I220, Autumn 202: Exercise 2

1. Given signature

$$\begin{aligned}\Sigma = (\{ \quad & \mathsf{nat}\},\\ \{ \quad & \mathsf{zero} :\to \mathsf{nat},\\ & \mathsf{succ} : \mathsf{nat} \to \mathsf{nat},\\ & \mathsf{plus} : \mathsf{nat} \times \mathsf{nat} \to \mathsf{nat}\})\end{aligned}$$

and algebras A, B, C for Σ, defined by

$$\begin{aligned}A(\mathsf{nat}) &= \mathbb{N}\\ A(\mathsf{zero} :\to \mathsf{nat}) &= 0\\ A(\mathsf{succ} : \mathsf{nat} \to \mathsf{nat})(n) &= n+1\\ A(\mathsf{plus} : \mathsf{nat} \times \mathsf{nat} \to \mathsf{nat})(n,m) &= n+m\end{aligned}$$

$$\begin{aligned}B(\mathsf{nat}) &= \mathbb{N}\\ B(\mathsf{zero} :\to \mathsf{nat}) &= 0\\ B(\mathsf{succ} : \mathsf{nat} \to \mathsf{nat})(n) &= (n+1) \bmod 8\\ B(\mathsf{plus} : \mathsf{nat} \times \mathsf{nat} \to \mathsf{nat})(n,m) &= (n+m) \bmod 8\end{aligned}$$

$$\begin{aligned}C(\mathsf{nat}) &= \{0,1,2,3,4,5,6,7\}\\ C(\mathsf{zero} :\to \mathsf{nat}) &= 0\\ C(\mathsf{succ} : \mathsf{nat} \to \mathsf{nat})(n) &= (n+1) \bmod 8\\ C(\mathsf{plus} : \mathsf{nat} \times \mathsf{nat} \to \mathsf{nat})(n,m) &= (n+m) \bmod 8\end{aligned}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers.

(a) Show that A is freely generated by $\{\mathsf{zero}, \mathsf{succ}\}$.

(b) Show that A is not generated by $\{\mathsf{zero}, \mathsf{plus}\}$.

(c) Show that B is not generated.

(d) Show that C is generated by $\{\mathsf{zero}, \mathsf{succ}\}$.
Show that C is not freely generated.

(e) Show that A is not a subalgebra of B.

(f) Show that A is not a subalgebra of C.

(g) Show that B is not a subalgebra of A.

(h) Show that C is not a subalgebra of A.

(i) Show that C is a subalgebra of B

(j) Find a homomorphism $h : A \to B$. Why is this homomorphism unique?

(k) Show that $h(A) \simeq C$

(l) Define the congruence $\equiv_h$.
Define the algebra $A/\equiv_h$

(m) Show that $A/\equiv_h \simeq C$.

(n) Find a homomorphism $g : B \to C$. Why is this homomorphism unique?

2. Look at example 2.11 from the book.
   (a) Show that C is a subalgebra of D.
   (b) Find a congruence $\equiv$ on D such that $D/\equiv \simeq C$.
   (c) Find a congruence $\equiv$ on D such that $D/\equiv \simeq B$.
   (d) Is the algebra D generated?
3. From the book: 3.1-1, 3.1-2, 3.1-4, 3.1-5.
4. From the book: 3.2-1, 3.2-2, 3.2-3, 3.2-5, 3.2-6, 3.2-7.
5. From the book: 3.3-2, 3.3-3, 3.3-4, 3.3-9.
6. From the book: 3.4-2, 3.4-4.