I220, Autumn 2003: Exercise 1

1. Exercises from the book
   - 2.2-1
   - 2.3-2
   - 2.4-1
   - 2.4-2 (only the part for 2.3-2)
   - 2.4-3 (except (i))
   - 2.4-6

2. Given the signature BA

$$\begin{array}{lrlcl}
\textbf{sorts} & \mathrm{P} & & & \\
\textbf{ops} & \bot : & & \to & P \\
 & \top : & & \to & P \\
 & \cap : & P \times P & \to & P \\
 & \cup : & P \times P & \to & P
\end{array}$$

   Let M be any non-empty set (for the sake of the example you may choose $M = \{1, 2, 3\}$), and let $\wp(M)$ be its power set, i.e., $\wp(M) = \{X : X \subseteq S\}$. In the example

$$\wp(M) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

   (a) There is a very natural way of using the set $\wp(M)$ as the carrier for an algebra A for the signature BA, i.e., the algebra A is such that $A(P) = \wp(M)$. Give an interpretation of the operation declarations of BA in A.

   (b) Check whether your interpretation makes the following equations hold in your algebra (X, Y, Z are elements of the carrier set – the members of the power set $\wp(M)$, i.e., the subsets of M):

$$\begin{array}{rlrl}
1. & X \cup Y = Y \cup X & 4. & X \cap Y = Y \cap X \\
2. & (X \cup Y) \cup Z = X \cup (Y \cup Z) & 5. & (X \cap Y) \cap Z = X \cap (Y \cap Z) \\
3. & X \cup \bot = X & 6. & X \cap \top = X
\end{array}$$

   If this is not the case, define another algebra with $\wp(M)$ as the carrier set, so that these equations hold.

   (c) Extend BA to BB by adding a new operation $\bar{\ } : P \to P$. Find an interpretation for this operation in the algebra you have defined, so that the following equations hold: 7. $X \cup \overline{X} = \top$ 8. $X \cap \overline{X} = \bot$

3. Let *bool* $= \{\mathbf{tt}, \mathbf{ff}\}$ be a two element set. Repeat the previous exercise with this set instead of $\wp(M)$, i.e., turn the set *bool* (not its power set!) into an algebra for the signature BA so that it satisfies the axioms 1-6 (notice that now the variables X, Y, Z of the equations will range over the only two possible elements of *bool*).

   Then give an interpretation of the additional operation from BB so that axioms 7-8 are satisfied.