Time Efficient Gossiping in Known Radio Networks

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Abstract. We study here the gossiping problem (all-to-all communication) in known radio networks, i.e., when all nodes are aware of the network topology. We start our presentation with a deterministic algorithm for the gossiping problem that works in at most n units of time in any radio network of size n. This is an optimal algorithm in the sense that there exist radio network topologies, such as: a line, a star and a complete graph in which the radio gossiping cannot be completed in less then n units of time. Furthermore, we show that there isn't any radio network topology in which the gossiping task can be solved in time $< \lfloor \log(n-1) \rfloor + 2$. We show also that this lower bound can be matched from above for a fraction of all possible integer values of n; and for all other values of n we propose a solution admitting gossiping in time $\lceil \log(n-1) \rceil + 2$. Finally we study asymptotically optimal $O(D)$-time gossiping (where D is a diameter of the network) in graphs with max-degree $\Delta = O(\frac{D^{1-1/(i+1)}}{\log^i n})$, for any integer constant $i \geq 0$ and D large enough.

1 Introduction

The two classical problems of information dissemination in computer networks are the *broadcasting* problem and the *gossiping* problem. In the broadcasting problem, we want to distribute a particular message from a distinguished *source* node to all other nodes in the network. In the gossiping problem, each node v in the network initially holds a message m_v, and we wish to distribute all messages m_v to all nodes in the network. In both problems, we very often like to minimise the time needed to complete the tasks.

A radio network can be modelled as an undirected graph $G = (V, E)$, where V represents the set of nodes of the network and a set E contains (unordered) pairs of vertices, s.t., a pair $(v, w) \in E$, for some $v, w \in V$ iff nodes v and w can communicate directly. We say that all neighbours of a vertex $v \in V$ form the *range* of v. One of the radio network properties is that at any time step each processor is either in the transmitting mode or in the receiving mode. We also assume that all processors work synchronously, and if a processor u transmits a message m at time step t, the message reaches all nodes within its range at the same time step. However any of its neighbours v will receive message m successfully iff u is the only node that transmits to v (has v in its range) at time step t. Otherwise a *collision* occurs and the message m does not reach the node v.

In this paper we focus on gossiping algorithms that use the entire information about the network topology. Such topology-wise communication algorithms are useful in radio networks that have relatively stable topology/infrastructure. And as long as no

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changes occur in the network topology during the actual execution of the algorithm, the task of gossiping is completed successfully. Another interesting aspect of deterministic communication in known radio networks is its close relation to randomised communication in unknown radio networks.

1.1 Previous Work

Most of the work devoted to radio networks is focused on the broadcasting problem. In the model with known radio network topology Gaber and Mansour [8] showed that the broadcasting task can be completed in time $O(D + \log^5 n)$, where D is the diameter of the network. In other work of Diks *et al.* [7] we find efficient radio broadcasting algorithms for (various) particular types of network topologies. However in the general case, it is known that the computation of an optimal (radio) broadcast schedule for an arbitrary network is $NP - hard$, even if the underlying graph of connections is embedded in the plane [2, 12]. The gossiping problem was not studied in the context of known radio networks until very recent work of Gąsieniec and Potapov [10]. In that paper one can find a study on the gossiping problem in known radio networks, where each node transmission is limited to unit messages. In this model they proposed several optimal and almost optimal $O(n)$-time gossiping algorithms in various standard network topologies, including: lines, rings, stars and free trees. They also proved that there exists a radio network topology in which the gossiping (with unit messages) requires $\Omega(n \log n)$ time. A similar work, in model with messages of logarithmic size, can be found in [1], where Bar-Yehuda *et al.* study randomised multiple point-to-point radio communication, and in [5], where Clementi *et al.* consider simultaneous execution of multiple radio broadcast procedures. So far, the gossiping problem was mostly studied in the context of *ad-hoc* radio networks, where the topology of connections is unknown in advance. In this model, Chrobak *et al.* [4] proposed fully distributed deterministic algorithm that completes the gossiping task in time $O(n^{3/2} \log^3 n)$. For small values of diameter D, the gossiping time was later improved by Gąsieniec and Lingas [9] to $O(nD^{1/2} \log^3 n)$. Another interesting $O(n^{3/2})$-time algorithm, a tuned version of the gossiping algorithm from [4] can be found in [14]. A study on deterministic gossiping in unknown radio networks with messages of limited size can be found in [3]. The gossiping problem in *ad-hoc* radio networks attracted also studies on efficient randomised algorithms. In [4], Chrobak et al. proposed a $O(n \log^4 n)$ time gossiping procedure. This time was later reduced in [11] to $O(n \log^3 n)$, and very recently in [6] to $O(n \log^2 n)$.

1.2 Our Results

We start here with a proof that in any known radio network of size n, the gossiping task can be completed in at most n units of time. Please note here that we are interested in exact complexity (as oppose to the asymptotic complexity) since the design of an $O(n)$-time gossiping procedure is rather trivial. Our new algorithm is optimal in the sense that there exist radio network topologies including: lines, stars, but also complete graphs, in which the gossiping task cannot be completed in time $< n$. Further, we show that there isn't any radio network topology in which the gossiping task can be solved in time $<$

$\lfloor \log(n-1) \rfloor + 2$. We also show that this bound can be matched from above for a fraction of all possible integer values of n; and for all other values of n we propose a solution admitting gossiping in time $\lceil \log(n-1) \rceil + 2$. The second part of the paper is devoted to efficient gossiping in arbitrary graphs (as oppose to the worst case) graphs. This work is done in relation to [8], where the authors proposed an asymptotically optimal $O(D + \log^5 n)$-time broadcasting procedure in known radio networks. For obvious reasons, the diameter D is a natural lower bound for both the broadcasting and gossiping problems. In this paper we study a non-trivial class of graphs in which the gossiping can be done in time $O(D)$, i.e., in optimal asymptotic time. We first show that the gossiping can be performed in time $(2D-1)\Delta + 1$ in graphs with the maximal degree Δ. This result admits $O(D)$-time gossiping in all graphs with $D = O(1)$. Later we show how to perform gossiping in time $O(D)$ in any graph with $\Delta = O(\frac{D^{1-1/(i+1)}}{\log^i n})$, for any integer constant $i \geq 0$ and D large enough.

2 The Gossiping Algorithm in Time $\leq n$

Let the graph $G = (V, E)$ be the underlying graph of connections in a given radio network, where $n = |V|$ stands for the size (number of nodes) of the network. We assume that initially each node $v \in V$ holds a unique message m_v. The gossiping task (all-to-all communication) is performed in discrete *units of time*, also called *time steps*. At any unit of time, a node can be in one of the two transmission modes: either in the *receiving mode* or in the *transmitting mode*. A gossiping algorithm is understood to be a well defined transmission/reception procedure, for each node of the network. We assume here that during a single time step a transmitting node is allowed to send a combined message which includes all messages that it received so far. The running time of a gossiping algorithm is the smallest number of time steps in which the all-to-all communication is completed.

We say that a radio network has a radius k if there exists at least one node in V which is at distance $\leq k$ from all other nodes in the network, and there isn't any node in V at distance $< k$ from all other nodes in the network. We call this special node a *central node* and we denote it by c. Let m_v be a gossip message originated in a node $v \in V$. At any time step of the gossiping process M_v denotes the set of messages acquired by v until now. E.g., initially $M_v = \{m_v\}$. We also distinguish a set T of nodes that have not transmitted (at all) since the beginning of the gossiping process. Initially $T = V$. In a radio network with a radius k and a central node c, we can partition the set of nodes T into disjoint subsets $N_0, N_1, .., N_k$, such that the set $N_i = \{v : dist_G(v, c) = i\}$, for $i = 0, 1, 2, ..., k$, where $dist_G(u, v)$ stands for a length of a shortest path between nodes u and v in G. In other words, N_0 contains only the central node c, N_1 contains all neighbours of the central node, etc. A reasoning presented here is based on the notion of a *minimal covering set*. The minimal (in terms of inclusion) covering set C_i is a subset of N_i, s.t., every node in N_{i+1} is connected to some node in C_i, and a removal of any node from C_i destroy this property.

Observation 1 *Each node $v \in C_i$ is connected to some node $u \in N_{i+1}$, such that u is not connected to any other node in $C_i - \{v\}$.*

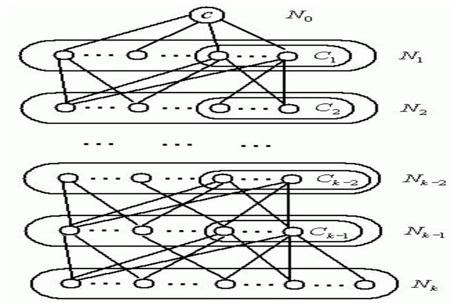

Fig. 1. Radio network with central node c and radius k

2.1 The Outline of the Algorithm

The general idea of the gossiping algorithm is as follows. Initially we point out that in any radio network with radius 1 and size n the gossiping task can be completed in time n. In radio networks with larger radii we show that there always exist four distinct nodes v, v', w and w', s.t., v can transmit (its current content) M_v to v' and w can transmit M_w to w' in the same time step, and that the removal of both v and w does not disconnect the remaining part of the network. We call this approach a *2-vertex reduction*. The gossiping algorithm uses the new approach for the purpose of moving all n messages to some connected subnetwork (of the original network) with radius 1 and size $n - 2k$, in exactly k units of time. This is followed by the gossiping in the subnetwork in at most $n - 2k$ units of time. Finally, we use the last k units of time to distribute the messages to all other nodes in the network by reversing the transmission direction in the initial k units of time. This means that the gossiping in any radio network of size n can be completed in time $\leq n$.

2.2 Gossiping in Radio Networks with Radius 1

We show here that in any radio network topology with radius 1 the gossiping algorithm can be completed in time n, where n is the size of the network. W.l.o.g., we assume that $n \geq 2$. According to the definition of a radio network with radius 1, we know that there exists a central node c which is at distance ≤ 1 from any other node in the network. In this case, we can transmit all messages m_v, s.t., $v \in V - \{c\}$ to the central node c, one by one, in time $n - 1$. When this stage is completed, $M_c = \{m_v : v \in V\}$. And we need one more time step to disseminate the content of M_c to all other nodes in the radio network.

2.3 Gossiping in Radio Networks with Higher Radii

We show here an efficient reduction of a network with an arbitrary radius k to its connected subnetwork with radius 1. This reduction is based on the concept of 2-vertex reduction, introduced at the beginning of this section.

Reduction 2⟶1. We prove here that any radio network with radius 2 can be efficiently reduced (by a sequence of 2-vertex reductions) to its subnetwork with radius 1. Given a radio network with radius 2. Initially, we choose a central node c and we construct sets N_0, N_1, N_2 and a covering set C_1. As long as $|C_1| \geq 2$, we can transmit successfully the contents of at least two nodes $v, w \in N_2$ to some nodes $v', w' \in C_1$ at each time step, thanks to the covering property of C_1. Please note that after each transmission nodes v, w are "removed" temporarily from the network and the content of the covering set C_1 is recomputed. When eventually $|C_1| < 2$ two cases apply. When $|C_1| = 0$ the radius of the subnetwork becomes 1, and the reduction is completed. If $|C_1| = 1$, we call the node remaining in C_1 an *essential node* e_1. Let Y_1 be the set of nodes in N_1 that are not neighbours of e_1. If both $|Y_1|, |N_2| > 0$, we can always chose $v \in Y_1$ and $w \in N_2$, where $v' = c$ and $w' = e_1$. If eventually $|N_2| = 0$ also $|C_1| = 0$, the reduction is completed. Alternatively when $|Y_1| = 0$, the essential node e_1 becomes a new central node in a subnetwork with radius 1.

Reduction k+1⟶k. We prove here that any radio network with radius $k + 1$ can be efficiently reduced (by a sequence of 2-vertex reductions) to its subnetwork with radius k, for any $k \geq 2$.

In a radio network with radius $k+1$, we initially choose a central node c and we construct the layer sets $N_0, N_1, N_2, \ldots, N_k, N_{k+1}$ and the covering sets $C_1, C_2, C_3, \ldots, C_{k-1}, C_k$. Let $\tilde{C}_i$ be a set $N_i - C_i$, for all $i = 1, 2, 3, \ldots, k-1, k$. Note, that as long as $|C_k| \geq 2$, we can transmit successfully the contents of at least two nodes $v, w \in N_{k+1}$ to some nodes $v', w' \in C_k$ during each time step, thanks to the covering property of C_k. When eventually $|C_k| < 2$ two cases apply. When $|C_k| = 0$, the radius of the subnetwork becomes k, and the reduction is completed. If $|C_k| = 1$, we call the node remaining in C_k an *essential node* e_k. By Y_k we denote the set of nodes in N_k that are not neighbours of e_k. If now $|\tilde{C}_1| + |\tilde{C}_2| + |\tilde{C}_3| +, \ldots, |\tilde{C}_{k-2}| + |\tilde{C}_{k-1}| + |Y_k| > 0$ and $|N_{k+1}| > 0$ we match any transmission from a node $v \in N_{k+1}$ to a node $v' \in C_k$ with a transmission from a node $w \in \tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3 \cup, \ldots, \tilde{C}_{k-2} \cup \tilde{C}_{k-1} \cup Y_k$ to its neighbour w' in the layer closer to the central node. This reduction process will terminate when eventually either: the set N_{k+1} is empty, which means that the radius of the subnetwork has been reduced to k; or at some point $|\tilde{C}_1| + |\tilde{C}_2| + |\tilde{C}_3| +, \ldots, |\tilde{C}_{k-2}| + |\tilde{C}_{k-1}| + |Y_k| = 0$. But then we know that the essential node e_k is a neighbour of all nodes in the set $N_k - \{e_k\}$. In this case, the essential node e_k becomes a new central node since its distance from any other node in the network is $\leq k$. The following theorem follows:

Theorem 1. *The gossiping task can be solved in any radio network of size n in at most n units of time.*

3 Optimal Topology for Gossiping in Radio Networks

In this section we present a radio network topology in which radio gossiping can be performed in time $\lceil \log(n-1) \rceil + 2$. We later present a simple argument that radio gossiping cannot be completed in time $< \lfloor \log(n-1) \rfloor + 2$. We conclude this section with presentation of more complex topology that allows to perform radio gossiping in time $\lfloor \log(n-1) \rfloor + 2$, for a fraction of all possible integer values of n.

3.1 Upper Bound

A topology of "gossiping-friendly" radio network supports the following transmission strategy. Initially collect (gather) all gossip messages in one distinguished node c and then distribute the messages to all other nodes as quickly as its is possible (i.e., in a single time unit). In order to achieve this goal we select a node (center) $c \in V$ and we connect it with any other node in V. The remaining $n-1$ vertices in $V - \{c\}$ are organised in an *optimal broadcasting tree* (OBT) used in the matching model, see, e.g., [13], with a root r.

We point out here, that an OBT of size m rooted in node r can also serve the purpose of gathering messages (in radio network model) in root r in the optimal time $\lceil \log m \rceil$. This is done by reversing all transmissions in time. I.e., if during broadcasting (in matching model) node v transmits to node w in step $i = 1, .., \lceil \log m \rceil$, in gathering algorithm (in radio network) node w transmits all messages it collected so far in step $\lceil \log m \rceil - i + 1$. Using this observation we show that we can collect all messages from $V - \{c\}$ in root r in time $\lceil \log(n-1) \rceil - i + 1$. These messages are later passed onto the central node c, who has all messages now. In the last step node c transmits a combined message (containing all messages) to all other nodes in one time step (since c is connected to every other node in the network). And the gossiping process is completed in time $\lceil \log(n-1) \rceil + 2$.

Lemma 1. *There exists a radio network topology in which the gossiping task can be completed in time* $\lceil \log(n-1) \rceil + 2$, *for any integer* n.

3.2 Lower Bound

On the other hand, note that during each consecutive round knowledge (a number of possessed messages) in each node can at most double. This means that after step i knowledge of any node is limited to 2^i original messages. Thus, after initial $\lfloor \log(n-1) \rfloor$ steps of any gossiping algorithm in any radio network topology, non of the nodes is completely informed, since $2^{\lfloor \log(n-1) \rfloor} < n$. Note also that during the last round of the gossiping process the only nodes that are permitted to transmit, are those who already possess all messages, since a transmitting node cannot receive messages at the same time. The following lemma follows.

Lemma 2. *The completion of the gossiping task in any radio network requires at least* $\lfloor \log(n-1) \rfloor + 2$ *steps.*

3.3 Tightening the Gap

We have just presented both the upper and the lower bounds for the most suitable topology for radio gossiping. Please note that the upper bound and the lower bound meet each other when $n = 2^k + 1$, for any integer k. For all other values of n the gap between the bounds is 1. This poses an interesting question, i.e., which of the two: $\lfloor \log(n-1) \rfloor + 2$ or $\lceil \log(n-1) \rceil + 2$ is the correct exact bound? In this section we show that the latter one is not. We propose more sophisticated radio network topology in which, for large enough n, s.t., $n \leq 2^k + 2^{k-3} - O(2^{\frac{k}{2}})$, the gossiping can be done in time $k + 2 = \lfloor \log(n-1) \rfloor + 2$.

Consider a network H which is composed of three components, see Figure 2:

- a tree T_1 with a root r_1 (and its two exact copies r_2 and r_3 including adjacent edges),
- a tree T_2 with a root h_3, and
- a group of *special* nodes: three *roots* r_1, r_2, r_3, three *central nodes* c_1,c_2,c_3, and three *helpers* h_1,h_2,h_3.

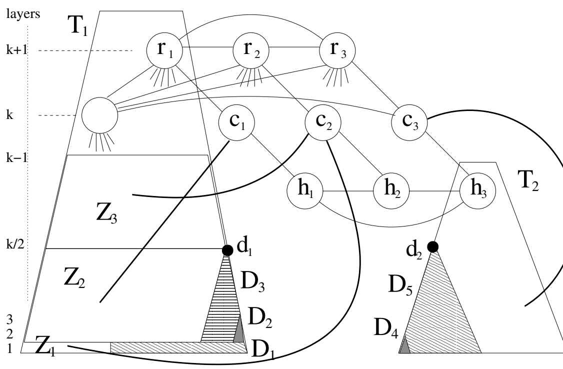

Fig. 2. The network topology for the optimal gossiping in time $\lfloor \log(n-1) \rfloor + 2$

The structure of trees T_1 and T_2 is very much based on the structure of the optimal broadcasting (in our case gathering) tree $\text{OBT}(d)$ which is of size 2^d and it contains $d+1$ (time) levels. The nodes in T_1 and T_2 that transmit in time step 1 during the gathering process are understood to be at layer 1, those that transmit in time step 2 at layer 2, etc, see Figure 2. Thus, layers in the trees T_1 and T_2 are enumerated in reverse order, comparing to the numbering of layers in standard OBTs. The tree T_1 is obtained from $\text{OBT}(k)$ via deletion of three sets of nodes D_1,D_2,D_3 (to be defined later) and all edges connected to them. The tree T_2 is obtained from $\text{OBT}(k-3)$ via deletion of two sets of nodes D_4, D_5 and all edges connected to these nodes. The content of each D_i, for $i = 1, .., 5$ is defined as follows:

- D_1 is a set of leaves in T_1 that are children of the nodes at layers $(\frac{k}{2}+1), .., k-1$,
- D_2 is a set of nodes, from layers $1, 2, 3$ and 4 in T_1 that belong to subtrees rooted in children and grandchildren of the root r_1,
- D_3 is a set of nodes of a subtree of T_1 rooted in node d_1, where d_1 is a child of the root r_1 at layer $k/2$,
- D_4 contains 3 nodes in T_2: two children x_1 and x_2 of the root h_3 at layer 1 and 2 respectively, and the child of x_2 at layer 1, and
- D_5 is a set of nodes in T_2 that form a subtree rooted in node d_2, where d_2 is a child of the root h_3 at layer $k/2$.

Another important component of the network is a set of special nodes. This set includes, the root r_1 of T_1 and its entirely equivalent copies r_2 and r_3, which are connected to the same nodes (as r_1 is) in the tree T_1. On the top of this, the roots r_1, r_2 and r_3 are mutually connected. The roots will be used to sent messages collected from the tree T_1 to the central nodes c_1, c_2 and c_3 in one time step (i.e., step $k+1$).

The set of special nodes includes also three central nodes c_1, c_2 and c_3. The central nodes share direct connections to all other nodes in the network H. In fact, the direct connection from the central nodes form a partition of other nodes. It means that after we gather all messages in each of the nodes c_1, c_2 and c_3, we are able to distribute the messages to all other nodes (i.e., complete the gossiping process) in a single time step. We show later how to inform all central nodes in at most $k+1$ rounds. In particular, the center c_1 is connected to all nodes in T_1 at layers $2, .., \frac{k}{2}$, to the root r_1, and the helper h_1. The center c_2 is connected to all nodes in T_1 at layer 1 and layers $(\frac{k}{2}+1), .., (k-1)$, the root r_2, and the helper h_2. The center c_3 is connected to all nodes in T_2 (including the helper h_3), the child of the root in T_1 at layer k, and the root r_3.

The last group of special nodes contains three helpers h_1, h_2 and h_3, where h_3 is the root of T_2. They are mutually connected and their purpose is to exchange original messages from the central nodes and to acquire messages gathered in the tree T_2.

Once the construction of the network H is completed we show that gossiping in H can be performed in $k+2 = \lfloor log(n-1) \rfloor + 2$ rounds. The expression $a \rightarrow b, c, d, ...$ is used to denote that a node a sends its all current knowledge to nodes $b, c, d, ...$ and $W_1 || W_2$ means that transmissions W_1 and W_2 are performed simultaneously.

Steps	Transmissions
(1)	$c_2 \rightarrow h_2 \mid\mid c_3 \rightarrow h_3 \mid\mid$ (all nodes at layer 1 in T_1 and T_2 transmit)
(2)	$r_1 \rightarrow r_2, r_3 \mid\mid h_2 \rightarrow h_3, h_1 \mid\mid$ (all nodes at layer 2 in T_1 transmit) $\mid\mid$ (all nodes at layer 1 in T_2 transmit)
(3)	$r_2 \rightarrow r_1, r_3 \mid\mid$ (all nodes at layer 3 in T_1 transmit) $\mid\mid$ (all nodes at layer 2 in T_2 transmit)
(4)	$r_3 \rightarrow r_1, r_2 \mid\mid$ (all nodes at layer 4 in T_1 transmit) $\mid\mid$ (all nodes at layer 3 in T_2 transmit)
...	
$(\frac{k}{2})$	$c_1 \rightarrow h_1 \mid\mid$ (all nodes at layer $\frac{k}{2}$ in T_1 transmit) $\mid\mid$ (all nodes at layer $\frac{k}{2} - 1$ in T_2 transmit)
$(\frac{k}{2}+1)$	$h_1 \rightarrow h_2, h_3 \mid\mid$ (all nodes at layer $\frac{k}{2}+1$ in T_1 transmit) $\mid\mid$ (all nodes at layer $\frac{k}{2}$ in T_2 transmit)
...	
(k-2)	(all nodes at layer $k-2$ in T_1 transmit) $\mid\mid$ (all nodes at layer $k-3$ in T_2 transmit)
(k-1)	$h_3 \rightarrow c_3 \mid\mid h_3 \rightarrow h_2 \mid\mid h_3 \rightarrow h_1 \mid\mid$ (all nodes at layer $k-1$ in T_1 transmit)
(k)	$h_1 \rightarrow c_1 \mid\mid h_2 \rightarrow c_2 \mid\mid$ (a node at layer k in T_1 transmits to r_1, r_2 and r_3)
(k+1)	$r_1 \rightarrow c_1 \mid\mid r_2 \rightarrow c_2 \mid\mid r_3 \rightarrow c_3$
(k+2)	c_1,c_2,c_3 transmit to all their neighbours.

During the gossiping process (in the network H) we first collect all messages in central nodes c_1, c_2 and c_3 in time $k+1 = \lfloor \log(n-1) \rfloor + 1$. The main idea behind the removal of sets D_1 through D_5 is to avoid collisions when the special nodes act, i.e., when they transmit and listen. The loss of nodes caused by removal of the sets D_1,D_2 and D_3 from T_1 is compensated by the nodes available in the tree T_2. In fact, the size of H formed of trees T_1 and T_2 and a few more special nodes is $2^k + 2^{k-3} - O(2^{k/2})$. This is due to the fact that the cardinality of each D_i, for $i = 1, ..., 5$ is $O(2^{k/2})$. The following lemma holds:

Lemma 3. *There exists a radio network topology in which the gossiping task can be completed in time* $\lfloor \log(n-1) \rfloor + 2$, *for any integer* $n = 2^k + 2^{k-3} - O(2^{k/2})$, *and* k *large enough.*

In particular, we conclude that we know how to build the optimal (in terms of gossiping) radio network topology for a fraction of all integer values of n.

4 Gossiping in Time $O(D)$

We discuss here a class of graphs admitting radio gossiping in time $O(D)$.

4.1 Gossiping in Time $(2D-1)\Delta + 1$

The general idea of the algorithm is as follows. Initially, we chose a central node c and we partition all nodes into disjoint subsets, *layers* l_i, where $0 \le i \le D$. This is followed by gathering stage when all (other $n-1$) messages are moved to the central node c, layer by layer. Finally, a combined message (including all original messages) is distributed from c to all other nodes, also layer by layer. In what follows we show that all messages that reside at layer l_k can be moved to a neighbouring layer l_{k-1} (or l_{k+1}) in at most Δ units of time.

Lemma 4. *All messages available at layer* l_k *can be moved to a neighbouring layer* l_{k-1} *in at most* Δ *units of time, where* $1 \le k \le D$.

Proof. We use here notation introduced in Section 2. Let $N_k^0 = l_k$, and C_{k-1}^0 (subset of l_{k-1}) be the minimal covering set for N_k^0. I.e., every node in N_k^0 is connected to some node in C_{k-1}^0, and removal of any node from C_{k-1}^0 destruct this property. Note, that every node $v \in C_{k-1}^0$ is connected to some node $u \in N_k^0$, s.t., u is not connected to any other node in $C_{k-1}^0 - \{v\}$; otherwise we could remove v from C_{k-1}^0. Thus, during a single time step, every node $v \in C_{k-1}^0$ receives a message m_u transmitted from its unique node $u \in N_k^0$. Later, node u is removed from N_k^0,, which means that a (virtual) degree of each node in C_{k-1}^0 is decreased by one. After removal of all us involved in the transmissions we end up with a new set N_k^1, and its new covering set $C_{k-1}^1 \subset C_{k-1}^0$. We repeat the whole process at most Δ times, since the degree of nodes in the covering set is decreased by 1 during each round of transmissions.

This means that the gossiping task in any radio network with diameter D and maximum degree Δ can be completed in time $\le (2D-1)\Delta + 1$, where $D\Delta$ comes from the gathering stage and $1 + (D-1)\Delta$ from the broadcasting stage.

Theorem 2. *In any graph* G, *with a diameter* D *and a constant maximum degree, the gossiping task can be completed in time* $O(D)$.

4.2 Gossiping in Graphs with Larger Max-Degree

An algorithm presented in this section is based on a concept of efficient broadcasting $O(D + \log^5 n)$-time procedure presented in [8]. We use here very similar partition of a (network) graph into clusters and super-levels. The main difference lies in greater complexity of the gossiping problem.

Cluster Graph and Tree of Clusters. Assume we have a graph $G = (V, E)$, where $|V| = n$ and a distinguished node $s \in V$. Assume also that the diameter of G is at most D. A layer l_i in G is formed by nodes that are at (same) distance i from s, for $i = 1, .., D$. All layers in G are grouped in x super-levels, s.t., the jth super-level is formed of layers $l_{\frac{D(j-1)}{x}+1}, .., l_{\frac{Dj}{x}}$, for $j = 1, .., x$. Each super-level is covered by the set of clusters, s.t., (1) each cluster has diameter $O(\frac{D \log n}{x})$, (2) the union of the clusters covers the super-level, and (3) the clusters graph can be coloured with $O(\log n)$ colours, where the clusters graph is obtained by treating each cluster as a node, and introducing an edge between two nodes if in the original graph there is some edge that connects nodes from the corresponding clusters or if the clusters share a common node. Note, that the number of clusters does not exceed n; otherwise we would be able to remove at least one (redundant) of them. It also follows from construction presented in [8] that each cluster at super-level i has a direct connection (an edge in the cluster graph) with some clusters at super-levels $i - 1$ and $i + 1$. This property allows to define a *tree of clusters*, which is a BFS tree rooted in a cluster that contain a distinguished node s. The broadcasting procedure proposed in [8] uses two types of information transfer in clusters, from the top layer through the bottom layer of a super-level. I.e., within each cluster we have either *slow* or *fast* transfers. The slow transfer is implemented by non-optimal broadcasting procedure, while the fast transfer is performed along a single path of length $\frac{D}{x}$. It is known, that transfers in the tree of clusters can be organised, s.t., on a path from any leaf to the root of the cluster tree there is at most $O(\log n)$ clusters operating slow transfers. In our gossiping algorithms the slow transfers are implemented by *limited gossiping* (defined below), and fast transfers (as in broadcasting) are performed along simple paths.

In our algorithm we use 3 types of communication procedures:

1. LIMITED GOSSIPING – where each node distributes its (currently possessed) message to all nodes within some radius r. Note that if $r = D$, the limited gossiping coincides with the gossiping problem. Note also that slow transfers are based on limited gossiping;
2. BETWEEN SUPER-LEVELS – where information residing at the top layer of a lower (further from the root cluster) super-level to the bottom layer of an upper super-level. This communication procedure is used when at least one cluster of the neighbouring super-levels is involved in slow transfers;
3. FAST TRANSFER – where information is moved across one cluster by fast pipelining along a single (simple) path.

The gossiping algorithm is implemented in 3 stages.

1. Initially messages in each cluster are collected in a distinguished node (possibly belonging to a fast route) in the top layer of each cluster. This is done by LIMITED GOSSIPING, where $r = \frac{D \log n}{x}$, i.e., maximal diameter of each cluster. Since the cluster graph can be coloured with $O(\log n)$ colours, all limited gossipings performed simultaneously in each cluster (at all super-levels of the cluster tree) can be preformed simultaneously with at most $\log n$-time slowdown. I.e., if $T^n_\Delta(D)$ stands for the time complexity of limited gossiping in a graph with n nodes, max-degree Δ, and diameter D, the contribution of the first stage to the time complexity of our gossiping algorithm is $O(T^n_\Delta(\frac{D \log n}{x}) \log n)$.
2. Messages from each cluster are delivered to the root cluster and in particular to the distinguished node s. During this stage the execution of three types of communication procedures is performed in separate (interleaved) time steps. I.e., LIMITED GOSSIPING in time steps $i = 0 \pmod 3$, BETWEEN SUPER-LEVEL in time steps $i = 1 \pmod 3$, and FAST TRANSFER in time steps $i = 2 \pmod 3$. Also, the execution of single rounds of communication procedures (within each type) is synchronised across all super-levels. E.g., the procedure LIMITED GOSSIPING starts and ends exactly at the same time in each cluster and each super-level. Moreover, when new messages arrive at the bottom of a super-level (e.g., delivered by the BETWEEN SUPER-LEVEL communication procedure) they are buffered at the bottom layer and allowed to traverse towards the upper layers only when the new round of LIMITED GOSSIPING or FAST TRANSFER is about to begin.
   A contribution to the time complexity of our gossiping algorithm of each the communication procedures is as follows:
   (a) Since executions of LIMITED GOSSIPINGs are synchronised across all super-levels, simultaneous execution of a single round of the LIMITED GOSSIPING procedure is done in time $T^n_\Delta(\frac{D \log n}{x})$. And since each message, traversing towards the root cluster, experiences at most $O(\log n)$ slow transfers (based on LIMITED GOSSIPING) the total contribution to the slowdown of each message is bounded by $O(T^n_\Delta(\frac{D \log n}{x}) \log n)$.
   (b) A single execution of one round of BETWEEN SUPER-LEVELS procedure can be implemented in time Δ. This is a consequence of Lemma 4. Since each message has to pass at most x borders between super-levels the contribution of this type of communication to the total time complexity is bounded by $x\Delta$.
   (c) A simultaneous execution of FAST TRANSFER in potentially many clusters on the same super-level results in a need of pipelined transmission of messages according to the color of a cluster. I.e., during one round of a fast transfer, messages that traverse along a path in clusters coloured with number 1 start their journey immediately, in all clusters coloured with number 2 messages are released three time steps later (in order to avoid collisions between layers), in clusters coloured with number 3 – six time steps later, etc. Thus finally in clusters coloured with the largest number – $O(\log n)$ time steps later. After message is released at the bottom layer it reaches the upper one in exactly $\frac{D}{x}$ time steps (a property of a fast transfer). Thus the contribution of this type of communication (across all super-levels) to the total time complexity is bounded by $O(x(\frac{D}{x} + \log n))$.

3. Eventually, after all messages are successfully gathered in the distinguished node s the combined message (containing all original messages) is broadcasted to all other nodes in the graph. This can be done by reversing the gathering process presented above, where the time complexity remains the same.

Theorem 3. *The time complexity of our gossiping procedure can be expressed by the recursive equation:* $T^n_\Delta(D) = O(T^n_\Delta(\frac{D\log n}{x})\log n + x(\Delta + \log n) + D)$, *where* x *is the number of super-levels in the Cluster Graph.*

After iteration of the recursive equation from Theorem 3 $i-1$ times, we obtain: $T^n_\Delta(D) = O(T^n_\Delta(\frac{D\log^i n}{x^i})\log^i n + x(\Delta + \log n)(\log^{i-1} + ... + \log n + 1) + D(\frac{\log^{2(i-1)} n}{x^{i-1}} + ... + \frac{\log^2 n}{x} + 1))$. After further substitution of the recursive component by the complexity $(2D-1)\Delta + 1$ (see the gossiping algorithm presented in Section 4.1) and taking $x = D^{\frac{1}{i+1}}\log n$, we get:

Corollary 1. $T^n_\Delta(D) = O(D)$, *for all graphs with* $\Delta = O(\frac{D^{1-1/i+1}}{\log^i n})$ *and* $D = \Omega(\log^{i+1} n)$, *for all constant integers* $i \geq 0$.

References

1. R. Bar-Yehuda, A. Israeli, A. Itai, Multiple Communication in Multihop Radio Networks, SIAM Journal on Computing, 22(4), 1993, pp 875-887.
2. I. Chlamtac and S. Kutten, On broadcasting in radio networks-problem analysis and protocol design, IEEE Transactions on Communications 33, 1985, pp 1240-1246.
3. M. Christersson, L. Gąsieniec and A. Lingas, Gossiping with bounded size messages in ad-hoc radio networks, 29th Int. Colloq. on Automata, Lang. and Prog., ICALP'02, pp 377-389.
4. M. Chrobak, L. Gąsieniec and W. Rytter, Fast Broadcasting and Gossiping in Radio Networks, Journal of Algorithms 43(2), 2002, pp 177-189.
5. A.E.F. Clementi, A. Monti, R. Silvestri, Distributed multi-broadcast in unknown radio networks, 20th ACM Symp. on Principles of Distributed Computing, PODC'01, pp 255-264.
6. A. Czumaj and W. Rytter, Broadcasting Algorithms in Radio Networks with Unknown Topology, 44th Symposium on Foundations of Computer Science, FOCS'03, pp 492-501.
7. K. Diks, E. Kranakis, A. Pelc, The impact of knowledge on broadcasting time in radio networks, 7th European Symposium on Algorithms, ESA'99, pp 41-52.
8. I. Gaber and Y. Mansour, Broadcast in radio networks, 6th Annual ACM-SIAM Symposium on Discrete Algorithms, SODA'95, pp 577-585.
9. L. Gąsieniec and A. Lingas, On adaptive deterministic gossiping in ad hoc radio networks, Information Processing Letters 2(83), 2002, pp 89-94.
10. L. Gąsieniec and I. Potapov, Gossiping with unit messages in known radio networks, 2nd IFIP International Conference on Theoretical Computer Science, TCS'02, pp 193-205.
11. D. Liu and M. Prabhakaran, On Randomized Broadcasting and Gossiping in Radio Networks, 8th Int. Conf. on Computing and Combinatorics, COCOON'02, pp 340-349.
12. A. Sen and M.L. Huson, A new model for scheduling packet radio networks, 15th Annual Joint Conference of the IEEE Computer and Communication Societies, 1996, pp 1116-1124.
13. P.J. Slater, E.J.Cockayne and S.T. Hedetniemi, Information Dissemination in Trees, SIAM Journal on Computing, 10, 1981, pp 892-701.
14. Y. Xu, An $O(n^{1.5})$ deterministic gossiping algorithm for radio networks, Algorithmica, 36(1), 2003, pp 93-96.