inducing from K), so we consider atoms $A \in \mathrm{At}(\mathbb{T}_D)$ with $A \notin K$, that is, $D^+ \not\models A$.

Let $A = Btc \in Ins$, i.e., Btc is an instance of LS Btx of some axiom (4.4) F. Since $D^+ \not\models Btc$, so $D^+ \models \neg Btc$ and $D^+ \models B_i ce$ for some instance of some conjunct $B_i xy$ in RS of F. Hence $B_i ce \in K$, and $Btc \in A^-_G(K)$ since $B_i ce \in A_G(Btc)$ by 4.5.6.

If $A = Bd \notin Ins$, then $A_G(Bd) = \{\overline{B}d\}$ by 4.5.7. Since $D^+ \not\models Bd$, so $\overline{B}d \in K$ and $Bd \in A^-_G(K)$. These two cases, with point **i**, establish $V_G \setminus K \subseteq A^-_G(K)$.

iii. For the opposite inclusion, assuming $A \in A^-_G(K)$, there are two cases.

If $A = Btc \in Ins$ then $Btc \in A^-_G(K)$ means that $B_i ce \in K$ for some instance $B_i ce$ of some $B_i xy$ in the RS of the axiom $Btx \leftrightarrow ...B_i xy$. Then $D^+ \models B_i ce$, so $D^+ \not\models Btc$ and $Btc \notin K$.

If $A = Bd \notin Ins$ then $A_G(Bd) = \{\overline{B}d\}$ by 4.5.7 and since $Bd \in A^-_G(K)$, so $\overline{B}d \in K$, which means that $D^+ \not\models Bd$, so that $Bd \notin K$.

These two cases give $\mathrm{At}(\mathbb{T}_D) \cap A^-_G(K) \subseteq V_G \setminus K$. If $v \in A^-_G(K)$ is $eq(\_,\_)$ or $\overline{eq}(\_,\_)$, then partial solutions to the subgraphs in point **i** give $v \in V_G \setminus K$, so that $A^-_G(K) \subseteq V_G \setminus K$. With **ii**, this gives $A^-_G(K) = V_G \setminus K$, i.e., $K \in Ker(G)$.

iv. Since $D^+ \models Bd \Leftrightarrow K \models Bd$ for each atom $Bd \in \mathrm{At}(\mathbb{T}_D)$, and $D^+ \models s = t \Leftrightarrow K \models eq(s,t)$ for $s, t \in \mathbb{T}_D$, the equivalence $D^+ \models \phi \Leftrightarrow K \models \phi$ holds for arbitrary formula ϕ of the language of Γ.

v. The so defined kr is injective, because two different models $D^+_1 \neq D^+_2$, over any given set D, obviously give two different $kr(D^+_1) \neq kr(D^+_2)$, since by definition $K_1 \neq K_2$.

(md) An injection $md : Ker(G) \to Mod_D(\Gamma)$ is obtained as follows.

i. Given a $K \in Ker(G)$, we note first that points 2 and 3 of Definition 4.5 ensure well-definedness of the function $i : \mathbb{T}_D \to D$, given by $i(t) = d$, for $d \in D$ such that $eq(t,d) \in K$. By 4.5.3 such a d is unique for each $t \in \mathbb{T}_D$, while by 4.5.2, the restriction $i|_D$ is the identity on D.

Subgraphs (r), (s) and (t) from 4.5.4 ensure that $eq(\_,\_)$ is an equivalence on D, while (c) that $eq(s,t)$ entails $eq(fs, ft)$ for each function symbol f. Hence i is a well-defined quotient mapping.

ii. We define the structure $M = md(K)$ on D by interpreting all function symbols using i. For each function f and $a \in D^{ar(f)}$, define its interpretation in M by $f^M a = i(fa) = d \in D$, where $\{eq(fa,d)\} = V_{C(fa)} \cap K$, for the subgraph $C(fa)$ from 4.5.3.

For each atom $Bd \in \mathrm{At}(\mathbb{T}_D)$, we define $M \models Bd$ iff $Bd \in K$. This is well-defined, since subgraphs 4.5.4.(c) ensure that for all $t, s \in \mathbb{T}_D$, if $eq(t,s) \in K$ then $Bt \in K \Leftrightarrow Bs \in K$. Suppose that Γ contains two distinct axioms (4.4), $Btx \leftrightarrow ...$ and $Bsy \leftrightarrow ...$, while $eq(ta,d) \in K$ and $eq(sb,d) \in K$ for some $a \in \mathbb{T}_D^{ar(t)}$, $b \in \mathbb{T}_D^{ar(s)}$. Then $eq(ta,sb) \in K$ by the subgraphs (s), (t) from Definition 4.5.4, and $Bta \in K \Leftrightarrow Bsb \in K$ by (c). Hence, $Bta \in K \Leftrightarrow Bsb \in K \Leftrightarrow Bd \in K$, so $M \models Bta \Leftrightarrow M \models Bsb \Leftrightarrow M \models Bd$.

iii. For any atomic equality axiom $s = t \in \Gamma$, subgraph $\circlearrowleft\bullet \to eq(sd,td)$ from 4.5.5 forces $eq(sd,td) \in K$ for every instance $sd, td \in \mathbb{T}_D$ of s,t. By **i**, for each such instance $i(sd) = i(td)$, so that $M \models s = t$. Similarly, for the axiom $s \neq t \in \Gamma$, the subgraph $eq(sd,td) \leftrightarrows \overline{eq}(sd,td) \leftarrow \bullet\circlearrowright$, for each instance $sd, td \in \mathbb{T}_D$, forces $eq(sd,td) \notin K$, so $i(sd) \neq i(td)$, giving $M \models s \neq t$.

iv. Consider now any axiom (4.4) $F \in \Gamma$, with Btx in LS, and any $d \in D^{ar(Bt)}$. If $M \models Btd$, i.e., $Btd \in K$, then $A_G(Btd) \cap K = \emptyset$ since $K \in Ker(G)$, so that $B_i dc \notin K$ for all $B_i xy$ in the RS of F and $c \in D^{ar(B_i d)}$. Thus $M \not\models B_i dc$, i.e., $M \models \neg B_i dc$ for all $B_i dc$ in the RS, so M satisfies the implication from left to right of F.

If $M \not\models Btd$ then $Btd \notin K$ and $Btd \in A^-_G(K)$, since $K \in Ker(G)$. This means that for some $B_i xy$ in the right side of F and $c \in D^{ar(B_i d)}$, $B_i dc \in K$, since such $B_i dc$ form $A_G(Btd)$ by point 6 of Definition 4.5. Thus M satisfies also the right to left implication of (4.4), so $M \models F$.

v. Since $M \models Bd \Leftrightarrow K \models Bd$ for each atom Bd and, by **ii**, $M \models s = t \Leftrightarrow K \models eq(s,t)$ for each $s, t \in \mathbb{T}_D$, the equivalence $M \models \phi \Leftrightarrow K \models \phi$ holds for arbitrary formula ϕ of the language of Γ.

vi. Two different kernels $K_1 \neq K_2$ of G differ for at least one atom Bd. This follows because membership of atoms in a kernel K determines uniquely this kernel, as can be seen inspecting the subgraphs in Definition 4.5. For instance, restriction of any kernel K to the atoms in any subgraph

4.5.4.(c), i.e., to Pt, Ps and all $eq(t_i, s_i)$, induces unique values to all remaining (auxiliary) vertices of this subgraph, which therefore must coincide with their (non)membership in K. By Definition 4.5.7, each $K \in Ker(G)$ determines $Bd \in K$ or $\overline{B}d \in K$ also for $Bd \notin Ins$.

Thus $K_1 \neq K_2$ implies for some atom $Bd \in (K_1 \setminus K_2) \cup (K_2 \setminus K_1)$. The respective models differ then on Bd by **ii**, $md(K_1) \neq md(K_2)$, so md is injective. □

Consequently, kernels represent exactly models of a theory by Fact 4.9 below, which follows from Fact 4.8 in the same way Fact 2.7.1–2 follows from Fact 2.5.

Fact 4.9 (cf.2.7) *In FOL:*

1. *For an arbitrary GNF theory* $\Gamma : Mod(\Gamma) \leftrightarrows GMod(\Gamma)$.
2. *For an arbitrary theory* $T : Mod(T) \leftrightarrows GMod(GNF(T))$.

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