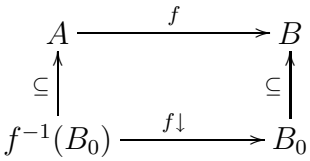
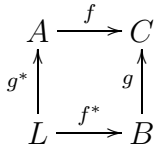
INF223 Categories for Software Engineering
Spring 2006

— Obligatory Exercise Set 2 , Deadline 24.04.2006 (lecture) —

1. Show that the following "inverse image" diagram is a pullback in the category **SET** where $f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}$:



2. We generalise the last exercise: Consider a pullback diagram in an arbitrary category **C**



   Prove that, if g is mono, so is g^*. (That is, f^* can be seen as a restriction of f!)

3. We consider a category **C** with products. The following law for the "tupling" of morphisms hold according to the definition of "tupling":

   - $\langle f, g\rangle; \pi_A = f$, $\langle f, g\rangle; \pi_B = g$ for $f : C \to A$, $g : C \to B$,

   and the law

   - $\langle \pi_A, \pi_B\rangle = id_{A\times B}$,

   can be proved by the uniqueness of "tupling".

   Prove that "tupling" is also compatible with composition in the following way:

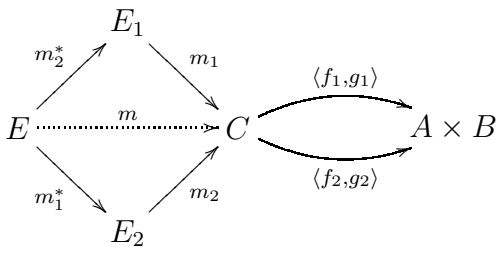
   (a) $k; \langle f, g\rangle = \langle k; f, k; g\rangle$ for $f : C \to A$, $g : C \to B$, $k : D \to C$,

4. We consider a category **C** with finite limits.

   Categorically, "intersection" is obtained by a pullback of mono's (see Exercise 2 above). You are asked to show that "intersection" and "tupling" are strongly related: Let $(E_1, m_1 : E_1 \to C)$ be the equalizer of the parallel pair $f_1, f_2 : C \to A$ of morphisms and $(E_2, m_2 : E_2 \to C)$

be the equalizer of the pair $g_1, g_2 : C \to B$. Further let $(E, m_2^* : E \to E_1, m_1^* : E \to E_2)$ be the pullback of the span $E_1 \xrightarrow{m_1} C \xleftarrow{m_2} E_2$ of the two equalisers (mono's).

Prove, that $(E, m : E \to C)$ with $m = m_2^*; m_1 = m_1^*; m_2$ is the equalizer of the pair $\langle f_1, g_1 \rangle, \langle f_2, g_2 \rangle : C \to A \times B$ of tuples.



Hint: The following results are assumed to be given:

- The composition of two mono's is mono as well,
- equalisers are mono's,
- Exercises 2 and 3 above.

Remark: Note, that we actually prove that the set consisting of the two equations $f_1(x) = f_2(x), g_1(x) = g_2(x)$ is "semantically" equivalent to the single equation $\langle f_1, g_1 \rangle(x) = \langle f_2, g_2 \rangle(x)$.