Planar graphs in Homotopy Type Theory

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Abstract

We consider a characterization of planar graphs in Homotopy Type Theory (HoTT). Using basic concepts from HoTT, such as univalence, one can define a type of graphs, such that equality (in the sense of the identity type) between graphs coincides with isomorphism. For planarity, we take inspiration from topological graph theory, in particular, combinatorial embeddings of graphs into surfaces. A proof-assistant for dependent type theory with HoTT support (Agda) is used to verify the correctness of this work in progress.

Introduction. In Graph theory, a graph is *planar* when it can be embedded into the plane. There are many characterizations of planar graphs [2, 4], e.g. forbidden minors $K_{3,3}$ and K_5). For our definition, we are taking inspiration from Topological Graph Theory [1] where one works with combinatorial embeddings which represent embeddings of graphs into surfaces up to isotopy. A graph is planar if and only if it can be embedded into the sphere: if one has an embedding into the sphere, one can obtain an embedding into the plane, by puncturing the sphere and applying stereographic projection. We get a representation of planar graphs, up to isotopy, as a combinatorial embedding into the sphere, and keeping track of where the sphere was punctured.

In the following, we elaborate some of the concepts needed in order to formalize[1] the above characterization of planar graph in HoTT [3]. In particular, the notion of combinatorial embedding and which of those induce embeddings into the sphere. To start with, we fix a notion of graphs. In what follows, G and H are graphs and x, y or z are variables for nodes.

Graphs. We concern ourselves about *simple* and *undirected* graphs which can be formalized as the following type[2,3,4]:

$$\mathsf{Graph} :\equiv \sum_{\mathsf{N}:U} \sum_{\mathsf{E}:\mathsf{N}\to\mathsf{N}\to U} \mathsf{isSet}(\mathsf{N}) \times \prod_{x,y:\mathsf{N}} \mathsf{isProp}(\mathsf{E}\,x\,y) \times \prod_{x,y:\mathsf{N}} (\mathsf{E}\,x\,y \to \mathsf{E}\,y\,x).$$

Graph Homomorphisms. A *homomorphism* from G to H is a *pair* of functions (α, β) where $\alpha : \mathsf{N}_G \to \mathsf{N}_H$ acts on nodes and $\beta\,x\,y : \mathsf{E}_G\,x\,y \to \mathsf{E}_H(\alpha\,x)(\alpha\,y)$. When both functions are equivalences[5], such a map is called an *isomorphism*. As is typical of HoTT, the identity type on graphs is equivalent to the type of isomorphism between graphs[6].

Cyclic Orders. A *cyclic order* on a set A is a ternary relation R on A such that the binary relation $\mathsf{R}a$ is a total order for every $a : \mathsf{A}$ and for $a, b, c : \mathsf{A}$, if $a \neq b$, $\mathsf{R}abc$ implies $\mathsf{R}bca$.

Combinatorial Embeddings. A *combinatorial embedding* of a graph is a cyclic order on the star[7] of each node.

$$\mathsf{CombinatorialEmbedding}(G) :\equiv \prod_{x:\mathsf{N}_G} \mathsf{CyclicOrder}(\mathsf{Star}\,G\,x).$$

[1] Extra care is taken to choose types such that their identity types coincide with the natural notion of equivalences of the mathematical objects.

[2] For any type A, $\mathsf{isSet}(A) :\equiv \Pi_{x,y:A}\Pi_{p,q\,:\,x=y}\,(p = q)$.

[3] For any type A, $\mathsf{isProp}(A) :\equiv \Pi_{x,y:A}\,(x = y)$.

[4] A *node* is of type N_G and an *edge* between nodes x, y is of type $\mathsf{E}_G\,x\,y$.

[5] Equivalence of $\alpha : \mathsf{N}_G \to \mathsf{N}_H$ is bijection and for $\beta\,x\,y$ is bi-implication.

[6] The natural map $\Pi_{G,H:\mathsf{Graph}}\,(G = H) \to (G \cong H)$ is an equivalence.

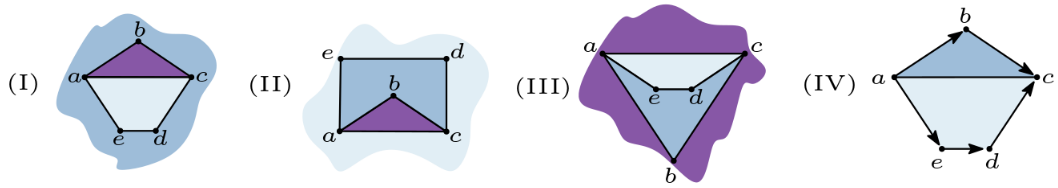
[7] $\mathsf{Star}_G(x) :\equiv \Sigma_{y:\mathsf{N}_G}\,\mathsf{E}_G\,y\,x$.

A combinatorial embedding defines an embedding of the graph into a closed surface but *which* surface is left implicit by the definition. However, the surface can be reconstructed from a combinatorial embedding by the notion of a face. We then recognize when the resulting surface is a sphere by using the fact the sphere is simply connected.

Cyclic Graphs. Identifying Fin_n with the set of $\{0, \cdots, n-1\}$, consider the function $S : \mathsf{Fin}_n \to \mathsf{Fin}_n$ which maps $(n-1 \mapsto 0,\ i \mapsto i+1)$. This function is an equivalence. From the function S, we construct the graph $\mathsf{C_n}$ for $n \geq 3$ with nodes Fin_n and edges from i to $S\ i$ for all $i : \mathsf{Fin}_n$. The type of *cyclic graphs* is the connected components of $(\mathsf{C_n}, S)$ in the type of all such structures[8]. A *cycle* in a graph G is cyclic graph H along with a homomorphism $H \to G$.

Corners. On the star x, a *corner* is a relation between two edges e_1,e_2 which satisfies there is no other edge in between. The corner is denoted by $e_1 \prec_{\mathsf{Star}_G(x)} e_2$.

Faces. A *face* is a cycle where all consecutive edges are corners and each corner occurs at most once. For example, the graph (I) below with the indicated combinatorial embedding has three faces with three, four, and five edges respectively.



Spherical Graphs. A combinatorial embedding of a graph is *spherical* if any walk[9] w_1 can be obtained by *deforming along faces* any other walk w_2 with the same endpoints. For example in (IV), the walk $a-b-c$ can be deformed into the walk $a-e-d-c$ along the triangular face abc and the face $acde$.

Planar Graphs. As spherical combinatorial embeddings, (I) (II) and (III) are all equivalent. To distinguish these as embeddings into the plane, we can keep track of a face, designated as the *outer face*. We require for technical reasons that planar graphs are connected.

$$\mathsf{Planar}(G) :\equiv \sum_{e:\mathsf{CombinatorialEmbedding}_G} \mathsf{Spherical}\,G\,e \times \mathsf{Face}\,G\,e \times \mathsf{Connected}\ G.$$

Conclusions. The predicate Planar is not a proposition, in fact we expect to have elements representing all embeddings of the graph into the plane, identified up to isotopy. We would like to compare this approach with other characterizations, of planarity. While we here focus on planar graphs, it is also possible to consider graphs with rotation system as a way to specify any closed orientable surface. Constructing surfaces in this way, using higher inductive types, is another interesting line of investigation.

References

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[8] $\mathsf{CyclicGraph} :\equiv \Sigma_{A:\mathsf{Graph}} \Sigma_{\varphi:A\to A} \Sigma_{n:\mathbb{N}} \parallel (A, \varphi) = (\mathsf{C_n}, S) \parallel$.

[9] A *walk* from x to y is a sequence of edges $e_1 : \mathsf{E}_G\,x\,x_1, \cdots, e_i : \mathsf{E}_G\,x_{i-1}\,x_i, e_i : \mathsf{E}_G\,x_i\,x_{i+1}, \cdots, e_n : \mathsf{E}_G\,x_k\,y$.