On the effectiveness of the incremental approach to minimal chordal edge modification[☆,☆☆]

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Abstract

Because edge modification problems are computationally difficult for most target graph classes, considerable attention has been devoted to inclusion-minimal edge modifications, which are usually polynomial-time computable and which can serve as an approximation of minimum cardinality edge modifications, albeit with no guarantee on the cardinality of the resulting modification set. Over the past fifteen years, the primary design approach used for inclusion-minimal edge modification algorithms is based on a specific incremental scheme. Unfortunately, nothing guarantees that the set $\mathcal{E}$ of edge modifications of a graph G that can be obtained in this specific way spans all the inclusion-minimal edge modifications of G. Here, we focus on edge modification problems into the class of chordal graphs and we show that for this the set $\mathcal{E}$ may not even contain any solution of minimum size and may not even contain a solution close to the minimum; in fact, we show that it may not contain a solution better than within an $\Omega(n)$ factor of the minimum. These results show strong limitations on the current favored algorithmic approach to inclusion-minimal edge modification and suggest that further developments might be better using other approaches.

Keywords: chordal graphs, edge modification, incremental algorithms

1. Introduction

Edge modification problems have been widely studied, both because they ask a very natural theoretic question (how close is a graph to satisfying a given property?) and because they have proven quite useful for solving real-world problems [1, 2, 3, 4]. Editing a graph, which is the most general of the edge modification problems, consists of modifying its edge set (adding some and

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deleting others) in order to obtain a graph in a target class. Ideally, one would like to compute the minimum number of modifications necessary to reach the target class. Unfortunately, most editing problems are difficult, even when the target class is very simple, such as threshold graphs [5] or the disjoint union of cliques [6].[1] Edge modification problems also usually remain difficult when only one kind of operation (addition or deletion) is allowed. As usual, there are several approaches that have been used to deal with this computational difficulty, such as parameterized complexity [7, 8, 9, 10], approximation algorithms [11] and heuristics [12, 13, 1, 14, 2].

For edge modification problems, the most flourishing of these approaches, in terms of the amount of work it has attracted, is the heuristic approach, probably because these heuristics are based on a theoretically well-founded relaxation of the problem: instead of asking for a set of modifications that has minimum cardinality, one can simply ask for a set of modifications that is minimal for inclusion. The immediate benefit of this is that, in most cases, this relaxation of the problem is polynomial-time solvable. As a counterpart, this approach does not give any guarantee on the number of modifications output by the algorithm relative to the minimum possible. This is not necessarily overly limiting as inclusion-minimal solutions are not bad, at least in the sense that they do not contain a better sub-solution, i.e. a proper subset of modifications that also reaches the target class. Moreover, inclusion-minimal algorithms often have a nice property which is exploited in practice: they often make choices during execution that, instead of being made deterministically, can be made randomly, thereby leading to a different solution each time the algorithm is run. Then, running the algorithm several times and keeping the best solution, one can hope that the resulting best solution is good enough, i.e. the number of modifications it contains is not much bigger than the minimum possible.

It turns out that in many cases the inclusion-minimal algorithms that have been developed are incremental: vertices of the graph are added to a solution graph one by one, and for each added vertex x, the algorithm computes an inclusion-minimal set of modifications M_x incident to x so that the new graph belongs to the target class. The rationale for using this incremental scheme, which we refer to as the *local incremental* approach, is that when the target graph class has the hereditary property and is stable under the addition of one universal vertex, which is most often the case for the classes considered in the literature, the solution at the end of the local incremental algorithm is guaranteed to be inclusion-minimal for the whole graph. This approach also simplifies the design of algorithms and yields low polynomial time complexities. In addition, it is naturally compatible with the randomized heuristic approach mentioned above: there are usually several possible choices for an inclusion-minimal set of modifications at each incremental step, and there is also a choice to be made for the order in which vertices are processed. Making these choices randomly provides a different solution at the end of each execution of the algorithm, which, as

[1] Graph terms are defined in Section 2.

explained above, is interesting for heuristic purposes. This local incremental approach was first introduced for modifying graphs into interval graphs [15, 12] and chordal graphs [13] and has since then been used many times for interval graphs themselves [16, 17] and for other target graph classes such as split graphs [18], comparability graphs [19], trivially perfect graphs [20], cographs [14, 21] and permutation graphs [22]. It has become such a common practice that for most of these classes, the local incremental approach is the only approach currently available for solving the inclusion-minimal modification problem.

Nevertheless, this local incremental approach raises a concern, which is the subject of this article. The concern is that the solutions that can be obtained using the local incremental approach are restricted, in the sense that the set $\mathcal{E}$ of such solutions does not necessarily contain all inclusion-minimal modifications of the input graph (see [20], for examples). This raises an immediate question: does the set $\mathcal{E}$ contain at least some good-enough solutions? More precisely, one might ask whether $\mathcal{E}$ always contains a minimum solution, or at least a solution that is not very far from the minimum, say within a constant factor. These are precisely the questions we address here in the particular case where the target is the class of chordal graphs. Rephrased in more algorithmic terms, we ask whether, for any input graph, there exists a choice of the processing order of the vertices and a choice of inclusion-minimal modification set at each incremental step such that the resulting inclusion-minimal chordal modification of the input graph at the end of the incremental scheme is minimum or within a constant factor of the minimum.

Our results. We consider three kinds of edge modification problems for the class of chordal graphs: pure completion (only additions of edges are allowed), pure deletion (only deletion of edges are allowed) and general editing (both additions and deletions of edges are allowed). We show very different behaviors for these three problems with regard to the local incremental approach.

For chordal deletion, we show that for any inclusion-minimal chordal deletion H of the input graph G, there exists an order on the vertices of G and at each incremental step a choice of an inclusion-minimal set of deletions such that the solution obtained at the end is exactly H. This means that for chordal deletion, the set of solutions that can be obtained by the local incremental approach is not restricted and contains all inclusion-minimal deletions of the input graph. Therefore, the questions of (1) the existence of a randomized algorithm based on the local incremental approach that has a non-null probability to discover a minimum cardinality deletion (note that deterministically computing a minimum deletion is NP-hard [23]) and (2) the existence of a deterministic algorithm based on this approach that guarantees a constant factor approximation, are open. Note however that our results do not imply that such algorithms do exist, only that it is possible they exist.

At the other extreme, we show that for both chordal completion and chordal editing there exist graphs for which the set of local incremental inclusion-minimal solutions do not contain any of minimum cardinality. Even worse, the number of modifications in the best local incremental inclusion-minimal

Table 1: Summary (with forward references to proofs)

Type of Modification	**Does the set of incremental solutions include a minimum cardinality solution?**	**How bad can the best incremental solution be relative to the minimum cardinality?**
Pure Completion	No (Theorem 2)	$\Omega(n)$ (Theorem 2)
Pure Deletion	Yes (Theorem 4)	1 (Theorem 4)
General Editing	No (Theorem 6)	$\Omega(n)$ (Theorem 6)

solutions is sometimes $\Omega(n)$ times larger than the number in a minimum cardinality solution. This shows that it is not possible to design an algorithm, either randomized or deterministic, that follows the local incremental approach and that guarantees an approximation ratio in the worst case better than $\Omega(n)$.

Overall, our results, which are summarized in Table 1, show some intrinsic and very sharp limitations of the local incremental approach for chordal edge-modification problems, which is currently the main algorithmic approach used for minimal edge-modification into classic graph classes, not only for chordal graphs. Interestingly, in the case of chordal graphs, these limitations appear very different depending on the type of edge modification considered. Therefore, our results call for a systematic study of the set of local incremental solutions for other graph classes and probably for a shift in the current algorithmic approach to minimal edge modifications into classic graph classes.

Outline of the paper. The paper is organized as follows. Section 2 provides an overview of the terminology. Section 3 introduces a family of graphs called fat-cycles and proves several key properties about local incremental inclusion-minimal solutions to the chordal modification of fat-cycles. Results for chordal graph completion (restricting the solution to adding edges) and chordal graph deletion (restricting the solution to deleting edges) when there are no restrictions on the structure of G are given in Sections 4 and 5 respectively. The more general case of incremental chordal graph editing is addressed in Section 6. Finally, Section 7 discusses future directions.

2. Preliminaries

All graphs considered here are finite and simple, meaning that they are undirected, unweighted, and do not include multiple edges or self-loops. We use G to represent an arbitrary graph, V its vertex set and E (or E_G) its edge set. We will also use the notation $G = (V, E)$ and let n represent the cardinality $|V|$. An edge between vertices x and y will be denoted by (x, y) or equivalently (y, x). The open neighborhood of x is denoted by $N(x)$ (or $N_G(x)$) and its closed neighborhood by $N[x] = N(x) \cup \{x\}$. The sub-graph of G induced by the set of vertices $X \subseteq V$ is denoted by $G[X] = (X, \{(x, y) \in E \mid x, y \in X\})$.

Table 2: Minimum Cardinality for Each Type of editing

Graph (reference)	G_1 in Fig. 2 (Theorem 1)	G_2 in Fig. 3 (Theorem 3)	G_3 in Fig. 4 (Theorem 5)
Pure Completion	3	$\Omega(n)$	$\Omega(n)$
Pure Deletion	$\Omega(n)$	1	$\Omega(n)$
General editing	3	1	6

A *clique* $K \subseteq V$ is a set of vertices that are pairwise adjacent. A *simplicial vertex* is a vertex x whose neighborhood $N(x)$ is a clique. A *chord* of a cycle is an edge that joins two non-consecutive vertices in the cycle. A *chordal graph* $G = (V, E)$ is a graph in which every cycle of length four or more has a chord.

Definition 1 (editing, completion, deletion). *A* chordal editing *H of a graph $G = (V, E_G)$ is a chordal graph $H = (V, E_H)$ on the same vertex set as G. The difference between E_G and E_H, represented by $M_{G,H} = E_H \triangle E_G$, is called the* modification set*. Where the graphs G and H are clear, we will simply use M to represent the modification set. Two standard variations on chordal editings are defined as follows.*

- *A* chordal completion *of G is a chordal editing H in which the modification set $M \cap E_G = \varnothing$. In the case of chordal completions, the edges in M are called* fill edges*.*
- *A* chordal deletion *of G is a chordal editing H for which the modification set $M \subseteq E_G$.*

Definition 2 (inclusion-minimal, minimum). *An* inclusion-minimal *editing (completion/deletion) H of G is one for which its modification set $M_{G,H}$ is minimal for inclusion among all chordal editings (completions/deletions) of G. In other words, no proper subset of $M_{G,H}$ yields a chordal modification of G.*
A minimum *editing (completion/deletion) H^{opt} of G is one for which its modification set $M_{G,H^{opt}}$ has minimum cardinality among all chordal editings (completions/deletions) of G.*

Clearly, editing is a generalization of both completion and deletion, as a completion or a deletion of a graph G is also an editing of G. In the case of chordal graphs, this generalization is strict in the sense that the minimum size of a chordal editing may be negligible compared to both the minimum size of a chordal completion and the minimum size of a chordal deletion (Column G_3 of Table 2). On the other hand, chordal completion and chordal deletion are incomparable in the sense that there are graphs for which the minimum size of a chordal completion is negligible compared to the minimum size of a chordal deletion (Column G_1 of Table 2), and vice versa (Column G_2 of Table 2).

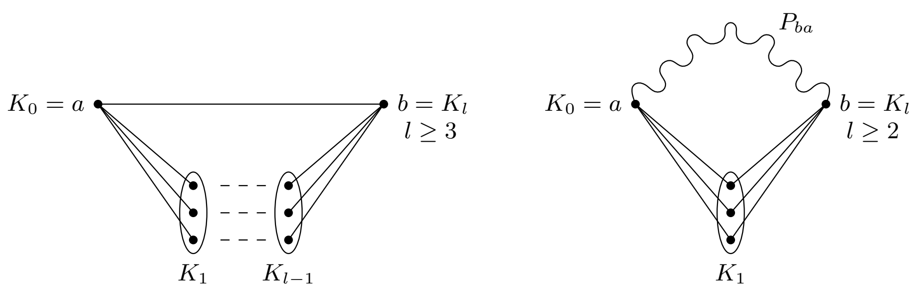

Figure 1: **(a).** Condition 1 of Lemma 1: $(a, b) \in E_H$ and $l \geq 3$.
Fig. 1. (b). Condition 2 of Lemma 1: $(a, b) \notin E_H$ and there exists a path P_{ba} from b to a in $H[K_l, \dots, K_{p-1}, K_0]$.

As explained previously, our goal is to determine the worst case performance of chordal edge modification algorithms using the main approach currently available to solve inclusion-minimal edge modification problems into graph classes that have the hereditary property. In the remainder of this article, we refer to this algorithmic approach as the *local incremental approach*. It is formally defined as follows.

Definition 3 (local incremental approach). *The* local incremental approach *to finding a chordal editing (completion/deletion) of a graph $G = (V, E_G)$ processes the vertices in some specified order $x_1, x_2, ..., x_n$, determining a series of disjoint modification sets $M_1, M_2, \dots, M_n$ at each incremental step such that $M_1 = \varnothing$ and the two following properties are satisfied for all $i \geq 2$, using notations $V_i = \{x_1, \dots, x_i\}$ and graph $G'_i = (V_i, E_{G[V_i]} \triangle \bigcup_{1 \leq j \leq i-1} M_j)$:*

1. *all modifications in M_i are incident to x_i, i.e. $M_i \subseteq \{x_i\} \times V_{i-1}$, and*
2. *M_i is an inclusion-minimal set of modifications such that the graph $H_i = (V_i, E_{G'_i} \triangle M_i)$ is chordal.*

The set $M = \bigcup_{1 \leq i \leq n} M_i$ is the modification set output for G and $H_n = (V, E_G \triangle M)$ is the corresponding inclusion-minimal chordal editing (completion/deletion) of G.

3. Fat cycles

In this section we introduce a family of graphs that will provide the basis for the extreme examples in subsequent sections.

Definition 4 (fat cycle, thin clique, thick clique). *A* fat cycle *is a graph G defined by a circular sequence*[2] *$K_0, K_1, \dots K_{p-1}$ of non-empty cliques such*

[2]This means that all indices in the sequence are taken modulo p.

that for any $i \in [\![0, p-1]\!]$ and any $x \in K_i$, the neighbors of x outside K_i are exactly the vertices in $K_{i-1} \cup K_{i+1}$. The cliques in a fat cycle are characterized based on their cardinality.

- *A* thin clique *K_i has only one vertex, that is $|K_i| = 1$.*
- *A* thick clique *K_i has more than one vertex, that is $|K_i| > 1$.*

In the remainder of this section we prove properties about chordal editings, completions, and deletions of sufficiently large fat cycles (at least four cliques) that have at least two thin cliques.

Lemma 1 (fat cycle editing in the presence of 2 thin cliques). *Let G be a fat cycle defined by the circular sequence $K_0, K_1, \dots K_{p-1}$ where $p \geq 4$, and suppose $K_0 = \{a\}$ and $K_l = \{b\}$ for $2 \leq l \leq p-1$ are two thin cliques. Let $\mu = min\{|K_i| \mid i \in [\![1, l-1]\!]\}$. In any editing H of G such that one of the two following mutually-exclusive conditions holds (see Figure 1 for an illustration):*

1. *$(a, b) \in E_H$ and $l \geq 3$, or*
2. *$(a, b) \notin E_H$ and there exists a path from b to a in $H[K_l, \dots, K_{p-1}, K_0]$*

the modification set $M_{G,H}$ contains at least μ distinct modifications, each of which is incident on a vertex in $\bigcup_{i \in [\![1, l-1]\!]} K_i$.

Proof. If $(a, b) \notin E_H$ (Condition 2 of the lemma), let P_{ba} be a chordless path from b to a in $H[K_l, \dots, K_{p-1}, K_0]$ and let $P_{ba} = b, a$ otherwise, i.e. if $(a, b) \in E_H$ (Condition 1 of the lemma). By definition of μ, for any $i \in [\![1, l-1]\!]$, K_i contains at least μ distinct vertices, which we denote by x_j^i, for $j \in [\![1, \mu]\!]$. For any $j \in [\![1, \mu]\!]$, consider the vertex set defined by $C_j = P_{ba} \cup \{x_j^1, \dots, x_j^{l-1}\}$. Observe that if $(a, b) \notin E_H$, then P_{ba} contains at least three vertices and hence C_j has at least four vertices. On the other hand, if $(a, b) \in E_H$ but $l \geq 3$, then C_j also has at least four vertices since there are at least two cliques K_1 and K_2 between K_0 and K_l.

As H is chordal, C_j does not induce a cycle in H, regardless of the choice of $1 \leq j \leq \mu$. But since P_{ba} is an induced path of H, it follows that H contains at least one modification of G among the vertices in C_j that is not between two vertices in P_{ba}. This modification is therefore incident to some vertex in $P_j = \{x_j^1, \dots, x_j^{l-1}\}$. As this holds for all $j \in [\![1, \mu]\!]$ and the vertices in P_j and $P_{j'}$ are distinct for $j \neq j'$, it follows that H contains at least μ modifications that are incident to at least one vertex of $\bigcup_{i \in [\![1, l-1]\!]} K_i$. ∎

Lemma 1 gives a general characterization of modification sets of a fat cycle G that has two thin cliques. The characterization is based on the presence or absence of an edge between the two thin cliques in the resulting chordal editing H. The result holds regardless of whether H is a pure completion, pure deletion, or editing. The next two corollaries to Lemma 1 refine the characterization based on the relative location of the two thin cliques.

Corollary 1 (fat cycle editing, consecutive thin cliques). *Let G be a fat cycle defined by the circular sequence $K_0, K_1, \ldots, K_{p-1}$ where $p \geq 4$ and suppose $K_0 = \{a\}$ and $K_{p-1} = \{b\}$ are two (consecutive) thin cliques in G. Let $\mu = min\{|K_i| \mid i \in [\![1, p-2]\!]\}$. Then*

1. *One of the minimum chordal editings of $G = (V, E_G)$ is the graph $H^{opt} = (V, E_G \triangle\, M_{G,H^{opt}})$ with $M_{G,H^{opt}} = \{(a, b)\}$, and*
2. *Any chordal editing $H = (V, E_G \triangle\, M_{G,H})$ of G for which $(a, b) \notin M_{G,H}$ has cardinality $|M_{G,H}| \geq \mu$. In other words, if the edge (a, b) is not deleted in H, then there are at least μ modifications in $M_{G,H}$.*

Proof. Since G is not chordal, $M_{G,H^{opt}}$ contains only one modification, and the resulting graph H^{opt} is clearly chordal, it follows that H^{opt} is a minimum chordal editing of G, proving Part 1. Since $p-1 \geq 3$, Part 2 follows immediately from Condition 1 of Lemma 1 with $K_0 = \{a\}$ and $K_l = K_{p-1} = \{b\}$. ∎

Note that, because a deletion is a particular editing, the statement of Corollary 1 remains true when changing the term *editing* to the term *deletion*. However, because this statement involves an edge deletion, this result does not apply to chordal completions of fat cycles.

Corollary 2 (fat cycle editing, non-consecutive thin cliques). *Let G be a fat cycle defined by the circular sequence $K_0, K_1, \ldots, K_{p-1}$ where $p \geq 4$ and suppose $K_0 = \{a\}$ and $K_l = \{b\}$ for $2 \leq l \leq p-2$ are two thin cliques in G. Let $\mu_1 = min\{|K_i| \mid i \in \{1, \ldots, l-1\}\}$, $\mu_2 = min\{|K_i| \mid i \in \{l+1, \ldots, p-1\}\}$, and $\mu = min\{\mu_1, \mu_2\}$. Any chordal editing $H = (V, E_G \triangle\, M_{G,H})$ of G for which $(a, b) \notin M_{G,H}$ has cardinality $|M_{G,H}| \geq \mu$. In other words, if the edge (a, b) is not a fill edge in H, then there are at least μ modifications in $M_{G,H}$.*

Proof. Note that since $(a, b) \notin M_{G,H}$ and since $(a, b) \notin E_G$, then $(a, b) \notin E_H$. Now if there exists a path from b to a in $H[K_l, \ldots, K_{p-1}, K_0]$, then by Condition 2 of Lemma 1 it must be that $|M_{G,H}| \geq \mu_1 \geq \mu$. On the other hand, if there does not exist any path from b to a in $H[K_l, \ldots, K_{p-1}, K_0]$, since each of the cliques $K_{l+1}, \ldots, K_{p-1}$ contains at least μ_2 vertices, then $G[K_l, \ldots, K_{p-1}, K_0]$ contains at least μ_2 paths from b to a such that the internal vertices of any two distinct paths are pairwise disjoint. It follows that $M_{G,H}$ must contain at least μ_2 deletions of edges, at least one for each path, and hence $|M_{G,H}| \geq \mu_2 \geq \mu$ in this case as well. ∎

4. Incremental chordal completion

In this section we address the effectiveness of the local incremental approach to chordal completion. We begin by showing that there are graphs that have a unique minimum chordal completion that is $\Omega(n)$ better than any other inclusion-minimal chordal completion of the same graph.

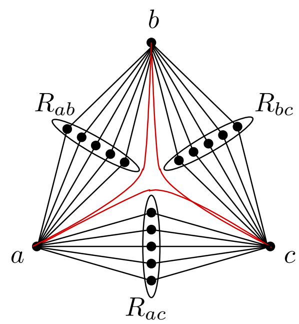

Figure 2: Graph G_1 is shown in black. It is a fat cycle $\{a\}, R_{ab}, \{b\}, R_{bc}, \{c\}, R_{ac}$. Each of the thick cliques R_{ab}, R_{bc}, and R_{ac} has $(n-3)/3$ vertices. When $n \geq 15$, G_1 admits a unique minimum chordal editing that adds the three fill edges shown in red.

Theorem 1. *For any $n \geq 15$, the graph G_1 depicted in Figure 2 has a unique minimum chordal editing which consists of the three fill edges in $M^{opt} = \{(a,b),(b,c),(a,c)\}$. Furthermore, any chordal editing of G_1 that does not contain all three fill edges in M^{opt} has cardinality at least $(n-3)/3 = \Omega(n)$.*

Proof.
Clearly the graph $H^{opt} = (V, E_{G_1} \triangle M^{opt})$, where $M^{opt} = \{(a,b),(b,c),(a,c)\}$, is chordal. Moreover, since $n \geq 15$, then $(n-3)/3 \geq 4$ and therefore the truth of the first sentence of the statement of the theorem will follow from the truth of the second sentence, which we will now prove.

First, let $\mu = |R_{ab}| = |R_{bc}| = |R_{ac}| = (n-3)/3$, and $H = (V, E_{G_1} \triangle M_{G_1,H})$ be a chordal editing of $G_1 = (V, E_{G_1})$ that does not include, without loss of generality, the edge (a,b). Now consider two possibilities: either there is a path from b to a in $H[\{b\} \cup R_{bc} \cup \{c\} \cup R_{ac} \cup \{a\}]$, or there is no such path. We will show that in either case $M_{G_1,H}$ contains at least $\mu = \Omega(n)$ modifications, thereby proving the theorem. If the path exists in H, then Condition 2 of Lemma 1 holds, and hence $|M_{G_1,H}| \geq \mu$. If, on the other hand, there is no such path, then the modification set $M_{G_1,H}$ must break all such paths in G_1. Label the vertices $R_{bc} = \{x_b^1, x_b^2, \ldots, x_b^\mu\}$ and $R_{ac} = \{x_c^1, x_c^2, \ldots, x_c^\mu\}$ and consider the μ paths from b to a defined by $P_j = b, x_b^j, c, x_c^j, a$, for $1 \leq j \leq \mu$. The only way to break the path P_j is to delete at least one of its edges. Since the P_j paths are pairwise disjoint relative to their sets of edges, it follows that at least μ edges must be deleted, one for each path. Hence, we have again that $M_{G_1,H}$ includes at least μ distinct modifications. The theorem follows. ■

Since the optimal editing of G_1 is a pure completion, Theorem 1 applies to completions as well. That is, we can say that, when $n \geq 15$, the graph G_1 depicted in Figure 2 has a unique minimum chordal completion, which consists of the three edges in M^{opt}, and moreover all other inclusion-minimal completions of G_1 have cardinality $\Omega(n)$. The next result shows that the unique minimum chordal completion of G_1 cannot be obtained, or even approached to within a factor better than $\Omega(n)$ by the local incremental approach, which is the purpose of this section. Note that Theorem 2 is stated in terms of editing instead of completion because we will use it in a more general context in the sequel. But since completion is a particular case of editing, it implies the same result for chordal completion.

Theorem 2. *It is impossible to obtain the unique minimum editing of the graph G_1 using the local incremental approach. Furthermore, all inclusion-minimal editings of G_1 that can be obtained by the local incremental approach have cardinality at least $\Omega(n)$, as compared to $|M^{opt}| = 3$.*

Proof. First, note that it follows from Theorem 1 that the unique minimum completion of G_1 has the modification set $M^{opt} = \{(a,b),(a,c),(b,c)\}$. By way of contradiction, assume $x_1, x_2, \cdots, x_n$ is a sequence of the vertices in V that produces the fill edges M^{opt} incrementally and without loss of generality, assume a comes before b and b comes before c in the sequence. Observe that the only chordless cycles in G_1 of length greater than 3 go through all of a, b, and c. Thus, in an inclusion-minimal incremental approach no fill edge can be added before c is processed. But then the local incremental approach will never add the fill edge (a,b) using the given sequence. This is a contradiction since the unique minimum completion of G_1 includes the edge (a,b).

The fact that all completions of G_1 that can be obtained incrementally have cardinality at least $\Omega(n)$ follows immediately from the second sentence in the statement of Theorem 1. ∎

5. Incremental chordal deletion

In this section we turn our attention to the effectiveness of the local incremental approach to chordal deletion. As we will see, the results for chordal deletion are substantially different than those for chordal completion. We begin, however, with a result that is similar to that of Theorem 1.

Theorem 3. *For any $n \geq 6$, the graph G_2 depicted on Figure 3 has a unique minimum chordal editing, which consists of deleting the single edge in $M^{opt} = \{(a,b)\}$. Furthermore, all the other inclusion-minimal editings of G_2 have cardinality $\Omega(n)$.*

Proof. Since the fat cycle $\{a\}, \{b\}, R_b, R_a$ satisfies the premise for Corollary 1, it follows from Part 1 that $H^{opt} = (V, E_{G_2} \triangle M^{opt})$ with $M^{opt} = \{(a,b)\}$ is a minimum chordal editing of G_2. Moreover, Part 2 of Corollary 1 guarantees that any other inclusion-minimal editing H of G_2 has $|M_{G_2,H}| \geq |R_a| = (n-2)/2 = \Omega(n)$, since $|R_a| = |R_b| = \min\{|R_a|, |R_b|\}$. The uniqueness of M^{opt} follows. ∎

Since the optimal editing for G_2 is a deletion, Theorem 3 applies to deletions as well. That is, we can say that any graph G_2 that has the structure depicted

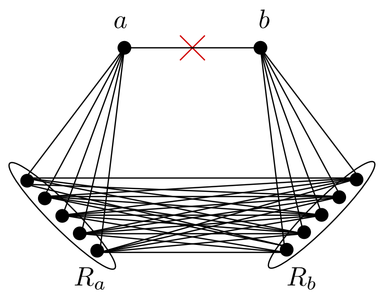

Figure 3: Graph G_2 is shown in black. It is a fat cycle $\{a\}, \{b\}, R_b, R_a$. The thick cliques R_a and R_b each have $(n-2)/2$ vertices, where $n \geq 6$. For $x \in R_a$ and $y \in R_b$, $(x, y) \in E_{G_2}$. When $n \geq 6$, G_2 admits a unique minimum editing that deletes the edge (a, b).

in Figure 3 has a unique minimum chordal deletion, which consists of deleting the single edge (a, b), and moreover all the other inclusion-minimal deletions of G_2 have cardinality $\Omega(n)$. We already used this result to fill in column G_2 of Table 2 and we will use it in the next section in the more general context of chordal editing.

We now come back to the main purpose of this section and we show that, unlike the negative result about chordal completion that we obtained in the previous section, for any graph G it is in fact *possible* to produce a minimum chordal deletion of G using the local incremental approach. The result follows from the fact that every inclusion-minimal chordal deletion of a graph can be produced using the local incremental approach, which is stated by the following lemma.

Lemma 2. *For any inclusion-minimal chordal deletion $H = (V, E_G \triangle M_{G,H})$ of a graph $G = (V, E_G)$, there exists an order $x_1, x_2, \ldots, x_n$ of the vertices in V and choices for inclusion-minimal deletion sets M_i at each incremental step i, such that the resulting modification set $M = \bigcup_{1 \leq i \leq n} M_i$ is precisely $M_{G,H}$.*

Proof. Consider a reverse perfect elimination ordering π of the vertices of H, i.e. an order $x_1, x_2, \ldots, x_n$ such that for all $i \in [\![1, n]\!]$, x_i is simplicial in $H[x_1, x_2, \ldots, x_i]$. It is well known since [24] that a graph is chordal if and only if it admits such an ordering. To prove the theorem, it is sufficient to prove that for all $i \in [\![2, n]\!]$, $N_{H[x_1,\ldots,x_i]}(x_i)$ is an inclusion-minimal deletion of the neighborhood $N_{G[x_1,\ldots,xi]}(x_i)$ making $H[x_1, \ldots, x_{i-1}, x_i]$ chordal, i.e. the modification set defined by H at step i satisfies the constraints of the local incremental approach. This is what we do now.

Assume for the sake of contradiction that this is not true for all $i \in [\![2, n]\!]$ and consider the smallest index l such that this does not hold, i.e. $N_{H[x_1,\ldots,x_l]}(x_l)$ is not an inclusion-minimal chordal deletion of the neighborhood of $N_{G[x_1,\ldots,xl]}(x_l)$. As $N_{H[x_1,\ldots,x_l]}(x_l)$ is clearly a chordal deletion of the neighborhood of x_l (H is chordal, which is a hereditary property), it follows that the reason why the property is not satisfied for index l is that $N_{H[x_1,\ldots,x_l]}(x_l)$ is a deletion of the neighborhood $N_{G[x_1,\dots,xl]}(x_l)$ but is not inclusion-minimal. Then, consider a modified neighborhood $N_l \supsetneq N_{H[x_1,\dots,x_l]}(x_l)$ for x at step l that is obtained from an inclusion-minimal deletion of the neighborhood of x.[3]

Now, consider the graph H' defined by $E_{H'} = E_H \cup (N_l \setminus N_{H[x_1,\dots,x_l]}(x_l))$. Since N_l is obtained by deleting some edges in G, and so are the neighborhoods at all other incremental steps, by definition, then H' is obtained from G by only deleting edges. Moreover, since $N_l \supsetneq N_{H[x_1,\dots,x_l]}(x_l)$, H' is a deletion of G which is strictly included in the one giving H. We will now show that H' is chordal, which will give an immediate contradiction with the fact that H is an inclusion-minimal chordal deletion of G.

To prove that H' is chordal, we exhibit a reverse perfect elimination ordering of its vertices, denoted σ. As we noted above, by definition, $H'[x_1, x_2, \dots, x_l]$ is chordal and thus admits a reverse perfect elimination ordering denoted $\sigma_l = x_{\sigma_l(1)}, x_{\sigma_l(2)}, \dots, x_{\sigma_l(l)}$. The beginning of σ is exactly σ_l and the rest of σ is defined as being the same as π. More explicitly, we have

$$\sigma = x_{\sigma_l(1)}, x_{\sigma_l(2)}, \dots, x_{\sigma_l(l)}, x_{l+1}, x_{l+2}, \dots, x_n.$$

In order to show that σ is a reverse perfect elimination ordering, we must show that for any $i \geq l+1$, x_i is simplicial in $H'[x_{\sigma_l(1)}, x_{\sigma_l(1)}, \dots, x_{\sigma_l(l)}, x_{l+1}, \dots, x_i] = H'[x_1, x_2, \dots, x_i]$. Note that the only difference between $H'[x_1, x_2, \dots, x_i]$ and $H[x_1, x_2, \dots, x_i]$ is that H does not contain the edges of $N_l \setminus N_{H[x_1,\dots,x_l]}(x_l)$, while H' does. Therefore, H' contains all the edges of H plus some others and these others are not incident to x_i. It follows that, since the neighborhood of x_i is a clique in $H[x_1, x_2, \dots, x_i]$, so it is in $H'[x_1, x_2, \dots, x_i]$. This shows that σ is a reverse perfect elimination ordering and therefore that H' is a chordal graph.

Recall that we pointed out earlier that H' is obtained by deleting some edges from G and that the set of edges of H' strictly contains that of H. This is a contradiction of the fact that H is an inclusion minimal chordal deletion of G. We therefore conclude that for all $i \in [\![2, n]\!]$, $N_{H[x_1,\dots,x_i]}(x_i)$ is an inclusion-minimal deletion of the neighborhood $N_{G[x_1,\dots,x_{i-1}]}(x_i)$ of x_i in $H[x_1, \dots, x_{i-1}] + x_i$, which achieves the proof of the lemma. ■

Lemma 2 gives us the following result for minimum chordal deletions.

Theorem 4. *For every minimum deletion $H^{opt} = (V, E_G \triangle\, M^{opt}_{G,H})$ of a graph $G = (V, E_G)$ there is an ordering of the vertices in V and a set of inclusion-minimal incremental choices $M_2, \dots, M_n$ that produces H^{opt}.*

Proof. The theorem follows immediately from Lemma 2. ■

[3]Note that the inclusion relationship between the set of deleted edges and the neighborhood obtained is reversed: if the set of deleted edges defining N_l is included in the one defining N, then N_l contains N.

With Theorem 4 we have shown that for every graph G, there exists a choice of the order in which to process the vertices of G and a choice of an inclusion-minimal set of deletions at each step so that the resulting chordal deletion of G has minimum cardinality. This is in contrast to Theorem 2, which showed that there are some graphs for which there do not exist any such choices that result in a chordal completion of G whose cardinality is within a factor smaller than $\Omega(n)$ of the minimum. However, note that the positive result we obtained for deletion does not provide any algorithm to determine the choices that will result in a minimum chordal deletion. Moreover, recall that making these choices deterministically in polynomial time is out of reach, unless P=NP, since the problem is NP-hard. Nevertheless, this result suggests exploring two exciting possibilities. The first is the possibility of designing an algorithm that is able to choose at random an inclusion-minimal set of deletions at each step of the local incremental approach. Starting with a random order of the vertices of G, this would give a non-null probability of discovering a minimum chordal deletion of G, which would be very nice for the heuristic approach. A second possibility is to deterministically, and in polynomial time, determine an order of the vertices and choices of deletions at each step such that the resulting inclusion-minimal chordal deletion of G is guaranteed to have a cardinality within a constant ratio of the minimum.

6. Incremental chordal editing

Graphs G_1 and G_2 introduced in previous sections are examples of graphs where the best chordal editing is a pure chordal completion and a pure chordal deletion, respectively. Then the question naturally arises as to whether there are graphs for which the best chordal editing requires both adding fill edges and deleting other edges? The graph G_3 shown in Figure 4 is such a graph, as stated by Theorem 5 below. Moreover, for G_3 the number of modifications obtained in the best pure completion and the best pure deletion are both $\Omega(n)$ times larger than the number needed for a chordal editing.

Theorem 5. *For any $n \geq 42$, the graph G_3 depicted in Figure 4 has a unique minimum chordal editing, which consists of the six modifications in $M^{opt} = \{(a_1, b_1), (b_1, c_1), (a_1, c_1), (a_1, a_2), (b_1, b_2), (c_1, c_2)\}$. Furthermore, any chordal editing of G_3 that does not contain one or more of the modifications in M^{opt} has cardinality $\Omega(n)$.*

Proof. We denote the vertices of four induced sub-graphs as follows:

- $V_1 = \{a_1, b_1, c_1\} \cup R_{a_1 b_1} \cup R_{b_1 c_1} \cup R_{a_1 c_1}$,
- $V_a = \{a_1, a_2\} \cup R_{a_1} \cup R_{a_2}$,
- $V_b = \{b_1, b_2\} \cup R_{b_1} \cup R_{b_2}$, and
- $V_c = \{c_1, c_2\} \cup R_{c_1} \cup R_{c_2}$.

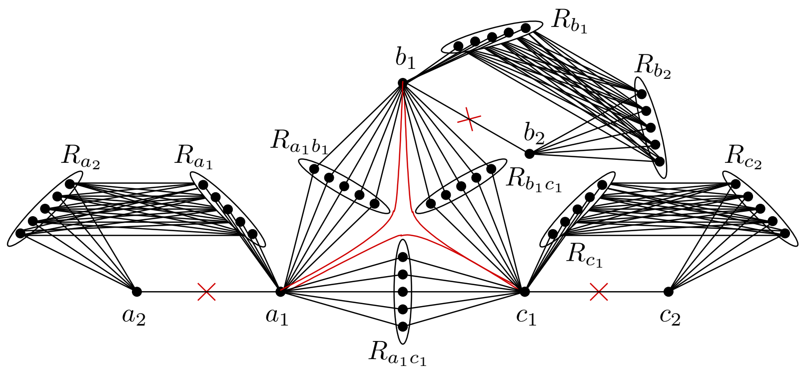

Figure 4: Graph G_3 is shown in black. The fat cycle $\{a_1\}, R_{a_1b_1}, \{b_1\}, R_{b_1c_1}, \{c_1\}, R_{a_1c_1}$ is combined with three fat cycles $\{x_1\}, \{x_2\}, R_{x_2}, R_{x_1}$ for $x \in \{a, b, c\}$. Each of the thick cliques has $(n-6)/9$ vertices. When $n \geq 42$, G_3 admits a unique minimum editing with the 3 deleted edges and 3 fill edges shown in red.

Note that $G_3[V_1]$ induces the graph G_1, which by Theorem 1 has a unique minimum editing with modification set $M_1^{opt} = \{(a_1, b_1), (b_1, c_1), (a_1, c_1)\}$. Likewise, each $G_3[V_t]$, $t \in \{a, b, c\}$, induces the graph G_2, which by Theorem 3 has a unique minimum editing with modification set $M_t^{opt} = \{(t_1, t_2)\}$. Also observe that any two of the sets V_1, V_a, V_b and V_c do not share a pair of vertices: pairwise, their intersections contain at most one vertex. It follows that any chordal editing of G_3 contains at least 6 modifications, as it must contain a chordal editing of each of the four induced graphs, whose set of modifications are always pairwise disjoint. Moreover, as the minimum editing of each of the four induced sub-graphs are unique, if there exists a chordal editing of G_3 containing exactly 6 modifications, then it is necessarily the one obtained from the union of the minimum editing of each of the four induced graphs, i.e. the editing whose set of modifications is $M^{opt} = M_1^{opt} \cup M_a^{opt} \cup M_b^{opt} \cup M_c^{opt}$. Finally, observe that the modification set M^{opt} indeed yields a chordal graph, which proves the first sentence of the statement of the theorem.

Consider now a chordal editing $H = (V, E_G \triangle M_{G,H})$ that does not contain one of the edges in M^{opt}. Note that since chordality is hereditary, it suffices to consider the impact of missing the modifications in each induced sub-graph separately. If a modification in $H^{opt}[V_1]$ is missing, then by Theorem 1 $M_{G,H}$, must include $\Omega(|V_1|) = \Omega(n)$ modifications. And if a modification in $H^{opt}[V_t]$ for $t \in \{a, b, c\}$ is missing, then by Theorem 3, $M_{G,H}$ must include $\Omega(|V_t|) = \Omega(n)$ modifications. Thus, the second sentence in the theorem is true. ∎

Note that we already used Theorem 5 to fill in column G_3 of Table 2. We now use it to answer the question of whether a minimum chordal editing can always be obtained in an incremental way. As for pure completion, the answer is negative. Actually, the graph G_1 depicted in Figure 2, for which we proved in Section 4 that it is impossible to obtain its unique minimum chordal completion by the local incremental approach (Theorem 2), already answers this question. As the unique minimum chordal editing of graph G_1 is the same as its unique minimum completion (see Theorem 1), then it cannot be obtained by the local incremental approach. Theorem 2 even proved that any chordal editing of G_1 that can be obtained by the local incremental approach has a cardinality that is a factor of at least $\Omega(n)$ of the minimum.

It is worth pointing out that such a situation is not limited to graphs whose minimum chordal editings are also minimum chordal completions. Even in the case where the minimum chordal editing of the graph requires both the addition of fill edges and the deletion of other edges, it may happen that the minimum chordal editing cannot be obtained using an incremental approach. The theorem below illustrates this with the graph G_3 depicted in Figure 4.

Theorem 6. *It is impossible to obtain the unique minimum chordal editing $H^{opt} = (V, E_{G_3} \triangle\, M^{opt})$ of the graph G_3 using the local incremental approach. Moreover, all inclusion-minimal editings of G_3 that can be obtained by the local incremental approach have cardinality at least $\Omega(n)$, as compared to $|M^{opt}| = 6$.*

Proof. Note that it follows from Theorem 5 that the unique minimum completion of G_3 has the modification set M^{opt}, and that $\{(a_1, b_1), (b_1, c_1), (a_1, c_1)\} \subset M^{opt}$. Then, the arguments in the proof of Theorem 2 still apply here and show that it is impossible that all these three modifications can be obtained together by the local incremental approach. The fact that all editings of G_3 that can be obtained by this approach have cardinality at least $\Omega(n)$ follows immediately from the second sentence in the statement of Theorem 5. ∎

In the remainder of this section, we turn to a new question slightly different from those considered so far. This question comes from the fact that there exists an algorithm [13] that computes an inclusion-minimal chordal editing (with both added and deleted edges) of a graph by the local incremental approach. Nevertheless, it proceeds in a way which contains an additional restriction compared to the general local incremental approach as we define it here. At each step of the incremental algorithm, when considering vertex x_i, [13] computes two inclusion-minimal editings of the neighborhood of x_i: one that is a pure completion and one that is a pure deletion. Then, the algorithm chooses one of them and moves to the next step. We call this approach the *step-wise-uniform approach*. It is a particular case of the local incremental approach to editing. Clearly, the step-wise-uniform approach has the capability of producing an inclusion-minimal editing that has both fill edges and edges to be deleted. The question naturally arises then of whether this restricted approach can compute editings of cardinality as small as the smallest ones that can be obtained by the more general local incremental approach, or whether there is a gain in using both additions and deletions in the same incremental step. We answer this question by exhibiting the graph G_4 depicted in Figure 5. A minimum chordal editing of G_4 can be obtained by the local incremental approach, but all the chordal editings that can be obtained by the step-wise-uniform approach have

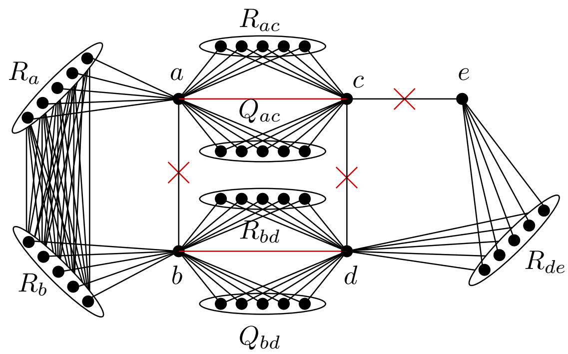

Figure 5: Graph G_4 is shown in black. Fat cycles $\{a\}, R_a, R_b, \{b\}$ and $\{d\}, \{c\}, \{e\}, R_{de}$ are combined with two fat cycles $\{x\}, R_{xy}, \{y\}, Q_{xy}$, for $x, y = a, c$ and $x, y = b, d$. Each of the thick cliques has $(n-5)/7$ vertices. When $n \geq 26$, G_4 admits a unique minimum editing with the 3 deleted edges and 2 fill edges shown in red.

a number of modifications that is at least $\Omega(n)$ times the minimum (Theorem 8 below). First, we show that the graph G_4 has a unique minimum editing.

Theorem 7. *For any $n \geq 26$, the graph G_4 depicted on Figure 5 has a unique minimum chordal editing, which consists of the 2 fill edges and 3 edges to be deleted in $M^{opt} = \{(a,c), (b,d), (a,b), (c,d), (c,e)\}$. Furthermore, any chordal editing of G_4 that omits any of the modifications in M^{opt} has cardinality at least $3 + (n-5)/7 = \Omega(n)$.*

Proof. We denote the vertices of four induced sub-graphs of G_4 as follows:

- $V_1 = \{a, b\} \cup R_a \cup R_b$,
- $V_2 = \{a, c\} \cup R_{ac} \cup Q_{ac}$,
- $V_3 = \{b, d\} \cup R_{bd} \cup Q_{bd}$, and
- $V_4 = \{c, d, e\} \cup R_{de}$.

First, observe that any two of the sets V_i and V_j, $i \neq j$ do not share a pair of vertices; in other words their intersection $V_i \cap V_j$ contains at most one vertex. Also observe that none of the induced subgraphs $G[V_i]$ is chordal. This implies that any chordal editing of G_4 contains at least one modification involving a pair of vertices from each V_i and that these modifications are distinct from each other. Therefore, any chordal editing $H = (V, E_{G_4} \triangle\ M_{G_4,H})$ of G_4 has $|M_{G_4,H}| \geq 4$. Moreover, we will show by contradiction that no chordal editing of G_4 has only four edges it its modification set.

Suppose $H = (V, E_{G_4} \triangle\ M_{G_4,H})$ is a chordal editing of G_4 such that $|M_{G_4,H}| = 4$. Since chordality is a hereditary property, necessarily, $M_{G_4,H}$ has exactly one modification in each of the induced graphs $G[V_i]$. Note that $G_4[V_1]$ induces the graph G_2, which by Theorem 3 has a unique minimum editing with modification set $M_1^{opt} = \{(a,b)\}$. Furthermore, from Corollary 2, each of $G[V_2]$ and $G[V_3]$ has a unique editing of cardinality 1, which consists of adding the fill edges (a, c) and (b, d), respectively. It follows that if $|M_{G_4,H}| = 4$, then $\{(a, b), (a, c), (b, d)\} \subset M_{G_4,H}$. From Lemma 1 applied to $G[V_4]$ with $K_0 = \{d\}$ and $K_l = \{e\}$, any editing of $G[V_4]$ that does not delete both edges (c, d) and (c, e) has cardinality at least $|R_{de}| \geq 3$. Since deleting only edge (c, d) or only edge (c, e) gives two distinct editings of $G[V_4]$ of cardinality 1, it follows that these are the only editings of $G[V_4]$ of cardinality 1. But neither of these two single edge modifications gives a chordal graph when associated with the three previous modifications, namely $\{(a, b), (a, c), (b, d)\}$, a contradiction. It follows that no chordal editing of G_4 has only 4 modifications.

Figure 5 exhibits an editing $H^{opt} = (V, E_G \triangle\ M^{opt})$ of G_4 with only five modifications, which is then a minimum cardinality editing of G_4. We will now show the second part of the statement of Theorem 7 holds, which will then imply that H^{opt} is the unique minimum cardinality editing of G_4 (indeed, note that since $n \geq 26$, then $3 + (n - 5)/7 \geq 6$).

To this purpose, let us consider an editing $H = (V, E_{G_4} \triangle\ M_{G_4,H})$ of G_4 that omits a modification in M^{opt}, and consider the cardinality of $M_{G_4,H}$ when once or more of the edges in M^{opt} is missing. We begin by observing that it must be that $\{(a, b), (a, c), (b, d)\} \subset M_{G_4,H}$, since otherwise $|M_{G_4,H}| \geq 3 + (n - 5)/7$:

- If $(a, b) \notin M_{G_4,H}$, then by Theorem 3 and the fact that chordality is hereditary, $|M_{G_4,H} \cap (V_1 \times V_1)| \geq (n - 5)/7$ and so $|M_{G_4,H}| \geq 3 + (n - 5)/7$, since, as explained above, each of V_2, V_3, V_4 contains at least one modification, which are pairwise distinct.
- If $(a, c) \notin M_{G_4,H}$ or $(b, d) \notin M_{G_4,H}$, then by Corollary 2 and the fact that chordality is hereditary, $|M_{G_4,H} \cap (V_2 \times V_2)| \geq (n-5)/7$ or $|M_{G_4,H} \cap (V_3 \times V_3)| \geq (n - 5)/7$, and again $|M_{G_4,H}| \geq 3 + (n - 5)/7$.

Consider now if $(c, d) \notin M_{G_4,H}$. Since $\{(a, b), (a, c), (b, d)\} \subset M_{G_4,H}$, but $(c, d) \notin M_{G_4,H}$, it must be that H has a path $P = a, c, d, b$ from a to b and Condition 2 of Lemma 1 holds for the fat cycle that is $\{b\}, R_b, R_a, \{a\}, \{c\}, \{d\}$ with $K_0 = \{b\}$ and $K_l = \{a\}$. Thus, by Lemma 1, $M_{G_4,H}$ contains at least $\min\{K_a, K_b\} = (n - 5)/7$ modifications incident to some vertices of $K_a \cup K_b$ and since $\{(a, b), (a, c), (b, d)\} \subset M_{G_4,H}$, then $M_{G_4,H} \geq 3 + (n - 5)/7$. Consequently, it must be that $\{(a, b), (a, c), (b, d), (c, d)\} \subset M_{G_4,H}$. Now the same argument can be used to show that by Lemma 1, if $(c, e) \notin M_{G_4,H}$, then $|M_{G_4,H}| \geq 3 + (n - 5)/7$. This proves the second part of the statement of Theorem 7 and since $3 + (n - 5)/7 \geq 6$this implies that H^{opt} is the unique minimum editing of G_4. ■

Theorem 8. *It is impossible to obtain the unique minimum chordal editing $H^{opt} = (V, E_{G_4} \triangle\ M^{opt})$ of the graph G_4 using a step-wise-uniform inclusion-minimal incremental approach, while it is possible to obtain H^{opt} if the modification set at each step can include both fill edges and edges to be deleted. Moreover, all editings of G_4 that can be obtained using a step-wise-uniform incremental approach have cardinality $\Omega(n)$, as compared to $|M^{opt}| = 5$.*

Proof. To see that it is possible to obtain H^{opt} incrementally with non-uniform modification set choices, consider an incremental sequence of vertices consistent with the following order of the cliques in G_4.

$$\{b\}, R_b, R_a, \{a\}, R_{ac}, Q_{ac}, \{c\}, R_{bd}, Q_{bd}, \{d\}, R_{de}, \{e\}$$

Then the modification set can be built by:

- deleting edge (a, b) when vertex a is processed,
- adding fill edge (a, c) when vertex c is processed,
- adding fill edge (b, d) and deleting (c, d) when vertex d is processed, and
- deleting edge (c, e) when vertex e is processed.

These inclusion-minimal choices build H^{opt} incrementally. Note, however, that the modification set when d is processed includes both a fill edge and an edge to be deleted.

We prove the remainder of the theorem by showing that any chordal editing $H = (V, E_{G_4} \triangle M_{G_4,H})$ obtained using step-wise-uniform, inclusion-minimal choices omits at least one of the modifications in M^{opt}, which, from Theorem 7, implies that $|M_{G_4,H}| = \Omega(n)$. Let H be an editing obtained using step-wise-uniform, inclusion-minimal choices and consider the last vertex α to be processed among those in $\{a, b, c, d\}$. Note that the modifications involving α in M^{opt} include both an edge addition and an edge deletion. Before α is processed, none of the adjacencies involving α can have been modified since α was not yet inserted in the graph. Similarly, after α is processed, since their two endpoints have been inserted, none of the adjacencies involving α can be changed anymore. Therefore, all the modifications of the adjacencies involving α are done exactly when α is processed and since H is step-wise-uniform, it follows that these modifications miss at least one of the modifications involving α in M^{opt}. Thus, from Theorem 7, $|M_{G_4,H}| = \Omega(n)$. ∎

7. Future work

The results in this paper demonstrate that the local incremental approach to chordal editing and chordal completion is guaranteed to produce, for some graphs, modification sets that are $\Omega(n)$ times worse than the minimum cardinality chordal editing, or chordal completion, for the same graph. This is a very sharp limitation on the quality of the solutions obtained by the local incremental approach.

The first crucial question that comes from our work is whether these limitations also hold for other target graph classes such as interval graphs, cographs, permutation graphs, comparability graphs, where the local incremental approach is often the only currently available approach to inclusion-minimal edge modification. Our results also suggest that developing other algorithmic approaches to inclusion-minimal edge modifications, for example not based on an incremental scheme, may be a key to substantially improving the quality of heuristic solutions for minimum edge modification problems.

In contrast, however, our results further show that in the case of chordal deletion, the local incremental approach may be able to produce good approximations of minimum chordal deletion. This gives rise to compelling questions about developing provably good approximation algorithms for chordal graph deletion using the local incremental approach. Note that for minimum chordal completion, it is already known that the minimum number k of fill edges can be approximated in polynomial time by a factor $8k$ [11] (i.e. the number of fill edges returned by the algorithm is at most $8k^2$) but cannot be approximated to within a constant factor [25] under the Small Set Expansion conjecture, which is a complexity conjecture implying P≠NP.[4] At the same time, to the best of our knowledge the question of approximating the minimum number of modifications in a chordal editing is still open. Here, we showed that this cannot be accomplished to any better than $\Omega(n)$ of the optimal using the local incremental approach.

Interestingly, our results also suggest that chordal deletion has a behavior very different from chordal completion and chordal editing: it is possible that neither minimum completion nor minimum editing can be approximated to within a constant factor while minimum chordal deletion can be. As mentioned previously, the question of whether the latter can be accomplished using the local incremental approach is also open.

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