INF220, Autumn 2003: Exercise 5

1. A group is a structure with one sort and a binary associative operation with identity and inverse. We may specify groups in the logic EL by the signature

$$\begin{aligned}\Sigma = (\{S\}, \quad \{ & \_\circ\_ : S \times S \to S, \\ & 0 : \quad \to S, \\ & \_^{-} : S \to S \ \})\end{aligned}$$

   and axioms (sentences, formulas) $\Phi \subseteq EL(\Sigma)$,

$$\begin{aligned}\Phi = \{ \ \forall \ \{x : S, y : S, z : S\} \ . & \ x \circ (y \circ z) = (x \circ y) \circ z, \\ \forall \ \{x : S\} \ . & \ 0 \circ x = x, \\ \forall \ \{x : S\} \ . & \ x \circ 0 = x, \\ \forall \ \{x : S\} \ . & \ x \circ (x^{-}) = 0, \\ \forall \ \{x : S\} \ . & \ (x^{-}) \circ x = 0 \ \}.\end{aligned}$$

   Show that for any group $G \in \mathbf{Mod}_{\Sigma}(\Phi)$ the equations $\Psi \subseteq EL(\Sigma)$ hold, where

$$\begin{aligned}\Psi = \{ \ \forall \ \{x : S\} \ . & \ (x^{-})^{-} = x, \\ \forall \ \{x : S, y : S\} \ . & \ (x \circ y)^{-} = y^{-} \circ x^{-} \ \}.\end{aligned}$$

   In other words, show that each of the formulas in Ψ follows from the formulas Φ.

   Note that an algebra A satisfies a set of equations Φ if it satisfies each of the formulas in the set:

$$A \models \Phi \quad \Leftrightarrow \quad (\forall \varphi \in \Phi \cdot A \models \varphi). \tag{1}$$

   The class $\mathbf{Mod}_{\Sigma}(\Phi) = \{A \in \mathbf{Alg}(\Sigma) \mid A \models \Phi\}$.

2. From the book: 5.2-1, 5.2-2, 5.2-3, 5.2-4, 5.2-5.