Universitetet i Bergen
Det matematisk-naturvitenskapelige fakultet

Examination in	:	INF-227 Introduction to logic
Date	:	26 May 2009
Time	:	9:00 – 12:00
No. of pages	:	2
Auxialiary materials	:	None

- All results shown at the lectures/exercises can be used without further justification – but you must state precisely the results you are using.
- You may use the result of one exercise in the solution of another even if you did not answer the exercise you are using.
- The percentages at each subproblem indicate only approximate weight at grading and/or only anticipated difficulty/time needed for solving the problem.

1 Induction and SL (40%)

Let $\mathcal{L}^\Sigma$ be the propositional language, relatively to some set of variables Σ, with the connectives: $\wedge, \rightarrow, \neg$. Call a well-formed formula in this language "positive" if it does not contain the connective $\neg$.

1.1. Define inductively the set of positive formulae $\mathcal{P}^\Sigma \subset \mathcal{L}^\Sigma$.

1.2. Assume given the usual boolean tables for the three involved connectives. Define inductively the extension of an assignment $V : \Sigma \rightarrow \{\mathbf{1}, \mathbf{0}\}$, to all formulae $\overline{V} : \mathcal{L}^\Sigma \rightarrow \{\mathbf{1}, \mathbf{0}\}$.

1.3. Let V be the valuation given by $V(x) = \mathbf{1}$, for all $x \in \Sigma$. Show by appropriate induction that then also $\overline{V}(A) = \mathbf{1}$, for every positive formula $A \in \mathcal{P}^\Sigma$.

1.4. Give an argument (one, at most two sentences) why $\{\wedge, \rightarrow\}$ is not an adequate set.

1.5. Let $\bot$ be a new constant symbol with the semantic requirement that for every valuation $V : \overline{V}(\bot) = \mathbf{0}$. Show that $\{\rightarrow, \bot\}$ is an adequate set of connectives.

1.6. Write the formula $(a \rightarrow \neg(\neg b \vee c)) \vee (\neg b \rightarrow \neg a)$ in DNF and CNF. (Preferably, in a shortest possible way.)

2 FOL (40%)

Consider the three following closed formulae (i.e., *schemata*, where A, B stand for arbitrary formulae with at most x free):

a) $\forall x.(A(x) \rightarrow B(x))$ b) $(\forall x.A(x)) \rightarrow (\forall x.B(x))$ c) $(\exists x.A(x)) \rightarrow (\exists x.B(x))$

2.1. Which of these three formulae imply logically which other ones? For each such implication give a proof, while for each missing one give a counter-example.

2.2. Write each formula in Prenex Normal form – for b) and c) give two such forms.

2.3. Is any of these three formulae valid? Either give a proof (for the formulae for which the answer is 'yes') or provide a counter-example (when the answer is 'no').

2.4. Is any of these three formulae implied logically by the formula $F : \forall x(A(x) \to A(x))$? (Give a shortest available answer.)

3 Meta-argument **(20%)**

Let Σ be a (not necessarily finite) set and $\Gamma, \Delta \subseteq \Sigma$ range over its finite subsets. We consider two axiomatic systems for deriving expressions of the form $\Gamma \vdash \Delta$:

	system $\mathcal{A}$	system $\mathcal{B}$
AXIOMS	$\Gamma \vdash_{\mathcal{A}} \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$	$\{A\} \vdash_{\mathcal{B}} \{A\}$ iff $A \in \Sigma$
RULES		1) $\dfrac{\Gamma \vdash_{\mathcal{B}} \Delta}{\Gamma \cup \{B\} \vdash_{\mathcal{B}} \Delta}$ for any $B \in \Sigma$ 2) $\dfrac{\Gamma \vdash_{\mathcal{B}} \Delta}{\Gamma \vdash_{\mathcal{B}} \Delta \cup \{B\}}$ for any $B \in \Sigma$

3.1. Using appropriate induction(s) show that for any Γ, Δ we have: $\Gamma \vdash_{\mathcal{A}} \Delta$ iff $\Gamma \vdash_{\mathcal{B}} \Delta$.

3.2. Is the relation $\vdash_{\mathcal{A}}$ decidable? Give a precise argument. Is $\vdash_{\mathcal{B}}$ decidable?

3.3. If we allow Γ, Δ to be infinite, the equivalence from **3.1.** does not hold. Instead, there is only one-way implication. Say which one and explain why.

3.4. Is the relation $\vdash_{\mathcal{A}}$ decidable, when Γ, Δ may be infinite? Give a precise argument.

Good luck!
Michał Walicki

Gentzen's system for FOL

1. *Axioms* : $\Gamma \vdash_{\mathcal{G}} \Delta$ *whenever* $\Gamma \cap \Delta \neq \emptyset$

2. $\vdash\vee$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} A, B, \Delta}{\Gamma \vdash_{\mathcal{G}} A \vee B, \Delta}$

2′. $\vee\vdash$ $\quad \dfrac{\Gamma, A \vdash_{\mathcal{G}} \Delta \quad ; \quad \Gamma, B \vdash_{\mathcal{G}} \Delta}{\Gamma, A \vee B \vdash_{\mathcal{G}} \Delta}$

3. $\vdash\wedge$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} A, \Delta \quad ; \quad \Gamma \vdash_{\mathcal{G}} B, \Delta}{\Gamma \vdash_{\mathcal{G}} A \wedge B, \Delta}$

3′. $\wedge\vdash$ $\quad \dfrac{\Gamma, A, B \vdash_{\mathcal{G}} \Delta}{\Gamma, A \wedge B \vdash_{\mathcal{G}} \Delta}$

4. $\vdash\neg$ $\quad \dfrac{\Gamma, B \vdash_{\mathcal{G}} \Delta}{\Gamma \vdash_{\mathcal{G}} \neg B, \Delta}$

4′. $\neg\vdash$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} B, \Delta}{\Gamma, \neg B \vdash_{\mathcal{G}} \Delta}$

5. $\vdash\to$ $\quad \dfrac{\Gamma, A \vdash_{\mathcal{G}} B, \Delta}{\Gamma \vdash_{\mathcal{G}} A \to B, \Delta}$

5′. $\to\vdash$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} \Delta, A \quad ; \quad \Gamma, B \vdash_{\mathcal{G}} \Delta}{\Gamma, A \to B \vdash_{\mathcal{G}} \Delta}$

6. $\vdash\exists$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} \Delta, \exists x A, A^x_t}{\Gamma \vdash_{\mathcal{G}} \Delta, \exists x A}$ $\quad A^x_t$ *legal*

6′. $\forall\vdash$ $\quad \dfrac{A^x_t, \forall x A, \Gamma \vdash_{\mathcal{G}} \Delta}{\forall x A, \Gamma \vdash_{\mathcal{G}} \Delta}$ $\quad A^x_t$ *legal*

7. $\vdash\forall$ $\quad \dfrac{\Gamma \vdash_{\mathcal{G}} A^x_{x'}, \Delta}{\Gamma, \vdash_{\mathcal{G}} \forall x A, \Delta}$ $\quad x'$ *fresh*

7′. $\exists\vdash$ $\quad \dfrac{\Gamma, A^x_{x'} \vdash_{\mathcal{G}} \Delta}{\Gamma, \exists x A \vdash_{\mathcal{G}} \Delta}$ $\quad x'$ *fresh*

Problem 1 – solution

1.1. $\Sigma \subset \mathcal{P}$ and if $A, B \in \mathcal{P}$ then also $A \wedge B$ and $A \rightarrow B \in \mathcal{P}$.

1.2. $\overline{V}(x) = V(x)$ for all $x \in \Sigma$

$\overline{V}(A \wedge B) = \overline{V}(A) \underline{\wedge} \overline{V}(B)$ and $\overline{V}(A \rightarrow B) = \overline{V}(A) \underline{\rightarrow} \overline{V}(B)$, where the underlied symbols represent the functions from the respective boolean tables.

1.3. $\overline{V}(x) = V(x) = \mathbf{1}$ by assumption on V.

$\overline{V}(A \wedge B) = \overline{V}(A) \underline{\wedge} \overline{V}(B) \stackrel{\text{IH}}{=} \mathbf{1} \underline{\wedge} \mathbf{1} = \mathbf{1}$

$\overline{V}(A \rightarrow B) = \overline{V}(A) \underline{\rightarrow} \overline{V}(B) \stackrel{\text{IH}}{=} \mathbf{1} \underline{\rightarrow} \mathbf{1} = \mathbf{1}$

1.4. By the prvious point, any formula F over these two connectives with only a single variable x will, under the assignment $V(x) = \mathbf{1}$ evaluate to $\overline{V}(F) = \mathbf{1}$. Hence, it cannot possibly define unary negation.

1.5. Defining $\neg(x) = x \rightarrow \bot$, we verify easily that we obtain the boolean table for negation. Since $\{\rightarrow, \neg\}$ is adequate, so is the set $\{\rightarrow, \bot\}$ which allows to define $\neg$.

1.6. $(\neg a \vee (b \wedge \neg c)) \vee (\neg a \vee b) \Leftrightarrow \neg a \vee (b \wedge \neg c) \vee (\neg a \vee b)$

which is in DNF. But since $b \wedge \neg c \rightarrow b$, we can simplify to $\neg a \vee b$. This is then both DNF and CNF.

Problem 2 – solution

2.1. a $\rightarrow$ b and c, which can be verified using Gentzen proofs (and then referring to its soundness), e.g.:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A(x) \rightarrow B(x), A(x) \vdash_{\mathcal{G}} B(x)}}{\forall x(A(x) \rightarrow B(x)), A(x) \vdash_{\mathcal{G}} B(x)}}{\forall x(A(x) \rightarrow B(x)), A(x) \vdash_{\mathcal{G}} \exists x B(x)}}{\forall x(A(x) \rightarrow B(x)), \exists x A(x) \vdash_{\mathcal{G}} \exists x B(x)}$$

b $\not\rightarrow$ a (as seen many times at the lectures), and neither b $\rightarrow$ c. E.g., the structure M, with the interpretation domain $\{a, b\}$, $A^M = \{a\}$ and $B^M = \emptyset$, is such that $M \models b$ (since $M \not\models \forall x A(x)$), but $M \not\models c$. Also, $M \not\models a$.

c $\not\rightarrow$ b nor a. To falsfiy implication to a, let M have the domain as above, but with $A^M = \{a\}$ and $B^M = \{b\}$. Then $M \models c$ but $M \not\models a$. To falsify the implication to b, let N be as M only with an extended $A^N = \{a, b\}$. Then $N \models c$ but $N \not\models b$.

2.2. a) is in PNF.

b) $\exists x \forall y (A(x) \rightarrow A(y))$ or $\forall y \exists x (A(x) \rightarrow A(y))$

c) $\exists x \forall y (A(y) \rightarrow A(x))$ or $\forall y \exists x (A(y) \rightarrow A(x))$

2.3. a and c are not valid by the counter-examples for implications from b and b is not valid by the one for the implication from c - **2.1.**

2.4. Since none of these formulae are valid, none is logically implied by F, since $F \Rightarrow X$, i.e., $\forall M : M \models F \rightarrow X$, would imply validity of X, since $\forall M : M \models F$.

Problem 3 – solution

3.1. System $\vdash_{\mathcal{A}}$ contains no rules, so
1a) we show $\Gamma \vdash_{\mathcal{A}} \Delta \Rightarrow \Gamma \vdash_{\mathcal{B}} \Delta$ by induction on the total length l of the Γ and Δ. The shortest case is $l = 2$, i.e., $\{A\} \vdash_{\mathcal{A}} \{A\}$, which is also an axiom in $\{A\} \vdash_{\mathcal{B}} \{A\}$. The induction step for $l+1$ has two cases:

i) We have $\Gamma \cup \{A\} \vdash_{\mathcal{A}} \Delta$. Without loss of generality, we may assume that $\Gamma \vdash_{\mathcal{A}} \Delta$ (if it is A that occurs both in Γ and Δ, we may rewrite it as $\Gamma' \cup \{B\} \vdash_{\mathcal{A}} \Delta$, where $\Gamma' \cup \{B\} = \Gamma \cup \{A\}$ and $\Gamma' \vdash_{\mathcal{A}} \Delta$.) By IH, we have $\Gamma \vdash_{\mathcal{B}} \Delta$ and by an application of rule 1) obtain $\Gamma \cup \{A\} \vdash_{\mathcal{B}} \Delta$.

ii) We have $\Gamma \vdash_{\mathcal{A}} \Delta \cup \{A\}$ and, as in i), assume that $\Gamma \vdash_{\mathcal{A}} \Delta$ and by IH $\Gamma \vdash_{\mathcal{B}} \Delta$. An application of rule 2), yields $\Gamma \vdash_{\mathcal{B}} \Delta \cup \{A\}$.

1b) We show $\Gamma \vdash_{\mathcal{B}} \Delta \Rightarrow \Gamma \vdash_{\mathcal{A}} \Delta$ by induction on the length l of the proof. All axioms $\{A\} \vdash_{\mathcal{B}} \{A\}$ are obviously in $\vdash_{\mathcal{A}}$. For a proof of lenght $l+1$, IH allows us to assume that for the assumption of the last rule, $\Gamma \vdash_{\mathcal{B}} \Delta$, we also have $\Gamma \vdash_{\mathcal{A}} \Delta$. This means that $\Gamma \cap \Delta \neq \emptyset$. But then also $(\Gamma \cup \{B\}) \cap \Delta \neq \emptyset$ and $\Gamma \cap (\Delta \cup \{B\}) \neq \emptyset$, i.e., both are in $\vdash_{\mathcal{A}}$.

3.2. Membership in $\vdash_{\mathcal{A}}$ is decidable. In $\vdash_{\mathcal{A}}$ we have two finite sets $\Gamma = \{g_1, g_2, \ldots, g_n\}$ and $\Delta = \{d_1, d_2, \ldots, d_m\}$. Take g_1 and compare it to each element in Δ. If you find it there, i.e., find a $d_i = g_1$, stop with answer YES. If not, do the same with g_2, etc. until you find one of g_i's in Δ (then return YES), or else you empty the whole Γ – then return NO.

Since $\vdash_{\mathcal{A}} = \vdash_{\mathcal{B}}$, this shows that $\vdash_{\mathcal{B}}$ is decidable as well.

3.3. If Γ, Δ may be infinite, we only have implication 1b). Since any proof must be finite, the system $\mathcal{B}$ can generate only finite strings, i.e., only finite sets of formulae.

3.4. Take as Δ the set of codes of pairs $\langle M, w\rangle$ such that $M(w)$ halts. It is an infinite set (also, recursively enumerable). Now, if we could decide $\Gamma \vdash_{\mathcal{A}} \Delta$, we also could decide $\{\langle M, w\rangle\} \vdash_{\mathcal{A}} \Delta$, i.e., $\langle M, w\rangle \in \Delta$, which is the Halting problem. But it is undecidable, hence so is $\vdash_{\mathcal{A}}$.