I220, Autumn 2002: Exercise 1

1. Exercises from the book
   - 2.2-1
   - 2.3-2
   - 2.4-1
   - 2.4-2 (only the part for 2.3-2)
   - 2.4-3 (except (i))
   - 2.4-6

2. Let S be any non-empty set (for the sake of example, it may have 3 elements), and $\wp(S)$ be its power set, i.e., $\wp(S) = \{X : X \subseteq S\}$. Consider the signature

$$
\begin{array}{rll}
\Sigma = \ \mathcal{S}: & P & \\
\Omega: & 0: & \rightarrow P \\
 & 1: & \rightarrow P \\
 & \cap\ : & P \times P \rightarrow P \\
 & \cup\ : & P \times P \rightarrow P
\end{array}
$$

   (a) There is a very natural way of turning the set $\wp(S)$ into a Σ-algebra –i.e., of using it as a carrier on which one has defined all Σ-operations. How would you intepret the Σ-operations in order to endow $\wp(S)$ with the algebraic Σ-structure?

   (b) Check whether your interpretation makes the following equations hold in your algebra (X, Y, Z are variables referring to the elements of the carrier – the members of the power set $\wp(S)$, i.e., the subsets of S):

   1. $X \cup Y = Y \cup X$
   2. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
   3. $X \cup 0 = X$
   4. $X \cap Y = Y \cap X$
   5. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
   6. $X \cap 1 = X$

   If not, re-interpret Σ in $\wp(S)$ so that these equations hold.

   (c) Extend now Σ to Σ' by adding a new operation $\overline{} : P \rightarrow P$. Find an interpretation for this operation in the algebra you obtained in the previous point, so that the following equations hold:

   7. $X \cup \overline{X} = 1$
   8. $X \cap \overline{X} = 0$

3. Let *bool* $= \{\mathbf{tt}, \mathbf{ff}\}$ be a two element set. Repeat the previous exercise with this set instead of $\wp(S)$, i.e., turn this set (not its power set!) into a Σ algebra so that it satisfies the axioms 1-6 (notice that here the variables will range over the only two possible elements of *bool*), and then add an interpretation of the additional operation from Σ' so that axioms 7-8 are satisfied.