INF223 Category Theory and Diagrammatic Modelling
Spring 2011

— Exercise Set 2 , for group meeting 04.02.2011 —

1. In the script we have now the following about the category Graph:

   Definition 1 (Category Graph). *Small graphs and the homomorphisms between them constitute a category* Graph*:* **objects:** *all small graphs* G
   morphisms: *all homomorphisms* $\varphi : G \to H$ *between small graphs*
   composition: *The composition* $\varphi;\psi : G \to K$ *of two graph homomorphisms* $\varphi = (\varphi_0, \varphi_1) : G \to H$, $\psi = (\psi_0, \psi_1) : H \to K$ *is defined componentwise:* $(\varphi;\psi) \stackrel{def}{=} (\varphi_0;\psi_0, \varphi_1;\psi_1)$*, i.e.,* $(\varphi;\psi)_0 \stackrel{def}{=} \varphi_0;\psi_0$ *and* $(\varphi;\psi)_1 \stackrel{def}{=} \varphi_1;\psi_1$.
   Identity: *For any graph* G *the* ***identity homomorphism*** $id_G : G \to G$ *is given by a pair of identity maps:* $id_G \stackrel{def}{=} (id_{G_0}, id_{G_1})$*, i.e.,* $(id_G)_0 \stackrel{def}{=} id_{G_0}$ *and* $(id_G)_1 \stackrel{def}{=} id_{G_1}$.

   Proof. It sounds very strange that we have to give a proof for a definition. This is, however, very often the case. What we have to prove is that the structure we have defined/constructed has indeed the properties we have been claiming in the definition. In our case we would have to show that Graph is indeed a category, i.e., we have to show the following properties:

   - The componentwise composition of two graph homomorphisms provides indeed a graph homomorphism, i.e., the pair of maps $(\varphi_0;\psi_0, \varphi_1;\psi_1)$ preserves sources and targets.
   - The pair (id_{G_0}, id_{G_1}) of identity maps is indeed a graph homomorphism.
   - The composition in Graph satisfies the identity and the associativity law. Since composition and identities are defined componentwise both laws are inherited from the category Set.

   □

   Your task is to prove, in detail, all the four properties mentioned above.

2. In the last lecture on Tuesday we have defined the concept of a functor:

   Definition 2 (Functor). *A* ***functor*** $\varphi : \mathsf{C} \to \mathsf{D}$ *between two categories* C *and* D *is given by a graph homomorphism from* $gr(\mathsf{C})$ *into* $gr(\mathsf{D})$ *satisfying the following additional compatibility conditions:*

   - $\varphi_1(f;g) = \varphi_1(f);\varphi_1(g)$ *for any pair* $f : x \to y$, $g : y \to z$ *of composable morphisms in* C.
   - $\varphi_1(id_x) = id_{\varphi_0(x)}$ *for all objects* x *in* C.

Your task is to look for some more sample functors:

(a) Define a functor from Nat into Incl and validate the functor property. That is, you have to assign to each natural number n a set $e(n)$ such that $n \leq m$ implies $e(n) \subseteq e(m)$ for all natural numbers n, m. (This resembles a well-known trick to code natural numbers by sets.)

(b) Define a functor from Incl into Set and validate the functor property.

(c) Define a functor from Set into Rel and validate the functor property.

(d) Define a functor from Set^{op} into Mult and validate the functor property. (Hint: It may be helpful to google "preimage or inverse image of a set"?)

(e) If you have still time you may look for a functor from Rel into Mult and a functor into the other direction.