The Membership Problem for Regular Expressions with Unordered Concatenation and Numerical Constraints

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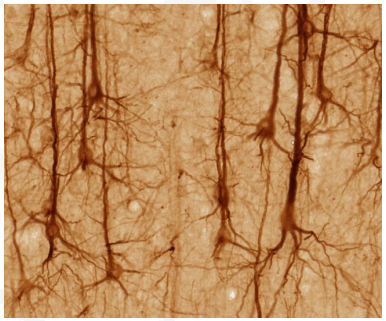
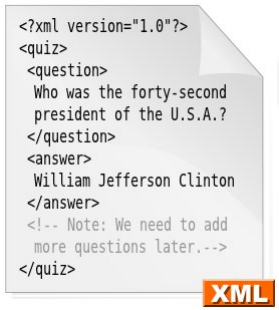
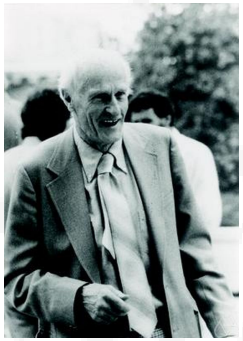
From Neurons to XML





From Neurons to Automata



- 1880s-1900s: The Neuron as basic unit (Ramòn y Cajal, Golgi)
- 1943: McCulloch and Pitts, "A logical calculus of ideas immanent in nervous activity"
- 1951/56: S.C. Kleene, "Representation of Events in Nerve Nets and Finite Automata"
- 1950s-70s: Tapes, words, events and languages

From Automata to XML

- 1970s: Ken Thompson: qed with reg.exp support (1971), grep (1973)
- 1980s-90s: Perl and SGML
- XML (1998), XML Schema (2001)

From xkcd.com

Numerical Constraints

- Expresses iterations (as the $*$) but with more control: bounds the number of iterations
- $abbac \in \|((a+b)^{1..40}c)\|$
- Standard in text-search regular expressions like grep, perl
- Used in Extensible Markup Language (XML) Schema.

Unordered Concatenation

- $\&(ya, basta)$ matches both yabasta and bastaya
- Defined in terms of permutations (reorderings)
- A word w matches an unordered concatenation $\&(r_1, \ldots, r_n)$ if:
  - $r'_1 \cdots r'_n$ is a permutation of $r_1 \cdots r_n$
  - and $w = w_1 \cdots w_n$ where w_1 matches r'_1, w_2 matches r_2 etc.
- $\{abc, acb, bca, cba\} = \|\&(a, \&(b, c))\|$

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- $\{abc, acb, bca, cba\} = \|\&(a, \&(b, c))\|$
- Not associative
- $\{abc, bac, cab, cba\} = \|\&(\&(a, b), c)\|$
- First used in Standard Generalized Markup Language (SGML), later in a limited form in XML Schema

Regular Expressions with Numerical Constraints and Unordered Concatenation

$$\mathbb{N} = \{1, 2, \ldots\} \text{ and } \mathbb{N}_1 = \{2, 3, 4, \ldots\} \cup \{\infty\}$$

$$R_\Sigma ::= R_\Sigma + R_\Sigma \mid R_\Sigma \cdot R_\Sigma \mid R_\Sigma^{\mathbb{N}..\mathbb{N}_1} \mid \&(R_\Sigma, \ldots, R_\Sigma) \mid \Sigma \mid \epsilon$$

$$(r^* = r^{1..\infty} + \epsilon.)$$

The language of a regular expression with numerical constraints and unordered concatenation

- $\|r_1 + r_2\| = \|r_1\| \cup \|r_2\|$
- $\|r_1 \cdot r_2\| = \|r_1\| \cdot \|r_2\|$
- $\|\&(r_1, \ldots, r_n)\| = \bigcup_{\sigma \in \mathsf{Perm}(\{1,\ldots,n\})} \|r_{\sigma_1}\| \cdots \|r_{\sigma_n}\|$
- $\|r^{l..u}\| = \bigcup_{i=l}^{u} \|r\|^i$ (where $A^0 = \{\epsilon\}$ and for $i > 0$, $A^i = A \cdot A^{i-1}$)
- $\|a\| = \{a\}$, when $a \in \Sigma \cup \{\epsilon\}$,

Numerical Constraints and Unordered Concatenation as Short-Hand

- $r^l : \underbrace{r \cdots r}_{l}$
- $r^{l..\infty} : r^l \cdot r^*$
- $r^{l..u} : r^l(r + \epsilon)^{u-l}$, when $u \neq \infty$
- $\&(r_1, \ldots, r_n)$: $r_1 \cdot \&(r_2, \ldots, r_n) + \cdots + r_n \cdot \&(r_1, \ldots, r_{n-1})$

Complexity of Membership for the Usual Regular Expressions

- Polynomial time
- Streaming / Linear time in word: DFAs
- Constructing DFAs is “PSPACE-hard”

1-Unambiguity

- Many applications only work with a subset, the *1-unambiguous* regular expressions.
- Construction of DFAs from 1-unambiguous regular expressions is polynomial-time.

With Numerical Constraints

- Regular expressions with numerical constraints: P
- Linear in the length of the word: Strongly 1-unambiguous
- Regular expressions with unordered concatenation: NP-complete

Goal

- Find a subset of the regular expressions with numerical constraints and unordered concatenation, such that membership can be decided in polynomial time
- For fixed regular expression: runtime linear in the length of the word.

Finite Automata with Counters (FACs)

- Based on classical finite automata
- A counter for each subexpression with numerical constraints
- A counter for each of the arguments to the unordered concatenations
- Counter States: $\gamma : \mathcal{C} \rightarrow \{0, 1, 2, \ldots\}$.
- min and max upper and lower limits for final value of each counter
- *update instruction* : $\psi : \mathcal{C} \rightarrow \{\text{inc}, \text{res}, \text{one}\}$
- inc for *increment*
- res for *reset* (to 0)
- one for setting to 1

An FAC is a Finite Automaton with Counters

An FAC is a tuple $(\Sigma, Q, \mathcal{C}, \mathcal{A}, \Phi, \min, \max, q^I, \mathcal{F})$.

- Σ is the alphabet, Q is the *states*, and $\mathcal{C}$ is the *counters*.
- $q^I \in Q$ is the *initial state*.
- $\mathcal{A} : Q - \{q^I\} \to \Sigma$: the letter matched when entering a non-initial state.
- Φ maps each state to a set of pairs of a state and an update instruction.

$$\Phi : Q \to \wp(Q \times (\mathcal{C} \to \{\text{inc}, \text{res}, \text{one}\})) .$$

- min and max are the counter conditions.
- $\mathcal{F} \subset Q \times (\mathcal{C} \to \{\text{res}\})$: the *final configurations*.

Satisfaction of update instructions

$$\gamma \models^{\max}_{\min} \varnothing$$

$$\gamma \models^{\max}_{\min} \psi_1 \cup \{p \mapsto \mathsf{inc}\} \quad \Leftrightarrow \quad \gamma \models^{\max}_{\min} \psi_1 \wedge \gamma(p) < \max(p)$$

$$\gamma \models^{\max}_{\min} \psi_1 \cup \{p \mapsto \mathsf{res}\} \quad \Leftrightarrow \quad \gamma \models^{\max}_{\min} \psi_1 \wedge \gamma(p) \geq \min(p)$$

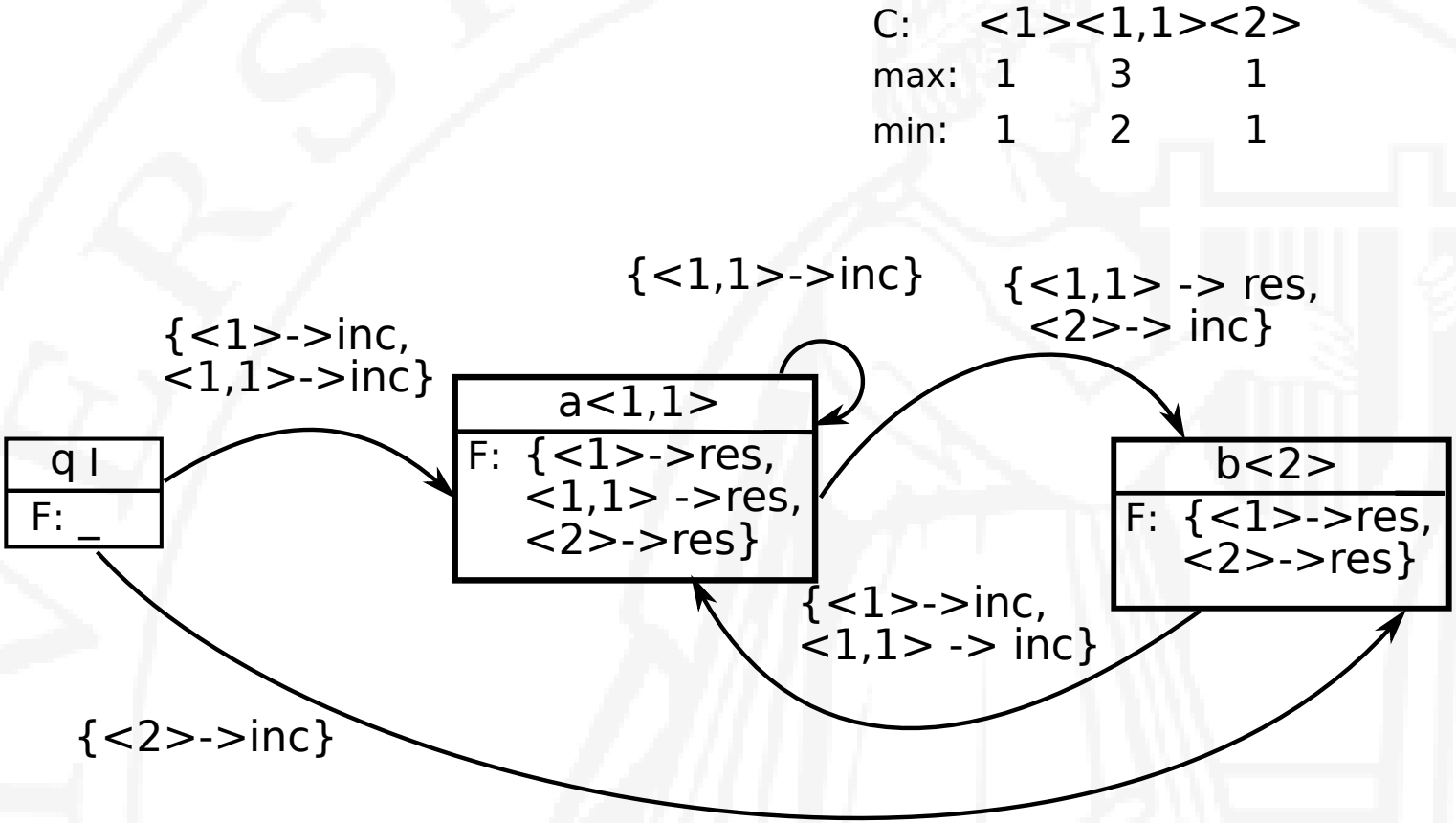
$$\gamma \models^{\max}_{\min} \psi_1 \cup \{p \mapsto \mathsf{one}\} \quad \Leftrightarrow \quad \gamma \models^{\max}_{\min} \psi_1 \wedge \gamma(p) \geq \min(p)$$

A counter can only be increased if the value is smaller than the maximum, while a value can only be reset if it's value is at least as large as the minimum.

Running an FAC

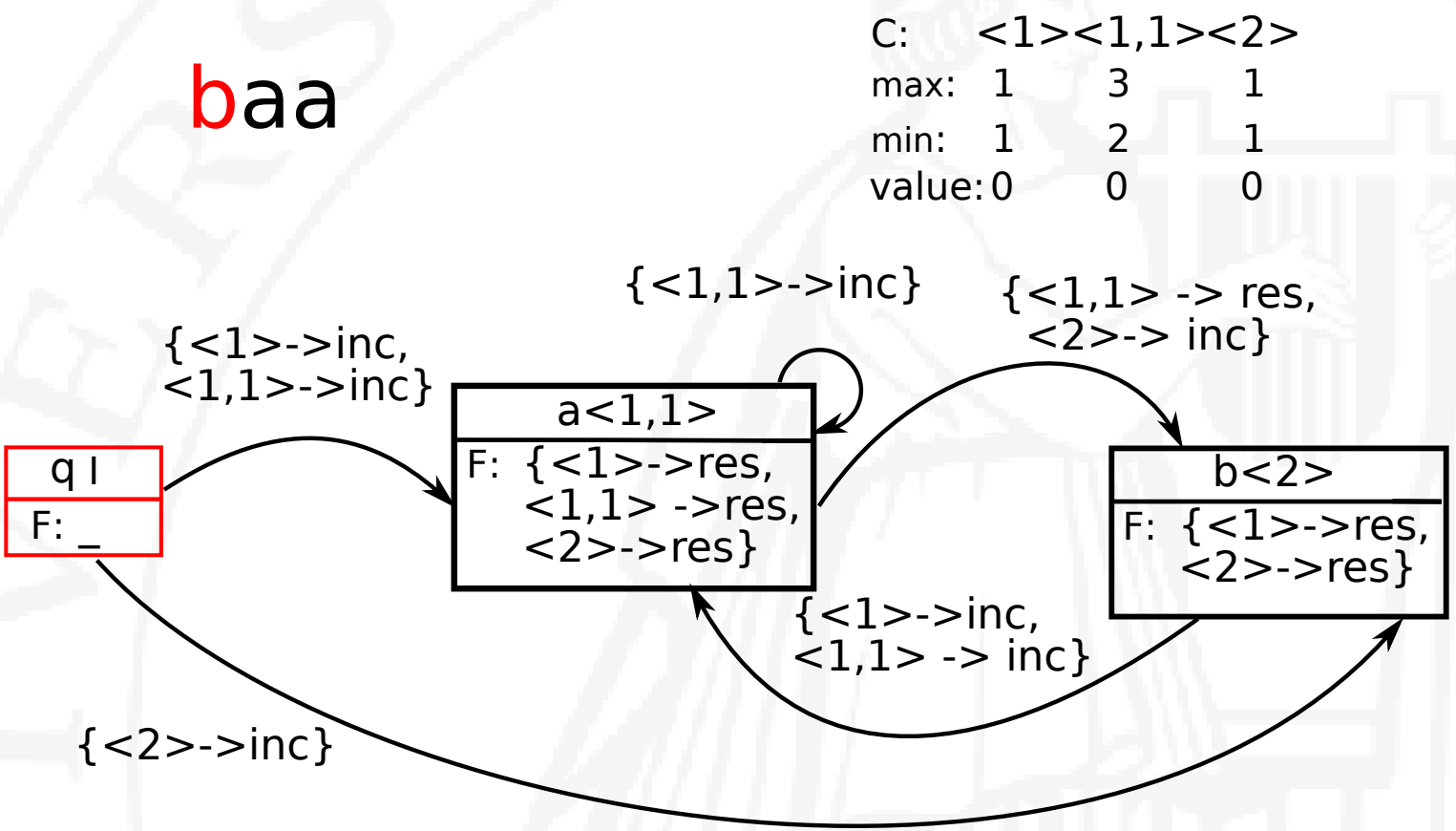
- A *configuration* of an FAC: a pair (q, γ)
- $q \in Q$
- $\gamma : \mathcal{C} \to \{0, 1, 2, \ldots\}$
- δ : the transition function
- δ is calculated from Φ, $\mathcal{A}$, min, and max.
- Initial configuration: $(q^I, \{c \mapsto 0 \mid c \in \mathcal{C}\})$
- (q, γ) is a final configuration if $\gamma \models^{\max}_{\min} \psi$ for some $(q, \psi) \in \mathcal{F}$.
- If δ never returns more than 1 configuration, the FAC is deterministic

FAC recognizing $\|\&(a^{2..3}, b)\|$



- qI initial
- The other states final

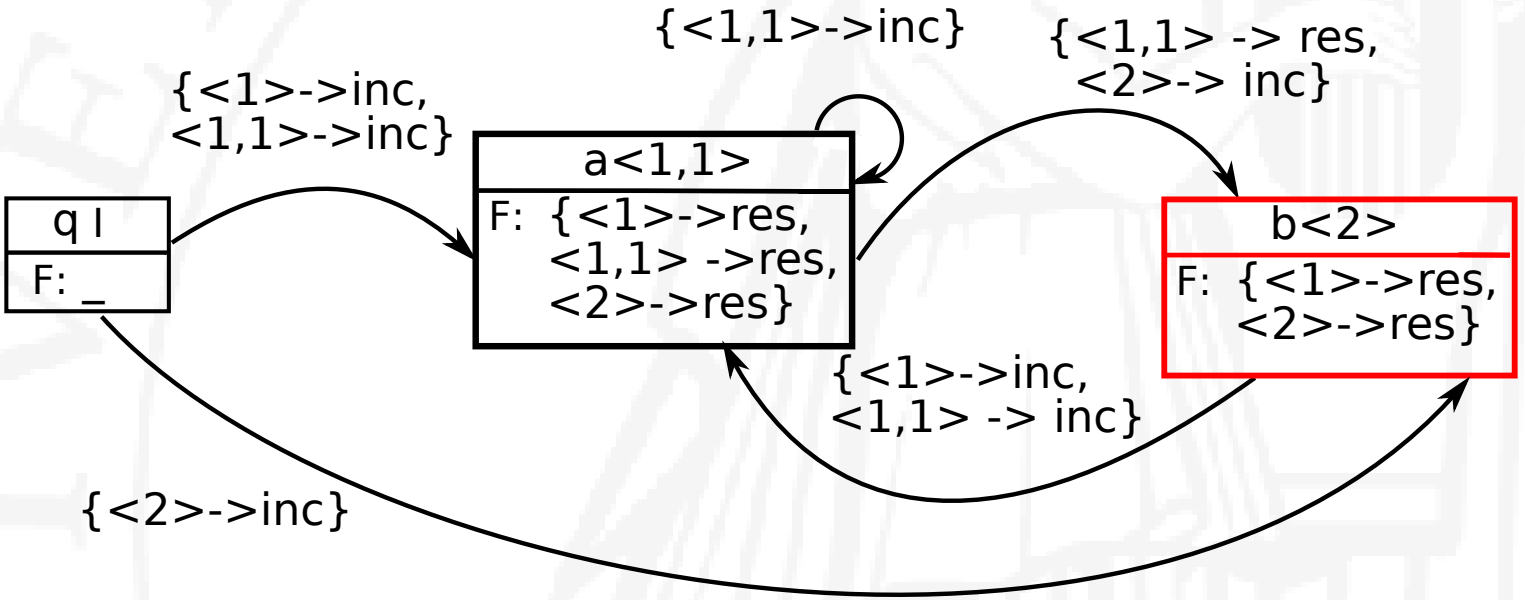
$baa \in \|\&(a^{2..3}, b)\|$?


baa
C: <1><1,1><2>
max: 1 3 1
min: 1 2 1
value: 0 0 0
{<1,1>->inc}
{<1,1> -> res,
<2>-> inc}
{<1>->inc,
<1,1>->inc}
a<1,1>
F: {<1>->res,
<1,1> ->res,
<2>->res}
q I
F: _
b<2>
F: {<1>->res,
<2>->res}
{<1>->inc,
<1,1> -> inc}
{<2>->inc}

$baa \in \|\&(a^{2..3}, b)\|$?

baa

C:	<1>	<1,1>	<2>
max:	1	3	1
min:	1	2	1
value:	0	0	1



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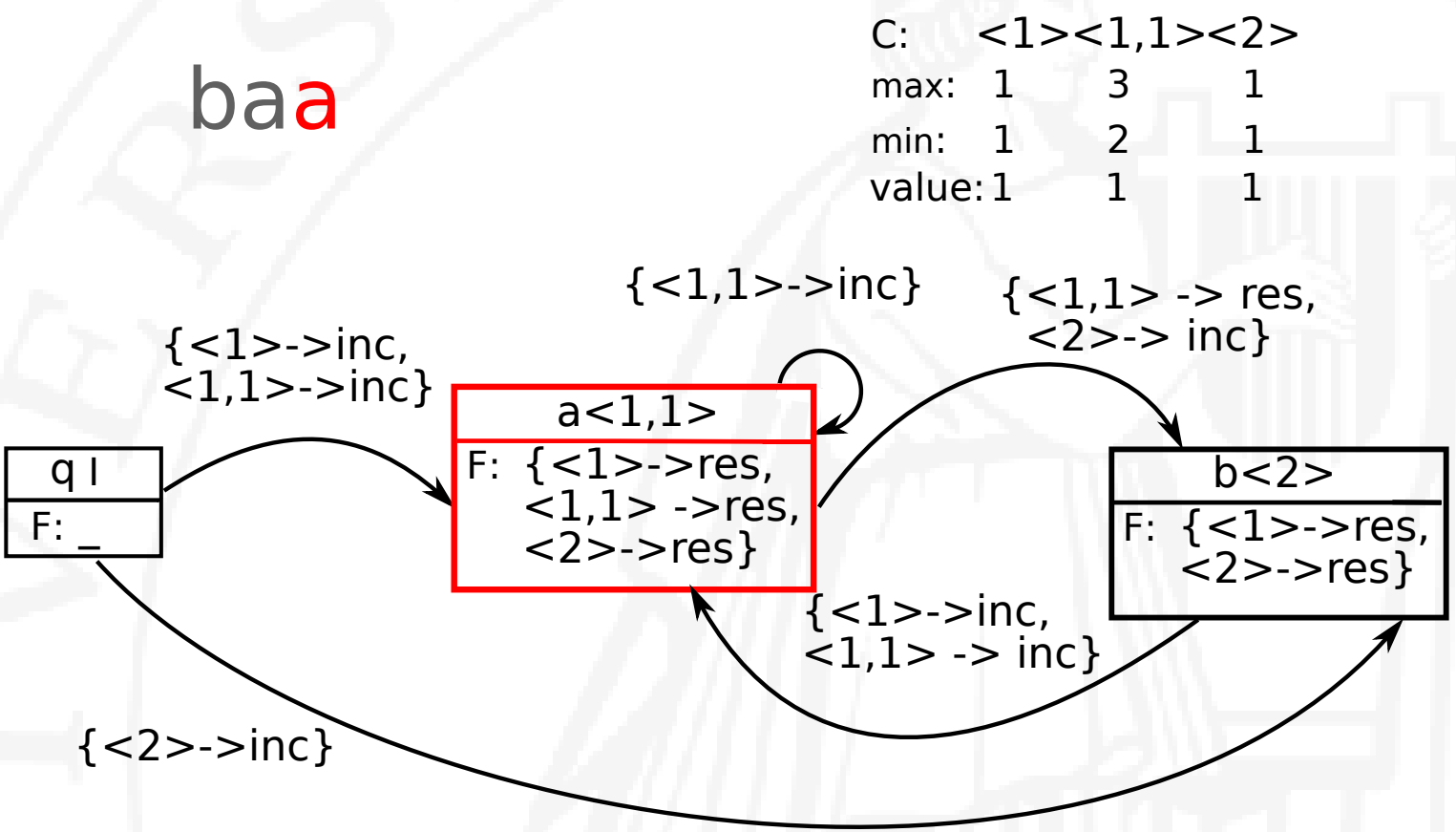
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max: 1 3 1
min: 1 2 1
value: 1 1 1
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{<1,1> -> res,
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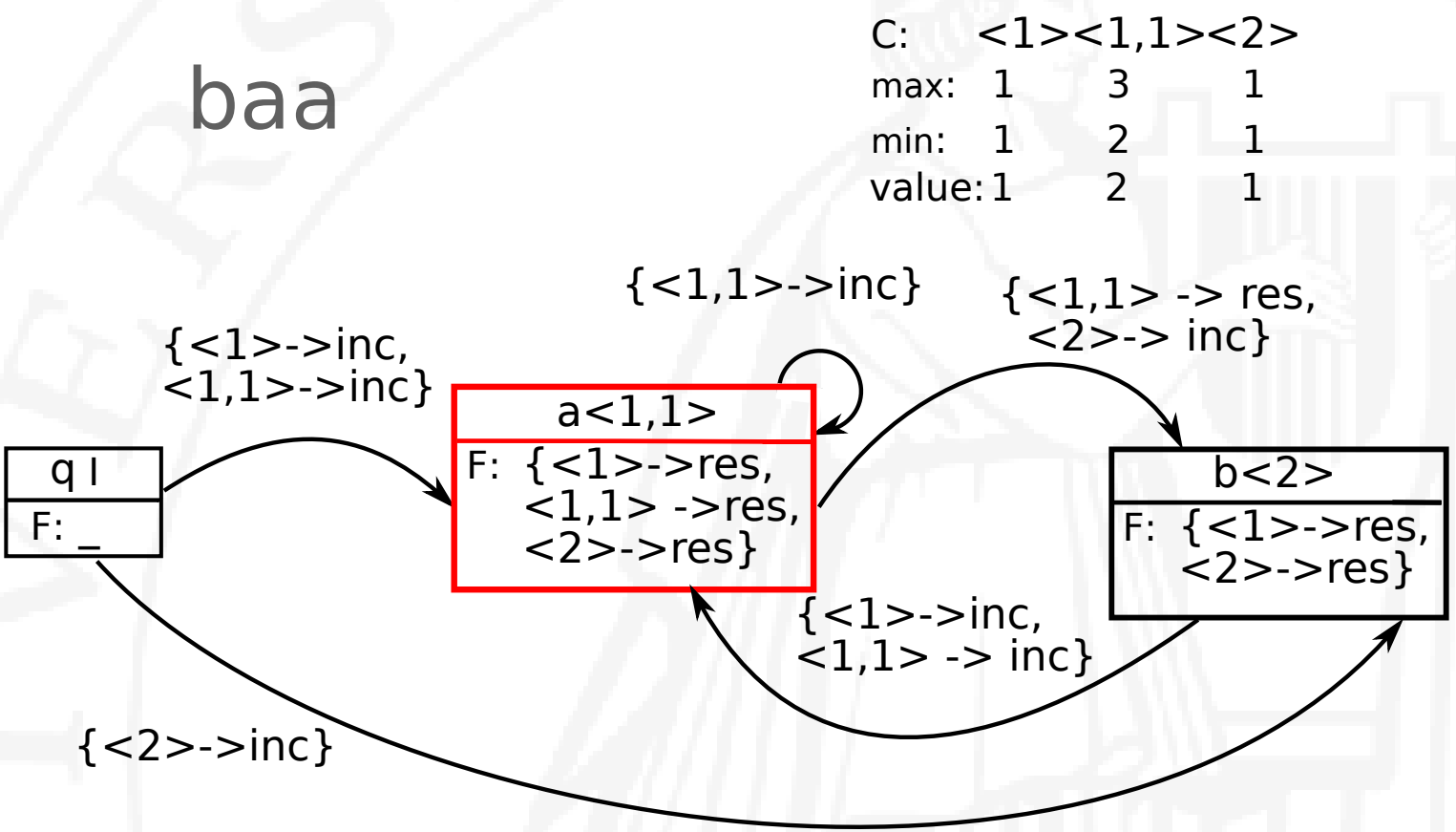
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min: 1 2 1
value: 1 2 1
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{<1,1> -> res,
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b<2>
F: {<1>->res,
<2>->res}
{<1>->inc,
<1,1> -> inc}
{<2>->inc}

- Construction based on *first, last, and follow sets*.
- first-set becomes $\Phi(q^I)$.
- last-set becomes the final configurations
- follow-set becomes the rest of Φ.
- Construction of first, last, and follow is polynomial-time

Result

- For any regular expression with numerical constraints and unordered concatenation, we can in polynomial time construct an FAC recognizing the language of the regular expression
- If the expression is *strongly 1-unambiguous*, the constructed FAC is deterministic
- A deterministic FAC can accept or reject a word in time linear in the length of the word.