Nowhere-dense graph classes and algorithms

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Geilo winter school
unless I attribute the results to somebody, they are by Nešetřil and Ossona de Mendez
they are also preparing a comprehensive book on the subject:
  Sparsity (Graphs, Structures, and Algorithms)
Plan of the lecture

- tell something about **nowhere-dense graph classes** and **graph classes with bounded expansion**
- and algorithms for them
- a little about data structures
- we will avoid mostly theoretical connections to logic and theory of homomorphisms
The term **bounded expansion** used in this talk has no immediate connection to

- edge/vertex expansion of graphs, or
- expanders.

I apologize for sticking with this somewhat unfortunate name.
Graph classes

- a graph class: a set (more precisely, proper class) of graphs closed on isomorphism
- we only consider finite graphs without loops and parallel edges
- “nowhere-dense” and “bounded expansion” are properties of graph classes
  - not of single graphs
  - e.g., the class of all planar graphs has bounded expansion
Plan of the lecture

- usually, I start by defining what “nowhere-dense” and “bounded expansion” means
- but then lot of time is spent by explaining the definitions
- so, let’s start with the algorithms
Subgraph problem

Problem

Input: graphs H and G.
Question: is H a subgraph of G?
General algorithms for subgraph problem

- NP-complete \((H = K_k\) is a special case).
- trivial algorithm in \(O(kn^k)\), where \(n = |V(G)|\) and \(k = |V(H)|\).
- less trivially in \(n^{\omega k/3}\) (Nešetřil and Poljak). Idea:
  - \(K_3 \subseteq G \iff E(G^2) \cap E(G) \neq \emptyset\)
  - \(G^2\) computed by matrix multiplication.
- Can \(f(k)n^{O(1)}\) algorithm exist (FPT)?
  - unlikely – \(W[1]\)-complete.
Restricting $G$

What if $G$ belongs to some special class of graphs?

- $G$ has tree-width at most $t$: $f(k, t)O(n)$.
- $G$ has maximum degree at most $d$: $f(k, d)O(n)$
- $G$ is planar: $f(k)O(n)$ (Eppstein)
- $G$ does not contain $K_t$ as a minor: $f(k, t)n^{O(1)}$ (Dawar, Grohe and Kreutzer)

Using:

- locality
- decompositions
Decompositions of graphs

Idea: partition $V(G)$ to a small number of parts, s.t. union of every $|V(H)|$ of them induces a graph with simple structure (e.g., bounded tree-width).
Decompositions of graphs

Definition

$(p, tw \leq t)$-coloring of $G$ is a coloring such that union of every $p$ color classes induces a graph of tree-width at most $t$.

Algorithm:

- find a $(k, tw \leq t(k))$-coloring of $G$ by $m(k)$ colors
- for each $k$ color classes $C_1, \ldots, C_k$, test whether $H \subseteq G[C_1 \cup \ldots \cup C_k]$.

Time complexity $O\left(c(k, n) + \binom{m(k)}{k} f(k, t(k)) n\right)$, where

- $c(k, n)$ is the complexity of finding the coloring
- $f(k, t)O(n)$ is the complexity of finding subgraph in graphs of tree-width at most $t$. 
Example: planar graphs

Theorem (Robertson and Seymour)

A planar graph of radius $r$ has tree-width at most $3r$.

- choose a vertex $v$
- let $C_i = \{u \in V(G) : d(u, v) \mod (p + 1) = i\}$

$C_0, C_1, \ldots, C_p$ give a $(p, \text{tw} \leq 3p)$-coloring by $p + 1$ colors.
Example: planar graphs

**Theorem**

For every $p$, a $(p, tw \leq 3p)$-coloring by $p + 1$ colors can be found in linear time for every planar graph.

Consequently,

**Theorem**

Testing whether $H \subseteq G$ can be done in $O(kf(k, 3k)n)$ for every planar graph $G$. 

Proper minor-closed classes

Theorem (DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan)

If $\mathcal{C}$ is a proper minor-closed class of graphs, then for every $p$, every $G \in \mathcal{C}$ has a $(p, tw \leq p - 1)$-coloring by $f_{\mathcal{C}}(p)$ colors.

- implies FPT for subgraph testing
- but complicated (based on minor structure theory).

Definition

A class of graphs $\mathcal{C}$ has low tree-width colorings if there exists a function $g$ such that for every $p$, every $G \in \mathcal{C}$ has a $(p, tw \leq p - 1)$-coloring by $g(p)$ colors.
we want a simpler algorithm for finding $(p, tw \leq p - 1)$-coloring

but tree-width is still a rather complicated parameter

can even simpler class of graphs be used instead?

$(2, tw \leq 1)$-coloring \ldots acyclic coloring

- union of any two color classes induces a forest
- no bichromatic cycles

star coloring

- union of any two color classes induces a star forest
- no bichromatic $P_4$
- needs at most quadratic number of colors wrt. acyclic coloring
Tree-depth

- **depth** of a rooted tree: maximum number of edges on a path to the root
- **closure** $\text{cl}(T)$ of a rooted tree $T$: for each $v$, add edges from $v$ to all vertices on the path from $v$ to the root
Definition

Tree-depth $\text{td}(G)$ of a connected graph $G$ is the minimum depth of a rooted tree $T$ such that $G \subseteq \text{cl}(T)$. Tree-depth of disconnected graph is the maximum of the tree-depths of its components.
Properties of tree-depth

1. \( \text{td}(G) = 0 \) ... isolated vertices; \( \text{td}(G) = 1 \) ... star forest
2. minor-monotone
3. \( \text{td}(G) \geq \text{pw}(G) \geq \text{tw}(G) \)
4. \( G \) connected: \( \text{td}(G) = 1 + \min \{ \text{td}(G - v) : v \in V(G) \} \)
5. \( \text{td}(K_n) = n - 1, \text{td}(P_n) = \lfloor \log_2 n \rfloor \)
Tree-depth and paths

Theorem

\[ \lceil \log_2 p \rceil \leq td(G) \leq \left( \frac{p+1}{2} \right), \text{ where } p \text{ is the number of vertices of the longest path in } G. \]

Proof.

\( P \subseteq G \) is a path on \( p \) vertices \( \Rightarrow G - V(P) \) does not contain any path on \( p \) vertices:

\[ td(G) \leq p + td(G - V(P)) \leq p + \left( \frac{p}{2} \right) \text{ by induction} \]
Tree-depth coloring

**Definition**

A (p, td ≤ t)-coloring of G is a coloring such that union of every p color classes induces a graph of tree-depth at most t.

**Definition**

A class of graphs C has low tree-depth colorings if there exists a function g such that for every p, every G ∈ C has a (p, td ≤ p – 1)-coloring by g(p) colors.

Does any non-trivial graph class have this property?
Tree-depth versus tree-width

Claim

There exists a function $g$ such that for every $t$ a $p$, every graph with tree-width at most $t$ has a $(p, td \leq p - 1)$-coloring by $g(t, p)$ colors.

We will prove a stronger result later. For now:

Corollary

If $G$ has a $(p, tw \leq t)$-coloring by $c$ colors, then it also has a $(p, td \leq p - 1)$-coloring by at most $cg(t, p)^{(c)}$ colors.
Proof of the Corollary.

- Let $\varphi$ be the $(p, \text{tw} \leq t)$-coloring.
- Let $C_1, C_2, \ldots, C_{c(p)}$ be all possible unions of $p$ color classes and let $\varphi_i$ be a $(p, \text{td} \leq p - 1)$-coloring of $G[C_i]$ by at most $g(t, p)$ colors.
  - and define $\varphi_i$ arbitrarily on $V(G) \setminus C_i$
- Assign each vertex $v$ the color
  \[
  \left(\varphi(v), \varphi_1(v), \ldots, \varphi_{c(p)}(v)\right)
  \]
  - any union of at most $p$ color classes in this coloring is a subset of some $C_i$
  - and thus also a subset of a union of at most $p$ color classes of $\varphi_i$
Corollary (of the Corollary)

A class of graphs has low tree-width colorings if and only if it has low tree-depth colorings.
How to find a coloring?

Greedy algorithm:
- remove a vertex $v$ of smallest degree, color the rest of the graph, then color $v$ by the smallest possible color

Reformulation: let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$.
- backdegree of $v_i$ is the number of its neighbors among $v_1, v_2, \ldots, v_{i-1}$
- coloring number of the ordering is the maximum of backdegrees of the vertices

Definition

Coloring number $col_1(G)$ is the minimum of coloring numbers of all possible orderings of $V(G)$.

Note: $\chi(G) \leq col_1(G) + 1$. 

Nowhere-dense graph classes and algorithms
Z. Dvořák
Introduction
Subgraph problem
Tree-depth
Orderings
Generalized coloring number
Bounded expansion
Nowhere-dense graph classes
Shallow minors
Closures
Orientations
What about acyclic coloring?

Arrangeability: let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$.

- $v_j$ is 2-backreachable from $v_i$ if $j < i$ and there exists a path $P$ of length at most two between $v_i$ and $v_j$, such that the internal vertex $v_m$ (if any) of $P$ satisfies $i < m$.
- 2-backdegree of $v$ is the number of vertices 2-backreachable from $v$
- arrangeability of the ordering is the maximum of 2-backdegrees of vertices

Definition

Arrangeability $\text{col}_2(G)$ is the minimum of arrangeabilities of all possible orderings of $V(G)$.
Arrangeability

2-backdegree of \( v \) is 4
What about acyclic coloring?

### Theorem

*G has an acyclic coloring by at most* \( \text{col}_2(G) + 1 \) *colors.*

### Proof.

- Color vertices in the order certifying the arrangeability, assign colors different from 2-backreachable vertices
- No bichromatic cycle:

![Diagram](image-url)

\( W \) is 2-backreachable from \( V \)
Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$.

- An $s$-backpath from $v_i$ to $v_j$ with $j < i$ is a path of length at most $s$ such that if $v_m$ is an internal vertex of $P$, then $i < m$.
- $v_j$ is $s$-backreachable from $v_i$ if there exists an $s$-backpath from $v_i$ to $v_j$.
- The $s$-backdegree of $v$ is the number of vertices $s$-backreachable from $v$.
- The $s$-coloring number of the ordering is the maximum of $s$-backdegrees of the vertices.

Definition

The $s$-coloring number $\text{col}_s(G)$ is the minimum of $s$-coloring numbers of all possible orderings of $V(G)$. 
Working with generalized coloring number

Problems:

- Does generalized coloring number give us low tree-depth colorings?
- How to determine it (and find the ordering)?
  - NP-complete.
  - How to approximate it?
Let \( v_1, v_2, \ldots, v_n \) be an ordering of \( V(G) \).

**Definition**

\( v_a \) is \((s, r)\)-backreachable from \( v_b \), if there exist indices 
\( a = i_0, i_1, \ldots, i_t = b \), where \( t \leq r \), and \( v_{i_j} \) is \( s \)-backreachable from \( v_{i_{j+1}} \) for \( 0 \leq j < t \).

If the ordering has \( s \)-coloring number \( d \), then at most 
\( d + d^2 + \ldots + d^r < (d + 1)^r \) vertices are 
\((s, r)\)-backreachable from any vertex.
Every graph has \((p, \text{td} \leq p - 1)\)-coloring by at most \((\text{col}_s(G) + 1)^s\) colors, where \(s = 2^{p-1}\).

Proof.

- colors different from \((s, s)\)-backreachable vertices
- union of every \(t \leq p\) color classes has \(\text{td} \leq t - 1\):
How to determine $s$-coloring number?

- greedy algorithm
  - choose vertices $v_n, v_{n-1}, \ldots$
  - always pick a vertex with smallest $s$-backdegree
- problem: picking $v_i$ may increase $s$-backdegrees of remaining vertices
  - it is possible to make a wrong choice
- solution: minimize a different parameter
Let \( v_1, v_2, \ldots, v_n \) be an ordering of \( V(G) \).

- the \textit{s-backconnectivity} of a vertex \( v_i \) is the maximum number of \textit{s}-backpaths from \( v_i \) that intersect only in \( v_i \)
- the \textit{s-admissibility} of the ordering is the maximum of the \textit{s-backconnectivities} of the vertices

\textbf{Definition}

The \textit{s-admissibility} \( \text{adm}_s(G) \) is the minimum of \textit{s-admissibilities} of all possible orderings of \( V(G) \).
Admissibility

Nowhere-dense graph classes and algorithms

Z. Dvořák

Introduction
Subgraph problem
Tree-depth
Orderings
Generalized coloring number
Bounded expansion
Nowhere-dense graph classes
Shallow minors
Closures
Orientations

Admissibility

arrangeability:

2-back degree = 5

2-back connectivity = 4
Admissibility

Observation: greedy algorithm correctly determines $\text{adm}_S(G)$
Problem: determining $s$-backconnectivity is NP-complete for $s \geq 5$.

- but, testing whether it is less than a given constant is in P, and
- can be approximated within the factor of $s$ (greedily)

Testing whether $\text{adm}_s(G) \leq a$ for fixed $a$ and $s$ can be implemented in $O(n)$ using further results.
Admissibility vs coloring number

Theorem

Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$, $c$ its $s$-coloring number and $a$ its $s$-admissibility. Then $a \leq c \leq a^s$.

Proof.

- let $T$ be the tree of shortest $s$-backpaths from $v_i$
- $\Delta(T) \leq a$
- hence, $T$ has at most $a^s$ leaves
Definition

A class of graphs $\mathcal{C}$ has bounded admissibilities if there exists a function $f$ such that for every $s$ and every $G \in \mathcal{C}$, $\text{adm}_s(G) \leq f(s)$.

So far, we proved the following.

Theorem (Zhu)

Any class of graphs with bounded admissibilities has low tree-depth colorings.

Which graph classes have bounded admissibilities?
For a graph $G$,

- the $k$-subdivision $sd_k(G)$ is the graph created from $G$ by subdividing every edge by exactly $k$ vertices.
- a $(\leq k)$-subdivision is a graph created from $G$ by subdividing every edge by at most $k$ vertices; not necessarily every edge the same number of times; some edges may remain unsubdivided.
- a $(\leq k)$-topological minor of $G$ is any $H$ such that some $(\leq k)$-subdivision of $H$ is a subgraph of $G$. 

Image: 2-subdivision of $K_4$
Admissibility of subdivisions

Theorem

If $H$ is a $(≤ s − 1)$-subdivision of a graph with minimum degree $d ≥ 3$, then $\text{adm}_s(H) ≥ d$.

Proof.

- let $v$ be the last vertex of degree at least three in the ordering
- the $s$-backconnectivity of $v$ is at least $d$
Theorem

If \( \text{adm}_s(G) > (16d)^s \), then \( G \) has a \((\leq s - 1)\)-topological minor \( H \) such that \( \delta(H) > d \).

Proof.

- Otherwise, average degree of any \((\leq s - 1)\)-topological minor is \( \leq 2d \).
- Greedy algorithm fails, with set \( M \) of unchosen vertices:
Subdivisions in high-admissibility graphs

- We have more than $(16d)^s|M|$ paths.
- Almost gives the topological minor, but the paths may overlap.
- We need to clean up the paths; consecutively by levels $(s - 1$ times)
- Assuming levels up to $i$ consist of disjoint paths:
  1. throw away paths ending with the next step: $-4d|M|
  2. if next step of $P$ ends in level $j < i$ of $Q$, throw away $P
     $ or $Q$: /3
  3. for each vertex reachable in level $i + 1$, choose one of
     the paths: /4$d$
- The resulting graph is too dense.
Subdivisions in high-admissibility graphs
Definition

Let $\nabla_s(G)$ be the largest minimum degree of an $(\leq s)$-topological minor of $G$.

We have

$$\nabla_{s-1}(G) \leq \text{adm}_s(G) \leq (16\nabla_{s-1}(G))^s.$$
Definition

A class of graphs $\mathcal{C}$ has **bounded expansion** if there exists a function $f$ such that for every $s$ and every $G \in \mathcal{C}$, $\nabla_s(G) \leq f(s)$.

Theorem (Zhu; D.)

A class has bounded expansion if and only if it has bounded admissibilities.

Recall: bounded admissibilities $\Rightarrow$ low tree-depth colorings.
Lemma

If $\delta(H) > d$, then every subdivision $H'$ of $H$ has $td(H') > d$.

Proof.

Consider the vertex $v$ of degree greater than $d$ appearing deepest in the tree certifying tree-depth of $H'$:
Low tree-depth colorings and subdivisions

Theorem

If \( sd_s(G) \) has an \((s + 2, \text{td} \leq s + 1)\) coloring by at most \( c \) colors, then \( \delta(G) \leq 2(s + 1)(\binom{c}{s+2}) \).

Proof.

- For \( e \in E(G) \), let \( S_e \) be the set of colors on the corresponding path (including endvertices).
- For \( S \subseteq \{1, \ldots, c\} \) of size \( s + 2 \), let \( G_S \) be the subgraph with edges \( \{e : S_e \subseteq S\} \).
- For some \( S \), \( G_S \) has average degree at least \( \delta(G)/(\binom{c}{s+2}) \), and contains \( G' \subseteq G_S \) with \( \delta(G') \geq \delta(G)/\left[2\binom{c}{s+2}\right] \).
- Observe \( \text{td}(sd_s(G')) \leq s + 1 \) and apply lemma.
Corollary

If a class of graphs has low tree-depth colorings, then it has bounded expansion.

I.e., the following are equivalent:

- bounded expansion
- bounded admissibilities
- having low tree-depth colorings
- having low tree-width colorings
Classes with bounded expansion

Theorem

Any proper class \( \mathcal{C} \) of graphs closed on topological minors has bounded expansion.

Proof.

- \( K_k \not\in \mathcal{C} \) for some \( k \)
- if \( H \in \mathcal{C} \), then \( \delta(H) \leq O(k^2) \) (Komlós)
- closed on topological minors: \( \overline{\nabla}_s(G) \leq O(k^2) \) for every \( G \in \mathcal{C} \) and \( s \geq 0 \).
Corollary

The following graph classes have bounded expansion:

- graphs with bounded maximum degree
- proper minor-closed graph classes, e.g.,
  - graphs with bounded tree-width
  - planar graphs
Remark on bounded tree-width

Bounded tree-width $\Rightarrow$ bounded expansion $\Rightarrow$ low tree-depth colorings, proving

**Claim**

*There exists a function $g$ such that for every $t$ a $p$, every graph with tree-width at most $t$ has a $(p, td \leq p - 1)$-coloring by $g(t, p)$ colors.*

as we promised before.
Other classes with bounded expansion

- graphs drawn in a fixed surface with a bounded number of crossings on each edge
- created by adding edges in mutual distance \( \omega(1) \) to graphs in any class with bounded expansion
- almost all graphs with linear number of edges
Generalizations of the subgraph problem

For all graph classes with bounded expansion (D., Král’, Thomas):

- testing first-order properties in linear time
  - e.g., having dominating set of size at most $k$ ($k$ fixed):
    $$(\exists x_1) \ldots (\exists x_k)(\forall y) \quad y = x_1 \lor \ldots \lor y = x_k \lor$$
    $$E(y, x_1) \lor \ldots \lor E(y, x_k).$$

- data structure for graphs with colored vertices and edges
  - linear-time initialization
  - change color of an element in $O(1)$
  - decide first-order query with bounded number of quantifiers in $O(1)$
A class of graphs \( C \) is nowhere-dense if there exists a function \( f \) such that for every \( s \), \( K_{f(s)} \) is not an \((\leq s)\)-topological minor of any graph in \( C \).

- Equivalently, for every \( s \), the set of \((\leq s)\)-topological minors of graphs in \( C \) does not contain all graphs.
- Bounded expansion \( \Rightarrow \) nowhere-dense
Properties of nowhere-dense classes

If $\mathcal{C}$ is nowhere-dense, then for every $\varepsilon > 0$, integer $s$ and $G \in \mathcal{C}$ with $n$ vertices:

- $\nabla_s(G) = O(n^\varepsilon)$, hence
- $\text{adm}_s(G) = O(n^\varepsilon)$, hence
- $G$ has $(s, \text{td} \leq s - 1)$ coloring by $O(n^\varepsilon)$ colors, hence
- we can test $H \subseteq G$ (for fixed $H$) in $O(n^{1+\varepsilon})$. 
Most results for graph classes with bounded expansion also holds for nowhere-dense graph classes (with $O(n^c)$ replacing constants). Exception:

**Problem**

*Are first-order properties FPT on nowhere-dense graph classes?*
Theorem

Assume that the subgraph problem is not FPT on the class of all graphs. If the subgraph problem is FPT on a class of graphs $\mathcal{C}$ closed on subgraphs, then $\mathcal{C}$ is nowhere-dense.

Proof.

For every $s \geq 0$,

$$ H \subseteq G \iff sd_s(H) \subseteq sd_s(G). $$
Examples of nowhere-dense classes

- locally bounded expansion, including
  - locally bounded tree-width
  - locally proper minor closed

**Definition**

A class of graphs $\mathcal{C}$ has **locally bounded expansion** if there exists a function $f$ such that for every $s, d \geq 0$ and every $G \in \mathcal{C}$, if $H$ is a subgraph of $G$ of radius at most $d$, then $\nabla_s(H) \leq f(s, d)$. 
The class

\[ \mathcal{C} = \{ G : \Delta(G) \leq \log \log |V(G)|, \operatorname{girth}(G) \geq \log \log |V(G)| \} \]

- is nowhere-dense: if \( \text{sd}_s(K_k) \in \mathcal{C} \), then \( k - 1 \leq \log \log |V(G)| \leq 3s \).
- does not have bounded expansion: unbounded minimum degree
Bounded expansion has many different characterizations.

But we still did not see the one that came the first chronologically.
Shallow minors

Definition

A depth $r$ minor of $G$ is a graph obtained from a subgraph of $G$ by contracting vertex-disjoint subgraphs of radius at most $r$. 

![Shallow minor example](image)
**Definition**

Let $\nabla_r(G)$ be the greatest average density $|E(H)|/|V(H)|$ of a depth $r$ minor $H$ of $G$.

**Remark:** Greatest Reduced Average Density, hence the $\nabla$ symbol.

**Theorem (D.)**

*For any $r \geq 0$ and any graph $G$,*

$$\nabla_{2r}(G) \leq 2\nabla_r(G) \leq 4(4\nabla_{2r})^{(r+1)^2}.$$

**Proof.**

Idea: split the spanning trees of the shallow minor on vertices of big enough degree.
Shallow minors and bounded expansion

Corollary

A class of graphs $\mathcal{C}$ has bounded expansion if and only if there exists a function $f$ such that for every $r$ and every $G \in \mathcal{C}$, $\nabla_r(G) \leq f(r)$.

We say that the expansion of $\mathcal{C}$ or of the graph $G$ is bounded by $f$. 

Introduction
Subgraph problem
Tree-depth
Orderings
Generalized coloring number
Bounded expansion
Nowhere-dense graph classes
Shallow minors
Closures
Orientations
Nowhere-dense graph classes and algorithms

Z. Dvořák

Introduction

Subgraph problem

Tree-depth

Orderings

Generalized coloring number

Bounded expansion

Nowhere-dense graph classes

Shallow minors

Closures

Orientations

Conclusions

Small separators

Definition

A set $S \subseteq V(G)$ is a separator if each component of $G - S$ has at most $2|V(G)|/3$ vertices.

Theorem (Plotkin, Rao and Smith)

*If a graph $G$ on $n$ vertices does not contain $K_h$ as depth $d \log_2 n$ minor, then $G$ has a separator of size at most $O(n/d + dh^2 \log n)$. Can be found in $O(|E(G)|n/d)$.*

Corollary

*If there exists $c \geq 0$ such that the expansion of $G$ is bounded by $O(r^c)$, then $G$ has a separator of size $(n \log n)^{1-1/(2c+2)}$. If the expansion of $G$ is bounded by a subexponential function, then $G$ has separator of sublinear size. Tight because of 3-regular expanders.*
Consequences of small separators

Corollary

If the expansion of $G$ is bounded by a subexponential function, then $G$ has sublinear tree-width.

Corollary (D., Norine)

For any function $f$ such that \( \limsup_{r \to \infty} \frac{\log \log f(r)}{\log r} < \frac{1}{3} \), there exists $c > 0$ such that the number of non-isomorphic graphs $G$ on $n$ vertices with expansion bounded by $f$ is at most $c^n$. 
Lemma

Let $H$ be the graph obtained from $G$ by blowing up each vertex to a clique of size $k$. Then $\overline{\Delta}_s(H)$ is bounded by a function of $\overline{\Delta}_s(G)$ and $k$.

Proof.

Let $F' \subseteq H$ be an $(\leq s)$-subdivision of a graph $F$ with $\delta(F) = \overline{\Delta}_s(H)$. Each $e \in E(F)$ has a path $P_e \subseteq F'$.

- $P_e$ does not contain twins, unless it is an edge
- merge twin branchpoints: min. degree $\geq (\delta - k + 1)/k$
- remove twins of branchpoints: average degree $A \geq (\delta - k + 1)/k - 2(k-1)$

Each $P_e$ now conflicts with $\leq (k - 1)s$ other paths. Choose largest subgraph where all paths are independent.
Blowing up vertices

Nowhere-dense graph classes and algorithms

Z. Dvořák

Introduction

Subgraph problem

Tree-depth

Orderings

Generalized coloring number

Bounded expansion

Nowhere-dense graph classes

Shallow minors

Closures

Orientations

- No twins in paths:

- Merge twin branch vertices:

  - Lose k-1 neighbors

  - Rest of the degree can decrease kX

- Remove twins of branches:

  - \( \leq (k-1) \cdot \Delta(Y) \)

  - Edges lost

- Each conflict with \( \leq (k-1) \cdot S \)
Lemma

Let $H$ be the graph obtained from $G$ by contracting a forest. Then $\nabla_s(H)$ is bounded by a function of $\nabla_{3s+5}(G)$.

Proof.

$$\nabla_s(H) \leq 2\nabla_{\lceil s/2 \rceil}(H) \leq 2\nabla_{3\lceil s/2 \rceil+1}(G) \leq 2f(\nabla_{3s+5}(G))$$
Theorem

For any fixed \( k \geq 0 \), the class of graphs that can be drawn in plane with at most \( k \) crossings on each edge has bounded expansion.

Proof.

Put vertices on crossings and subdivide the edges: planar graph, with bounded expansion. Blow up all vertices to cliques of size two, contract forest: creates crossings. Suppress vertices of degree two (at most \( 2k \) contractions of a forest).
Orientations with bounded degree

Claim

\[ \nabla_0(G) \leq d \text{ if and only if } G \text{ has an orientation with indegree at most } d. \]

- compare with: if \( \nabla_0(G) \leq d \), then \( G \) is \( 2d \)-degenerate (the reverse implication does not hold)
  - equivalently, \( G \) has an acyclic orientation with indegree \( \leq 2d \)
- we will now consider orientations whose acyclic versions correspond to generalized coloring numbers
Augmentations

Let $G$ be a directed graph. An unordered pair $\{u, v\}$ is

- a \textit{transitive pair} if $uw, wv \in E(G)$ for some $w \in V(G)$
- a \textit{fraternal pair} if $uw, vw \in E(G)$ for some $w \in V(G)$

Definition

A directed graph $H$ is an \textit{augmentation} of $G$ if the edge set of the underlying undirected graph of $H$ consists of the edges of $G$ and of all transitive and fraternal pairs in $G$.

I.e., add the transitive and fraternal pairs as edges and give them arbitrary orientations.
Density of augmentations

**Theorem**

If $H$ is the underlying undirected graph of an augmentation of $G$ and $d$ is the maximum indegree of $G$, then $\nabla_s(H)$ is bounded by a function of $d$ and $\nabla_{3s+5}(G)$.

**Proof.**

$H$ is a subgraph of graph obtained by replacing each vertex by $d + 1$ vertices and contracting a star forest:
Theorem

If $H$ is the underlying undirected graph of an augmentation of $G$ and $d$ is the maximum indegree of $G$, then $\nabla_s(H)$ is bounded by a function of $d$ and $\nabla_{3s+5}(G)$.

In particular, there exists an augmentation of $G$ with maximum indegree bounded by a function of $d$ and $\nabla_5(G)$. We call such an augmentation steady.
Short paths via augmentations

Lemma

Let $s \geq 2$, let $G$ be a graph, $G_0$ its orientation and $G_0, G_1, G_2, \ldots, G_s$ a sequence of augmentations. If the distance between $u$ and $v$ in $G$ is at most $(3/2)^{s-1} + 2$, then $u$ and $v$ are either adjacent or have a common in-neighbor in $G_s$.

Proof.

In each augmentation, the path of length $t$ gives rise to a path of length $\leq 2/3t + 2/3$.
Oracle for short paths

Fix \( s \geq 2 \). Find an orientation \( G_0 \) of \( G \) with bounded indegree and compute augmentations \( G_1, \ldots, G_s \).

- for each edge, remember the length of the shortest corresponding path
- plus one of the ways how it was created
- for bounded expansion classes, if steady augmentations are used:
  - linear-time preprocessing
  - by Lemma, \( O(1) \) queries for paths of length at most \( (3/2)^{s-1} + 2 \) (every vertex has only \( O(1) \) in-neighbors).
Dynamic version

- **idea:** maintain the augmentations when edges are added or removed
- **problem:** adding edge can create unbounded number of edges due to transitive pairs
- **solution:** only add edges for fraternal pairs (fraternal augmentation).
Lemma

Let $s \geq 0$, let $G$ be a graph, $G_0$ its orientation and $G_0, G_1, G_2, \ldots, G_s$ a sequence of fraternal augmentations. If the distance between $u$ and $v$ in $G$ is at most $s + 1$, then there exists a path $P = w_1w_2 \ldots w_t$ between $u = w_1$ and $v = w_t$ in $G_s$ of length at most $s + 1$, and an index $c \leq t$ such that the edges $w_1w_2, w_2w_3, \ldots, w_{c-1}w_c$ are oriented towards $u$ and the rest of the edges is oriented towards $v$.

To find the path, search in-neighbors up to distance $s + 1$ ($O(1)$ if steady augmentations are used)
Theorem (Brodal and Fagerberg)

For any \( d > 0 \) there exists \( D \) so that an orientation of a \( d \)-degenerate graph on \( n \) vertices with maximum indegree \( D \) can be maintained within the following time bounds:

- an edge can be added in \( O(\log n) \) (amortized)
- an edge can be removed in \( O(1) \)

Each vertex stores a list of in- and out-neighbors.
Maintaining the augmentations

- adding an edge results in $O(\log n)$ reorientations in $G_0$
- each reorientation adds or removes $O(1)$ edges in $G_1$
- $O(\log^2 n)$ reorientations in $G_1$, ... 

Gives $O(\log^s n)$ update time for maintaining paths of length at most $s$. 
Theorem

Let $s \geq 1$, let $G$ be a graph, $G_0$ its orientation and $G_0, G_1, G_2, \ldots, G_p$ a sequence of augmentations, where $p = 3(s + 1)^2$. Let $H$ be the underlying undirected graph of $G_p$. Then any proper coloring of $H$ gives an $(s, \text{td} \leq s - 1)$-coloring of $G$.

For bounded expansion classes, linear-time and the number of colors is bounded, if steady augmentations are used. Proof is lengthy and technical.
Theorem

Let $s \geq 1$, let $G$ be a graph, $G_0$ its orientation and $G_0, G_1, G_2, \ldots, G_p$ a sequence of augmentations, where $p = \binom{s+1}{2}$. Any proper coloring of $G_p$ is an $(s, \text{td} \leq \binom{2s}{2})$-coloring of $G$.

Proof.

- Show that no path on $2^s$ vertices uses $\leq s$ colors, since the subgraph induced by the path contains $K_{s+1}$.
- After $\binom{s}{2}$ augmentations we have two disjoint $K_s$;
- have directed Hamiltonian paths, starts $v_1$ and $v_2$;
- $v_1$ and $v_2$ adjacent or a common in-neighbor.
- Say $v_1$ has an in-neighbor $w$ outside of its clique,
- next $s$ augmentations to add $w$ to the clique.
Low tree-depth colorings via augmentations
Bounded expansion and nowhere-dense serve as good formalization of “structurally sparse” graphs.

Many natural graph classes have these properties.

Problems expressible in first-order logic can be solved efficiently for them.

Many other results for special classes of sparse graphs generalize to this setting.