# I220, Autumn 2003: Exercise 2

1. From the book: 2.6-7.

2. Given the signature $\mathsf{NN}$

$$\begin{array}{llcl} \textbf{sorts } \mathrm{N} & & & \\ \textbf{ops } 0: & & \to & N \\ s: & N & \to & N \\ \_+\_: & N \times N & \to & N \end{array}$$

   And algebras $A$ and $B$, defined by

$$\begin{array}{rcl} A(N) & = & \mathbb{N} = \{0, 1, 2, 3, \ldots\}, \\ A(0) & = & 0, \\ A(s)(a) & = & a + 1, \\ A(+)(a, b) & = & a + b, \end{array}$$

   and

$$\begin{array}{rcl} B(N) & = & \{\mathsf{false}, \mathsf{true}\}, \\ B(0) & = & \mathsf{false}, \\ B(s)(a) & = & \mathsf{not}\ a, \\ B(+)(a, b) & = & a\ \mathsf{xor}\ b = \begin{cases} \mathsf{false} & \text{if } a = b \\ \mathsf{true} & \text{if } a \neq b \end{cases} \end{array}$$

   Given variables $X$ where $X_N = \{x, y\}$, terms

$$s(s(x)) = y, \quad (1)$$
$$x + s(y) = s(x + y), \quad (2)$$

   and let $\alpha_1 : X \to A$ and $\alpha_2 : X \to A$ be assignments s.t.

$$\alpha_1(x) = 0, \qquad \alpha_2(x) = 16,$$
$$\alpha_1(y) = 2, \qquad \alpha_2(y) = 28.$$

   (a) Show that the assignment $\alpha_1 : X \to A$ makes the equations (1) and (2) hold, while $\alpha_2 : X \to A$ invalidates (1) but holds for (2).

   (b) Define two other assignements $\alpha_3 : X \to A$ and $\alpha_4 : X \to A$, such that $\alpha_3$ makes both equations (1) and (2) hold, while $\alpha_4$ invalidates (1).

   (c) Define ground substitutions $\sigma_1, \sigma_2, \sigma_3, \sigma_4 : X \to T_{\mathsf{NN}}$ corresponding to the assignemts $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. That is, $A(\sigma_{i,N}(x)) = \alpha_{i,N}(x)$ and $A(\sigma_{i,N}(y)) = \alpha_{i,N}(y)$ for $i = 1, 2, 3, 4$.

   (d) Let $Y$ be a collection of variables such that $Y_N = \{y\}$ and define the substitution $\sigma : X \to T_{\mathsf{NN}(Y)}$ by

$$\begin{array}{rcl} \sigma(x) & = & y, \\ \sigma(y) & = & s(s(y)). \end{array}$$

   Show that any assignment $\alpha : Y \to A$ will make the equations (1)$\sigma$ and (2)$\sigma$ hold, i.e., apply the substitution $\sigma$ to both terms in each equation, and prove that the new equations will hold for any assignment $\alpha : Y \to A$.

   (e) Define a homomorphism $h : A \to B$.
   Show that $h$ is unique.

(f) Let the assignments $\beta_1, \beta_2, \beta_3, \beta_4 : X \to B$ be defined by $\beta_{i,N} = h_N \circ \alpha_{i,N}$ for $i = 1, 2, 3, 4$.
Show that both equations (1) and (2) hold for $\beta_1, \beta_2, \beta_3$, and that (2) holds for $\beta_4$.
If (1) also holds for $\beta_4$, define an assignment $alpha_5 : X \to A$ such that the assignment $h \circ alpha_5 : X \to B$ invalidates equation (1). Show that $h \circ alpha_5$ still makes (2) hold.

(g) Define the algebra $F$ for the signature $\mathsf{NN}$ such that $F(N) = \{\#\}$.
Show that any assignment $\alpha : X \to F$ makes both equations (1) and (2) hold.
Show that $F$ is a final algebra for the class of all algebras for the signature $\mathsf{NN}$.

(h) Define an algebra $I$ for the signature $\mathsf{NN}$ by using expression trees as the carrier, i.e., let $0 :\to N$ be the tree with just a root node (and no children), let $s : N \to N$ create a tree with the argument as a substree, and let $\_ + \_ : N \times N \to N$ form a tree with the two argument trees as left and right child, respectively.
Show that neither of the equations (1) and (2) will hold for any assignment $\alpha : X \to I$.

(i) Show that the equation $(1)\sigma$ will hold in $I$ for any assignment $\alpha : Y \to I$.
Show that there is no substitution $\sigma' : X \to T_{\mathsf{NN}(Z)}$ for any collection of variables $Z$ that will make equation $(2)\sigma'$ hold for $I$.

(j) Show that there is a unique homomorphism from the algebra $I$ to any algebra for the signature $\mathsf{NN}$.
What is such an algebra called?