PLANAR DECOMPOSITIONS AND THE CROSSING NUMBER OF GRAPHS WITH AN EXCLUDED MINOR

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ABSTRACT. Tree decompositions of graphs are of fundamental importance in structural and algorithmic graph theory. Planar decompositions generalise tree decompositions by allowing an arbitrary planar graph to index the decomposition. We prove that every graph that excludes a fixed graph as a minor has a planar decomposition with bounded width and a linear number of bags.

The crossing number of a graph is the minimum number of crossings in a drawing of the graph in the plane. We prove that planar decompositions are intimately related to the crossing number. In particular, a graph with bounded degree has linear crossing number if and only if it has a planar decomposition with bounded width and linear order. It follows from the above result about planar decompositions that every graph with bounded degree and an excluded minor has linear crossing number.

Analogous results are proved for the convex and rectilinear crossing numbers. In particular, every graph with bounded degree and bounded tree-width has linear convex crossing number, and every $K_{3,3}$-minor-free graph with bounded degree has linear rectilinear crossing number.

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1. Introduction

The *crossing number* of a graph[1] G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in a drawing[2] of G in the plane; see [28, 49, 74] for surveys. The crossing number is an important measure of the non-planarity of a graph [70], with applications in discrete and computational geometry [47, 69] and VLSI circuit design [5, 38, 39]. In information visualisation, one of the most important measures of the quality of a graph drawing is the number of crossings [52–54].

Upper bounds on the crossing number are the focus of this paper. Obviously $\mathsf{cr}(G) \leq \binom{\|G\|}{2}$ for every graph G. A family of graphs has *linear*[3] crossing number if $\mathsf{cr}(G) \in \mathcal{O}(|G|)$ for every graph G in the family. For example, Pach and Tóth [50] proved that graphs of bounded genus[4] and bounded degree have linear crossing number. Our main result states that bounded-degree graphs that exclude a fixed graph as a minor[5] have linear crossing number.

Theorem 1.1. *For every graph H and integer Δ, there is a constant $c = c(H, \Delta)$, such that every H-minor-free graph G with maximum degree at most Δ has crossing number $\mathsf{cr}(G) \leq c\,|G|$.*

Theorem 1.1 implies the above-mentioned result of Pach and Tóth [50], since graphs of bounded genus exclude a fixed graph as a minor (although the dependence on Δ is different

[1]We consider graphs G that are undirected, simple, and finite. Let $V(G)$ and $E(G)$ respectively be the vertex and edge sets of G. Let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. For each vertex v of a graph G, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ be the neighbourhood of v in G. The *degree* of v is $|N_G(v)|$. Let $\Delta(G)$ be the maximum degree of a vertex of G.

[2]A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A *crossing* is a point of intersection between two edges (other than a common endpoint). A drawing with no crossings is *plane*. A graph is *planar* if it has a plane drawing.

[3]If the crossing number of a graph is linear in the number of edges then it is also linear in the number of vertices. To see this, let G be a graph with n vertices and m edges. Suppose that $\mathsf{cr}(G) \leq cm$. If $m < 4n$ then $\mathsf{cr}(G) \leq 4cn$ and we are done. Otherwise $\mathsf{cr}(G) \geq m^3/64n^2$ by the 'Crossing Lemma' [3, 38]. Thus $m \leq 8\sqrt{c}n$ and $\mathsf{cr}(G) \leq 8c^{3/2}n$.

[4]Let $\mathbb{S}_\gamma$ be the orientable surface with $\gamma \geq 0$ handles. An *embedding* of a graph in $\mathbb{S}_\gamma$ is a crossing-free drawing in $\mathbb{S}_\gamma$. A 2*-cell embedding* is an embedding in which each region of the surface (bounded by edges of the graph) is an open disk. The (*orientable*) *genus* of a graph G is the minimum γ such that G has a 2-cell embedding in $\mathbb{S}_\gamma$. In what follows, by a *face* we mean the set of vertices on the boundary of the face. Let $F(G)$ be the set of faces in an embedded graph G. See the monograph by Mohar and Thomassen [41] for a thorough treatment of graphs on surfaces.

[5]Let vw be an edge of a graph G. Let G' be the graph obtained by identifying the vertices v and w, deleting loops, and replacing parallel edges by a single edge. Then G' is obtained from G by *contracting* vw. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A family of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. $\mathcal{F}$ is *proper* if it is not the family of all graphs. A deep theorem of Robertson and Seymour [63] states that every proper minor-closed family can be characterised by a finite family of excluded minors. Every proper minor-closed family is a subset of the H-minor-free graphs for some graph H. We thus focus on minor-closed families with one excluded minor.

in the two proofs; see Section 6). Moreover, there are graphs with a fixed excluded minor and unbounded genus. For other recent work on minors and crossing number see [12, 30, 31, 33, 34, 44, 51].

Note that the assumption of bounded degree in Theorem 1.1 is unavoidable. For example, the complete bipartite graph $K_{3,n}$ has no K_5-minor, yet has $\Omega(n^2)$ crossing number [43, 56]. Conversely, bounded degree does not by itself guarantee linear crossing number. For example, a random cubic graph on n vertices has $\Omega(n)$ bisection width [15, 19], which implies that it has $\Omega(n^2)$ crossing number [25, 38].

The proof of Theorem 1.1 is based on *planar decompositions*, which are introduced in Sections 2 and 3. This combinatorial structure generalises tree decompositions by allowing an arbitrary planar graph to index the decomposition. We prove that planar decompositions and the crossing number are intimately related (Section 4). In particular, a graph with bounded degree has linear crossing number if and only if it has a planar decomposition with bounded width and linear order (Corollary 4.4). We study planar decompositions of: K_5-minor-free graphs (Section 5), graphs embedded in surfaces (Section 6), and finally graphs with an excluded minor (Section 7). One of the main contributions of this paper is to prove that every graph that excludes a fixed graph as a minor has a planar decomposition with bounded width and linear order. Theorem 1.1 easily follows.

1.1. **Complementary Results.** A graph drawing is *rectilinear* (or *geometric*) if each edge is represented by a straight line-segment. The *rectilinear crossing number* of a graph G, denoted by $\overline{\mathsf{cr}}(G)$, is the minimum number of crossings in a rectilinear drawing of G; see [2, 7, 14, 35, 40, 42, 57, 64, 66, 69]. A rectilinear drawing is *convex* if the vertices are positioned on a circle. The *convex* (or *outerplanar*, *circular*, or *1-page book*) *crossing number* of a graph G, denoted by $\mathsf{cr}^\star(G)$, is the minimum number of crossings in a convex drawing of G; see [16, 59, 67, 68]. Obviously $\mathsf{cr}(G) \leq \overline{\mathsf{cr}}(G) \leq \mathsf{cr}^\star(G)$ for every graph G. *Linear* rectilinear and *linear* convex crossing numbers are defined in an analogous way to linear crossing number.

It is unknown whether an analogue of Theorem 1.1 holds for rectilinear crossing number[6]. On the other hand, we prove that $K_{3,3}$-minor-free graphs with bounded degree have linear rectilinear crossing number.

Theorem 1.2. *For every integer Δ, there is a constant $c = c(\Delta)$, such that every $K_{3,3}$-minor-free graph G with maximum degree at most Δ has rectilinear crossing number $\overline{\mathsf{cr}}(G) \leq c\,|G|$.*

An analogue of Theorem 1.1 for convex crossing number does not hold, even for planar graphs, since Shahrokhi et al. [67] proved that the $n\times n$ planar grid G_n (which has maximum degree 4) has convex crossing number $\Omega(|G_n| \log |G_n|)$. It is natural to ask which property

[6]The crossing number and rectilinear crossing number are not related in general. In particular, for every integer $k \geq 4$, Bienstock and Dean [7] constructed a graph G_k with crossing number 4 and rectilinear crossing number k. It is easily seen that G_k has no K_{14}-minor. However, the maximum degree of G_k increases with k. Thus G_k is not a counterexample to an analogue of Theorem 1.1 for rectilinear crossing number.

of the planar grid forces up the convex crossing number. In some sense, we show that tree-width[7] is one answer to this question. In particular, G_n has tree-width n. More generally, we prove that every graph with large tree-width has many crossings on some edge in every convex drawing (Proposition 8.4). On the other hand, we prove that graphs with bounded tree-width and bounded degree have linear convex crossing number.

Theorem 1.3. *For all integers k and Δ, there is a constant $c = c(k, \Delta)$, such that every graph G with tree-width at most k and maximum degree at most Δ has convex crossing number $\overline{\mathsf{cr}}(G) \leq c\,|G|$.*

Again, the assumption of bounded degree in Theorem 1.3 is unavoidable since $K_{3,n}$ has tree-width 3.

2. Graph Decompositions

Let G and D be graphs, such that each vertex of D is a set of vertices of G (called a *bag*). Note that we allow distinct vertices of D to be the same set of vertices in G; that is, $V(D)$ is a multiset. For each vertex v of G, let $D(v)$ be the subgraph of D induced by the bags that contain v. Then D is a *decomposition* of G if:

- $D(v)$ is connected and nonempty for each vertex v of G, and
- $D(v)$ and $D(w)$ touch[8] for each edge vw of G.

Decompositions, when D is a tree, were introduced by Robertson and Seymour [61]. Diestel and Kühn [20][9] first generalised the definition for arbitrary graphs D.

Let D be a decomposition of a graph G. The *width* of D is the maximum cardinality of a bag. The number of bags that contain a vertex v of G is the *spread* of v in D. The *spread* of D is the maximum spread of a vertex of G. The *order* of D is the number of bags. D has *linear order* if its order is $\mathcal{O}(|G|)$. If the graph D is a tree, then the decomposition D is a *tree decomposition*. If the graph D is a cycle, then the decomposition D is a *cycle decomposition*. The decomposition D is *planar* if the graph D is planar. The *genus* of the decomposition D is the genus of the graph D.

A decomposition D of a graph G is *strong* if $D(v)$ and $D(w)$ intersect for each edge vw of G. The *tree-width* of G, denoted by $\mathsf{tw}(G)$, is 1 less than the minimum width of a strong tree decomposition of G. For example, a graph has tree-width 1 if and only if it is a forest. Graphs with tree-width 2 (called *series-parallel*) are planar, and are characterised as those graphs with no K_4-minor. Tree-width is particularly important in structural and algorithmic graph theory; see the surveys [9, 55].

For applications to crossing number, tree decompositions are not powerful enough: even the $n \times n$ planar grid has tree-width n. We show in Section 4 that planar decompositions

[7]Tree-width is a minor-closed parameter that is defined in Section 2.

[8]Let A and B be subgraphs of a graph G. Then A and B *intersect* if $V(A) \cap V(B) \neq \emptyset$, and A and B *touch* if they intersect or $v \in V(A)$ and $w \in V(B)$ for some edge vw of G.

[9]A decomposition was called a *connected decomposition* by Diestel and Kühn [20]. Similar definitions were introduced by Agnew [1].

are the right type of decomposition for applications to crossing number. It is tempting to define the 'planar-width' of a graph G to be the minimum width in a planar decomposition of G. However, by the following lemma of Diestel and Kühn [20], every graph would then have bounded planar-width. We include the proof for completeness.

Lemma 2.1 ([20])**.** *Every graph G has a strong planar decomposition of width 2, spread $|G|+1$, and order $\binom{|G|+1}{2}$.*

Proof. Let $n := |G|$ and say $V(G) = \{1, 2, \ldots, n\}$. Define a graph D with vertex set $V(D) := \{\{i, j\} : 1 \leq i \leq j \leq n\}$ and edge set $E(D) := \{\{i, j\}\{i + 1, j\} : 1 \leq i \leq n - 1, 1 \leq j \leq n\}$. Then D is a planar subgraph of the $n \times n$ grid; see Figure 1. For each vertex i of G, the set of bags that contain i is $\{\{i, j\} : 1 \leq j \leq n\}$, which induces a (connected) n-vertex path in D. For each edge ij of G, the bag $\{i, j\}$ contains i and j. Therefore D is a strong decomposition of G. The width is 2, since each bag has two vertices. Each vertex is in $n + 1$ bags. □

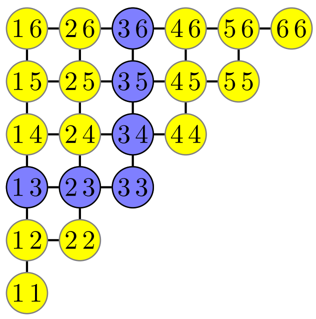

FIGURE 1. A strong planar decomposition of K_6 with width 2 and order 21; the subgraph $D(3)$ is highlighted.

The planar decomposition in Lemma 2.1 has large order (quadratic in $|G|$). The remainder of this paper focuses on planar decompositions with linear order.

Strong tree decompositions are the most widely studied decompositions in the literature. This paper focuses on decompositions that are not necessarily strong. One advantage is that every graph obviously has a decomposition isomorphic to itself (with width 1). On the other hand, if G has a strong decomposition D of width k, then

$$\|G\| \leq \binom{k}{2} |D| \quad . \tag{1}$$

It follows that if G has a strong decomposition isomorphic to itself then the width is at least $\sqrt{\frac{2\|G\|}{|G|}}$, which is unbounded for dense graphs, as observed by Diestel and Kühn [20]. Note that if G has a (non-strong) decomposition D of width k, then

$$\|G\| \leq k^2 \|D\| + \binom{k}{2} |D| \quad . \tag{2}$$

Every tree T satisfies the Helly property: every collection of pairwise intersecting subtrees of T have a vertex in common. It follows that if a tree T is a strong decomposition of

G then every clique[10] of G is contained in some bag of T. Other graphs do not have this property. It will be desirable (for performing k-sums in Section 3) that (non-tree) decompositions have a similar property. We therefore introduce the following definitions.

For $p \geq 0$, a *p-clique* is a clique of cardinality p. A *$(\leq p)$-clique* is a clique of cardinality at most p. For $p \geq 2$, a decomposition D of a graph G is a *p-decomposition* if each $(\leq p)$-clique of G is a subset of some bag of D, or is a subset of the union of two adjacent bags of D. An $\omega(G)$-decomposition of G is called an *ω-decomposition*. A p-decomposition D of G is *strong* if each $(\leq p)$-clique of G is a subset of some bag of D. Observe that a (strong) 2-decomposition is the same as a (strong) decomposition, and a (strong) p-decomposition also is a (strong) q-decomposition for all $q \in [2, p]$.

3. MANIPULATING DECOMPOSITIONS

In this section we describe four tools for manipulating graph decompositions that are repeatedly used in the remainder of the paper.

TOOL #1. CONTRACTING A DECOMPOSITION: Our first tool describes the effect of contracting an edge in a decomposition.

Lemma 3.1. *Suppose that D is a planar (strong) p-decomposition of a graph G with width k. Say XY is an edge of D. Then the decomposition D' obtained by contracting the edge XY into the vertex $X \cup Y$ is a planar (strong) p-decomposition of G with width $\max\{k, |X \cup Y|\}$. In particular, if $|X \cup Y| \leq k$ then D' also has width k.*

Proof. Contracting edges preserves planarity. Thus D' is planar. Contracting edges preserves connectiveness. Thus $D'(v)$ is connected for each vertex v of G. Contracting the edge XY obviously maintains the required properties for each $(\leq p)$-clique of G. □

Lemma 3.1 can be used to decrease the order of a decomposition at the expense of increasing the width. The following observation is a corollary of Lemma 3.1.

Corollary 3.2. *Suppose that D is a (strong) p-decomposition of a graph G with width k, and that D has a matching[11] M. The decomposition obtained from D by contracting M is a (strong) p-decomposition of G with width at most $2k$ and order $|D| - |M|$.* □

Lemma 3.3. *Suppose that a graph G has a (strong) planar p-decomposition D of width k and order at most $c|G|$ for some $c \geq 1$. Then G has a (strong) planar p-decomposition of width $c'k$ and order $|G|$, for some c' depending only on c.*

Proof. If $|D| \leq 3$ the result is trivial. Now assume that $|D| \geq 4$. Without loss of generality, D is a (3-connected) planar triangulation. Biedl et al. [6] proved that every 3-connected planar graph on n vertices has a matching of at least $\frac{n}{3}$ edges. Applying this result to D, and by Corollary 3.2, G has a (strong) planar p-decomposition of width at most $2k$

[10] A *clique* of a graph G is a set of pairwise adjacent vertices in G. The maximum cardinality of a clique of G is denoted by $\omega(G)$.

[11] A *matching* is a set of pairwise disjoint edges.

and order at most $\frac{2}{3}|D|$. By induction, for every integer $i \geq 1$, G has a (strong) planar p-decomposition of width $2^i k$ and order at most $(\frac{2}{3})^i|D|$. With $i := \lceil \log_{3/2} c \rceil$, the assumption that $|D| = c|G|$ implies that G has a (strong) planar p-decomposition of width $2^i k$ and order $|G|$. □

TOOL #2. COMPOSING DECOMPOSITIONS: Our second tool describes how two decompositions can be composed.

Lemma 3.4. *Suppose that D is a (strong) p-decomposition of a graph G with width k, and that J is a decomposition of D with width ℓ. Then G has a (strong) p-decomposition isomorphic to J with width $k\ell$.*

Proof. Let J' be the graph isomorphic to J that is obtained by renaming each bag $Y \in V(J)$ by $Y' := \{v \in V(G) : v \in X \in Y \text{ for some } X \in V(D)\}$. There are at most ℓ vertices $X \in Y$, and at most k vertices $v \in X$. Thus each bag of J' has at most $k\ell$ vertices.

First we prove that $J'(v)$ is connected for each vertex v of G. Let A' and B' be two bags of J' that contain v. Let A and B be the corresponding bags in D. Thus $v \in X_1$ and $v \in X_t$ for some bags $X_1, X_t \in V(D)$ such that $X_1 \in A$ and $X_t \in B$ (by the construction of J'). Since $D(v)$ is connected, there is a path $X_1, X_2, \dots, X_t$ in D such that v is in each X_i. In particular, each $X_i X_{i+1}$ is an edge of D. Now $J(X_i)$ and $J(X_{i+1})$ touch in J. Thus there is path in J between any vertex of J that contains X_1 and any vertex of J that contains X_t, such that every bag in the path contains some X_i. In particular, there is a path P in J between A and B such that every bag in P contains some X_i. Let $P' := \{Y' : Y \in P\}$. Then $v \in Y'$ for each bag Y' of P' (by the construction of J'). Thus P' is a connected subgraph of J' that includes A' and B', and v is in every such bag. Therefore $J'(v)$ is connected.

It remains to prove that for each ($\leq p$)-clique C of G,

(a) C is a subset of some bag of J', or

(b) C is a subset of the union of two adjacent bags of J'.

Moreover, we must prove that if D is strong then case (a) always occurs. Since D is a p-decomposition of G,

(1) $C \subseteq X$ for some bag $X \in V(D)$, or

(2) $C \subseteq X_1 \cup X_2$ for some edge $X_1 X_2$ of D.

First suppose that case (1) applies, which always occurs if D is strong. Then there is some bag $Y \in V(J)$ such that $X \in Y$ (since X is a vertex of D and J is a decomposition of D). Thus $C \subseteq Y'$ by the construction of J'. Now suppose that case (2) applies. Then $D(X_1)$ and $D(X_2)$ touch in D. That is, X_1 and X_2 are in a common bag of D, or $X_1 \in Y_1$ and $X_2 \in Y_2$ for some edge $Y_1 Y_2$ of D. If X_1 and X_2 are in a common bag Y, then since $C \subseteq X_1 \cup X_2$, we have $C \subseteq Y'$ by the construction of J'; that is, case (a) occurs. Otherwise, $X_1 \in Y_1$ and $X_2 \in Y_2$ for some edge $Y_1 Y_2$ of D. Then $C \cap X_1 \subseteq Y_1'$ and $C \cap X_2 \subseteq Y_2'$. Since $C \subseteq X_1 \cup X_2$ we have $C \subseteq Y_1' \cup Y_2'$; that is, case (b) occurs. □

TOOL #3. ω-DECOMPOSITIONS*:* The third tool converts a decomposition into an ω-decomposition with a small increase in the width. A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d.

Lemma 3.5. *Every d-degenerate graph G has a strong ω-decomposition isomorphic to G of width at most $d+1$.*

Proof. It is well known (and easily proved) that G has an acyclic orientation[12] such that each vertex has indegree at most d. Replace each vertex v by the bag $\{v\} \cup N^-_G(v)$. Every subgraph of G has a *sink*. Thus every clique is a subset of some bag. The set of bags that contain a vertex v are indexed by $\{v\} \cup N^+_G(v)$, which induces a connected subgraph in G. Thus we have a strong ω-decomposition. Each bag has cardinality at most $d+1$. □

Lemma 3.6. *Suppose that D is a decomposition of a d-degenerate graph G of width k. Then G has a strong ω-decomposition isomorphic to D of width $k(d+1)$.*

Proof. By Lemma 3.5, G has a strong ω-decomposition isomorphic to G of width d. By Lemma 3.4, G has a strong ω-decomposition isomorphic to D with width $k(d+1)$. □

In Lemma 3.6, the 'blow-up' in the width is bounded by a constant factor for the graphs that we are interested in: even in the most general setting, H-minor-free graphs are $c|H|\sqrt{\log|H|}$-degenerate for some constant c [36, 72, 73].

TOOL #4. CLIQUE-SUMS OF DECOMPOSITIONS: Our fourth tool describes how to determine a planar decomposition of a clique-sum of two graphs, given planar decompositions of the summands[13] Let G_1 and G_2 be disjoint graphs. Suppose that C_1 and C_2 are k-cliques of G_1 and G_2 respectively, for some integer $k \geq 0$. Let $C_1 = \{v_1, v_2, \dots, v_k\}$ and $C_2 = \{w_1, w_2, \dots, w_k\}$. Let G be a graph obtained from $G_1 \cup G_2$ by identifying v_i and w_i for each $i \in [1, k]$, and possibly deleting some of the edges v_iv_j. Then G is a *k-sum* of G_1 and G_2 *joined* at $C_1 = C_2$. An ℓ-sum for some $\ell \leq k$ is called a *$(\leq k)$-sum*. For example, if G_1 and G_2 are planar then it is easily seen that every (≤ 2)-sum of G_1 and G_2 is also planar, as illustrated in Figure 2.

Lemma 3.7. *Suppose that for integers $p \leq q$, a graph G is a $(\leq p)$-sum of graphs G_1 and G_2, and each G_i has a (strong) planar q-decomposition D_i of width k_i. Then G has a (strong) planar q-decomposition of width* $\max\{k_1, k_2\}$ *and order* $|D_1| + |D_2|$.

Proof. Let $C := V(G_1) \cap V(G_2)$. Then C is a $(\leq p)$-clique, and thus a $(\leq q)$-clique, of both G_1 and G_2. Thus for each i,

(1) $C \subseteq X_i$ for some bag X_i of D_i, or

[12]If each edge of a graph G is directed from one endpoint to the other, then we speak of an *orientation* of G with arc set $A(G)$. An orientation with no directed cycle is *acyclic*. For each vertex v of an orientation of G, let $N^-_G(v) := \{w \in V(G) : wv \in A(G)\}$ and $N^+_G(v) := \{w \in V(G) : vw \in A(G)\}$. The *indegree* and *outdegree* of v are $|N^-_G(v)|$ and $N^+_G(v)|$ respectively. A *sink* is a vertex with outdegree 0.

[13]Leaños and Salazar [37] recently proved some related results on the additivity of crossing numbers.

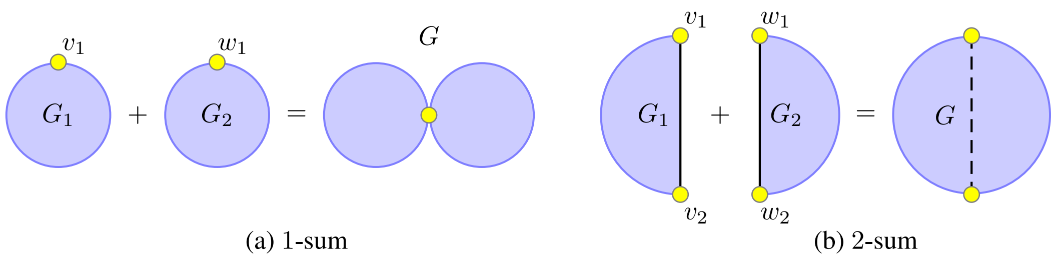

FIGURE 2. Clique-sums of planar graphs. In (b) we can assume that the edges v_1v_2 and w_1w_2 are respectively on the outerfaces of G_1 and G_2.

(2) $C \subseteq X_i \cup Y_i$ for some edge X_iY_i of D_i.

If (1) is applicable, which is the case if D_i is strong, then consider $Y_i := X_i$ in what follows.

Let D be the graph obtained from the disjoint union of D_1 and D_2 by adding edges X_1X_2, X_1Y_2, Y_1X_2, and Y_1Y_2. By considering X_1Y_1 to be on the outerface of G_1 and X_2Y_2 to be on the outerface of G_2, observe that D is planar, as illustrated in Figure 3.

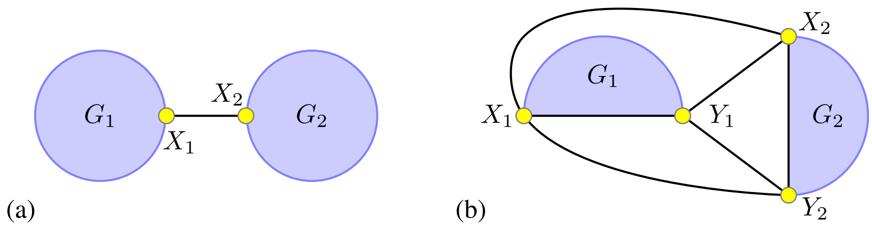

FIGURE 3. Sum of (a) strong planar decompositions, (b) planar decompositions.

We now prove that $D(v)$ is connected for each vertex v of G. If $v \notin V(G_1)$ then $D(v) = D_2(v)$, which is connected. If $v \notin V(G_2)$ then $D(v) = D_1(v)$, which is connected. Otherwise, $v \in C$. Thus $D(v) = D_1(v) \cup D_2(v)$. Since $v \in X_1 \cup Y_1$ and $v \in X_2 \cup Y_2$, and X_1, Y_1, X_2, Y_2 induce a connected subgraph ($\subseteq K_4$) in D, we have that $D(v)$ is connected.

Each $(\leq q)$-clique B of G is a $(\leq q)$-clique of G_1 or G_2. Thus B is a subset of some bag of D, or B is a subset of the union of two adjacent bags of D. Moreover, if D_1 and D_2 are both strong, then B is a subset of some bag of D. Therefore D is a q-decomposition of G, and if D_1 and D_2 are both strong then D is also strong. The width and order of D are obviously as claimed. □

A graph can be obtained by repeated (≤ 1)-sums of its biconnected components. Thus Lemma 3.7 with $p = 1$ implies:

Corollary 3.8. *Let G be a graph with biconnected components $G_1, G_2, \dots, G_t$. Suppose that each G_i has a (strong) planar q-decomposition of width k_i and order n_i. Then G has a (strong) planar q-decomposition of width* $\max_i k_i$ *and order* $\sum_i n_i$. □

Note that in the proof of Lemma 3.7, if $X_1 \subseteq X_2$ (for example) then, by Lemma 3.1, we can contract the edge X_1X_2 in D and merge the corresponding bags. The width is unchanged and the order is decreased by 1. This idea is repeatedly used in the remainder of the paper.

4. Planar Decompositions and the Crossing Number

The following lemma is the key link between planar decompositions and the crossing number of a graph.

Lemma 4.1. *Suppose that D is a planar decomposition of a graph G of width k. Then the crossing number of G satisfies*

$$\mathsf{cr}(G) \;\leq\; 2\,\Delta(G)^2 \sum_{X\in V(D)} \tbinom{|X|+1}{2} \;\leq\; k(k+1)\,\Delta(G)^2\,|D| \enspace .$$

Moreover, if $s(v)$ is the spread of each vertex v of G in D, then G has a drawing with the claimed number of crossings, in which each edge vw is represented by a polyline with at most $s(v)+s(w)-2$ bends.

Proof. By the Fáry-Wagner Theorem [29, 75], D has a rectilinear drawing with no crossings. Let $\epsilon > 0$. Let $R_\epsilon(X)$ be the open disc of radius ϵ centred at each vertex X in the drawing of D. For each edge XY of D, let $R_\epsilon(XY)$ be the union of all segments with one endpoint in $R_\epsilon(X)$ and one endpoint in $R_\epsilon(Y)$. For some $\epsilon > 0$,

(a) $R_\epsilon(X) \cap R_\epsilon(Y) = \emptyset$ for all distinct bags X and Y of D, and
(b) $R_\epsilon(XY) \cap R_\epsilon(AB) = \emptyset$ for all edges XY and AB of D that have no endpoint in common.

For each vertex v of G, choose a bag S_v of D that contains v. For each vertex v of G, choose a point $p(v) \in R_\epsilon(S_v)$, and for each bag X of D, choose a set $P(X)$ of $\sum_{v\in X} \deg_G(v)$ points in $R_\epsilon(X)$, so that if $\mathcal{P} = \cup\{P(X) : X \in V(D)\} \cup \{p(v) : v \in V(G)\}$ then:

(c) no two points in $\mathcal{P}$ coincide,
(d) no three points in $\mathcal{P}$ are collinear, and
(e) no three segments, each connecting two points in $\mathcal{P}$, cross at a common point.

The set $\mathcal{P}$ can be chosen iteratively since the boundary of each disc $R_\epsilon(X)$ is a circle containing infinitely many points, but there are only finitely many excluded points on the boundary of $R_\epsilon(X)$ (since a line intersects a circle in at most two points).

Draw each vertex v at $p(v)$. For each edge vw of G, a simple polyline

$$L(vw) = (p(v), x_1, x_2, \dots, x_a, y_1, y_2, \dots, y_b, p(w))$$

(defined by its endpoints and bends) is a *feasible* representation of vw if:

(1) $a \in [0, s(v)-1]$ and $b \in [0, s(w)-1]$,
(2) each bend x_i is in $P(X_i)$ for some bag X_i containing v,
(3) each bend y_i is in $P(Y_i)$ for some bag Y_i containing w,
(4) the bags $S_v, X_1, X_2, \dots, X_a, Y_1, Y_2, \dots, Y_b, S_w$ are distinct (unless $S_v = S_w$ in which case $a = b = 0$), and
(5) consecutive bends in $L(vw)$ occur in adjacent bags of D.

Since $D(v)$ and $D(w)$ touch, there is a feasible polyline that represents vw.

A drawing of G is *feasible* if every edge of G is represented by a feasible polyline, and no two bends coincide. Since each $|P(X)| = \sum_{v \in X} \deg(v)$, there is a feasible drawing. In particular, no edge passes through a vertex by properties (c)–(e), and no three edges have a common crossing point by property (e). By property (1), each edge vw has at most $s(v) + s(w) - 2$ bends.

Now choose a feasible drawing that minimises the total (Euclidean) length of the edges (with $\{p(v) : v \in V(G)\}$ and $\{P(X) : X \in V(D)\}$ fixed).

By properties (a), (b) and (2)–(5), each segment in a feasible drawing is contained within $R_\epsilon(X)$ for some bag X of D, or within $R_\epsilon(XY)$ for some edge XY of D. Consider a crossing in G between edges vw and xy. Since D is drawn without crossings, the crossing point is contained within $R_\epsilon(X)$ for some bag X of D, or within $R_\epsilon(XY)$ for some edge XY of D. Thus some endpoint of vw, say v, and some endpoint of xy, say x, are in a common bag X. In this case, charge the crossing to the 5-tuple (vw, v, xy, x, X). Observe that the number of such 5-tuples is

$$\sum_{X \in V(D)} \sum_{v,x \in X} \deg_G(v) \cdot \deg_G(x) \ .$$

At most four crossings are charged to each 5-tuple (vw, v, xy, x, X), since by property (4), each of vw and xy have at most two segments that intersect $R_\epsilon(X)$ (which might pairwise cross). We claim that, in fact, at most two crossings are charged to each such 5-tuple.

Suppose on the contrary that at least three crossings are charged to some 5-tuple (vw, v, xy, x, X). Then two segments of vw intersect $R_\epsilon(X)$ and two segments of xy intersect $R_\epsilon(X)$. In particular, $p(v) \notin R_\epsilon(X)$ and $p(x) \notin R_\epsilon(X)$, and vw and xy each have a bend in $R_\epsilon(X)$. Let (r_1, r_2, r_3) be the 2-segment polyline in the representation of vw, where r_2 is the bend of vw in $R_\epsilon(X)$. Let (t_1, t_2, t_3) be the 2-segment polyline in the representation of xy, where t_2 is the bend of xy in $R_\epsilon(X)$. Since at least three crossings are charged to (vw, v, xy, x, X), in the set of segments $\{r_1r_2, r_2r_3, t_1t_2, t_2t_3\}$, at most one pair of segments, one from vw and one from xy, do not cross. Without loss of generality, t_1t_2 and r_2r_3 are this pair. Observe that the crossing segments r_1r_2 and t_1t_2 are the diagonals of the convex quadrilateral $r_1t_2r_2t_1$. Replace the segments r_1r_2 and t_1t_2 by the segments r_1t_2 and t_1r_2, which are on opposite sides of the quadrilateral. Thus the combined length of r_1t_2 and t_1r_2 is less than the combined length of r_1r_2 and t_1t_2. Similarly, replace the segments r_3r_2 and t_3t_2 by the segments r_3t_2 and t_3r_2. We obtain a feasible drawing of G with less total length. This contradiction proves that at most two crossings are charged to each 5-tuple (vw, v, xy, x, X).

Thus the number of crossings is at most twice the number of 5-tuples. Therefore the number of crossings is at most

$$2 \sum_{X \in V(D)} \sum_{v,x \in X} \deg_G(v) \cdot \deg_G(x) \ \leq \ 2\,\Delta(G)^2 \sum_{X \in V(D)} \tbinom{|X|+1}{2} \ .$$

□

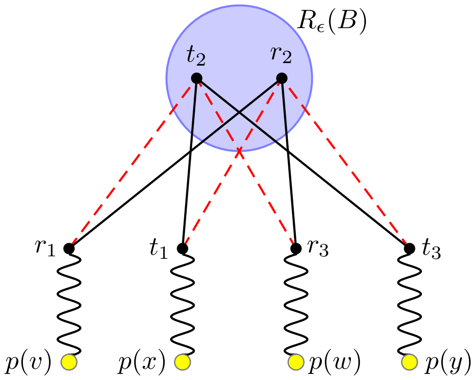

FIGURE 4. In the proof of Lemma 4.1, to shorten the total edge length the crossed segments are replaced by the dashed segments.

Note that the bound on the crossing number in Lemma 4.1 is within a constant factor of optimal for the complete graph. An easy generalisation of Lemma 2.1 proves that for all $n \geq k \geq 2$, K_n has a strong planar decomposition of width k and order at most $c(\frac{n}{k})^2$ for some constant c. Thus Lemma 4.1 implies that $\mathsf{cr}(K_n) \leq ck(k+1)\,\Delta(K_n)^2\,(\frac{n}{k})^2 \leq cn^4$, which is tight within a constant factor [57].

The following result is converse to Lemma 4.1.

Lemma 4.2. *Let G be a graph with n isolated vertices. Suppose that G has a drawing with c crossings in which q non-isolated vertices are not incident to a crossed edge. Then G has a planar decomposition of width 2 and order $\lceil \frac{n}{2} \rceil + q + c$, and G has a strong planar decomposition of width 2 and order $\lceil \frac{n}{2} \rceil + c + \|G\|$.*

Proof. First, pair the isolated vertices of G. Each pair can form one bag in a decomposition of width 2, adding $\lceil \frac{n}{2} \rceil$ to the order. Now assume that G has no isolated vertices.

We first construct the (non-strong) decomposition. Arbitrarily orient each edge of G. Let D be the planar graph obtained from the given drawing of G by replacing each vertex v by the bag $\{v\}$, and replacing each crossing between oriented edges vw and xy by a degree-4 vertex $\{v, x\}$. Thus an oriented edge vw of G is replaced by some path $\{v\}\{v, x_1\}\{v, x_2\} \dots \{v, x_r\}\{w\}$ in D. In particular, $D(v)$ and $D(w)$ touch at the edge $\{v, x_r\}\{w\}$. Moreover, $D(v)$ is a (connected) tree for each vertex v of G. Thus D is a decomposition of G. Each bag contains at most two vertices. The order is $|G| + c$. For each vertex v of G that is incident to some crossed edge, $\{v\}\{v, x\}$ is an edge in D for some vertex x. Contract the edge $\{v\}\{v, x\}$ in D and merge the bags. By Lemma 3.1, D remains a planar decomposition of width 2. The order is now $q + c$.

Now we make D strong. For each edge vw of G, there is an edge XY of D where $D(v)$ and $D(w)$ touch. That is, $v \in X$ and $w \in Y$. Replace XY by the path $X\{v, w\}Y$. Now each edge of G is in some bag of D, and D is strong. This operation introduces a further $\|G\|$ bags. Thus the order is $q + c + \|G\|$. For each non-isolated vertex v that is not incident to a crossed edge, choose an edge vw incident to v. Then $\{v\}\{v, w\}$ is an edge in D;

contract this edge and merge the bags. By Lemma 3.1, D remains a strong decomposition of G with width 2. The order is now $c + \|G\|$. □

Note the following special case of Lemma 4.2.

Corollary 4.3. *Every planar graph G with n isolated vertices has a strong planar decomposition of width 2 and order $\lceil \frac{n}{2} \rceil + \|G\| \leq 3|G| - 6$.* □

Lemmas 4.1 and 4.2 imply that the crossing number and planar decompositions are intimately related in the following sense.

Corollary 4.4. *Let $\mathcal{F}$ be a family of graphs with bounded degree. Then $\mathcal{F}$ has linear crossing number if and only if every graph in $\mathcal{F}$ has a planar decomposition with bounded width and linear order.* □

5. K_5-MINOR-FREE GRAPHS

In this section we prove the following upper bound on the crossing number.

Theorem 5.1. *Every K_5-minor-free graph G has crossing number*

$$\mathsf{cr}(G) \;\leq\; 8\,\Delta(G)^2\,(|G| - 2)\;.$$

Theorem 5.1 follows from Lemma 4.1 and Theorem 5.2 below, in which we construct ω-decompositions of K_5-minor-free graphs G. Since $\omega(G) \leq 4$ and each clique can be spread over two bags, it is natural to consider ω-decompositions of G with width 2.

Theorem 5.2. *Every K_5-minor-free graph G with $|G| \geq 3$ has a planar ω-decomposition of width 2 and order at most $\frac{4}{3}|G| - 2$, and at most $|G| - 2$ bags have cardinality 2.*

The proof of Theorem 5.2 is based on the following classical theorem of Wagner [76] and the two following lemmas, where V_8 is the graph obtained from the 8-cycle by adding an edge between each pair of antipodal vertices; see Figure 5(a).

Theorem 5.3 ([76]). *A graph is K_5-minor-free if and only if it can be obtained from planar graphs and V_8 by (≤ 3)-sums.*

Lemma 5.4. *Every K_5-minor-free graph G with $|G| \geq 3$ has a partition of $E(G)$ into three sets E^1, E^2, E^3 such that:*

- *each of E^1, E^2, E^3 has at most $|G| - 2$ edges,*
- *every triangle has one edge in each of E^1, E^2, E^3,*
- *if a subgraph H of G is isomorphic to V_8, then $E(H) \cap E^j$ is a perfect matching in H for all j.*

Moreover, if G is edge-maximal (with no K_5-minor), then every vertex is incident to an edge in E^j and an edge in E^ℓ for some $j \neq \ell$.

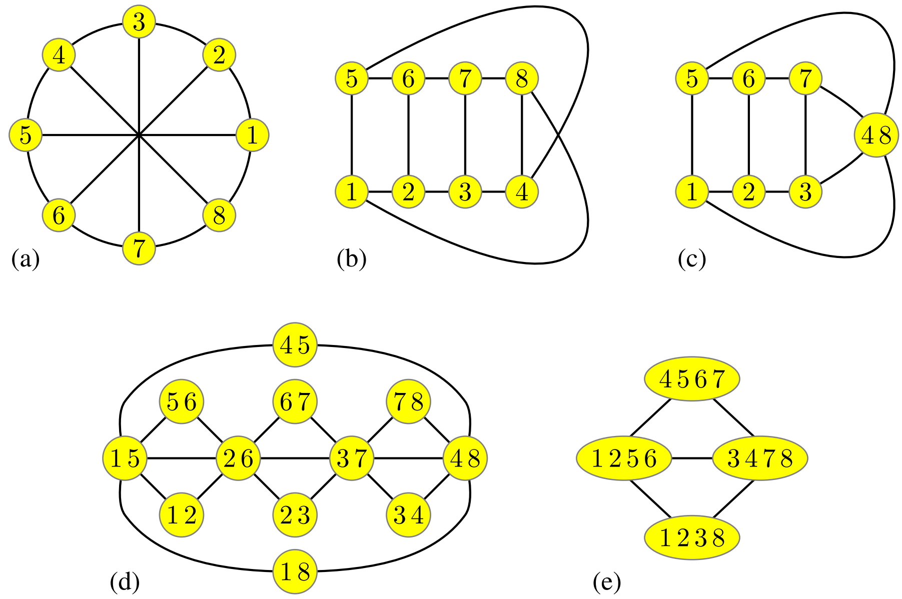

FIGURE 5. (a) The graph V_8. (b) Drawing of V_8 with one crossing.
(c) Planar ω-decomposition of V_8 with width 2 and order 7.
(d) Strong planar decomposition of V_8 with width 2 and order 12.
(e) Strong planar ω-decomposition of V_8 with width 4 and order 4.

Proof. By Theorem 5.3, we need only consider the following three cases.

Case (a). G is planar: Let G' be a planar triangulation of G. By the Four-Colour Theorem [60], G' has a proper vertex-colouring with colours a, b, c, d. Now determine a Tait edge-colouring [71]. Let E^1 be the set of edges of G' whose endpoints are coloured ab or cd. Let E^2 be the set of edges of G' whose endpoints are coloured ac or bd. Let E^3 be the set of edges of G' whose endpoints are coloured ad or bc. Since the vertices of each triangle are 3-coloured, the edges of each triangle are in distinct E^j. In particular, the edges of each face of G' are in distinct E^j. Each edge of G' is in two of the $2|G| - 4$ faces of G'. Thus $|E^j| = |G| - 2$. The sets $E^j \cap E(G)$ thus satisfy the first two properties for G. Since V_8 is nonplanar, G has no V_8 subgraph, and the third property is satisfied vacuously. Finally, if G is edge-maximal, then $G' = G$, each vertex v is in some face, and v is incident to two edges in distinct sets.

Case (b). $G = V_8$: Using the vertex-numbering in Figure 5(a), let $E^1 := \{12, 34, 56, 78\}$, $E^2 := \{23, 45, 67, 81\}$, and $E^3 := \{15, 26, 37, 48\}$. Each E^j is a matching of four edges. The claimed properties follow.

Case (c). G is a (≤ 3)-sum of two smaller K_5-minor-free graphs G_1 and G_2: Let C be the join set. By induction, there is a partition of each $E(G_i)$ into three sets E_i^1, E_i^2, E_i^3 with the desired properties. Permute the set indices so that for each edge e with endpoints in C, $e \in E_1^j \cap E_2^j$ for some j. This is possible because C is a (≤ 3)-clique in G_1 and G_2.

For each $j = 1, 2, 3$, let $E^j := E_1^j \cup E_2^j$. If $|C| \leq 2$, then $|E^j| \leq |E_1^j| + |E_2^j| \leq (|G_1| - 2) + (|G_2| - 2) = |G_1| + |G_2| - 4 \leq |G| - 2$, as desired. Otherwise, C is a triangle in G_1 and G_2, and $|E_1^j \cap E_2^j| = 1$. Thus $|E^j| \leq |E_1^j|+|E_2^j|-1 \leq (|G_1|-2)+(|G_2|-2)-1 = |G_1|+|G_2|-5 = |G|-2$, as desired. Each triangle of G is in G_1 or G_2, and thus has one edge in each set E^j.

Consider a V_8 subgraph H of G. Since V_8 is edge-maximal K_5-minor-free, H is an induced subgraph. Since V_8 is 3-connected and triangle-free, H is a subgraph of G_1 or G_2. Thus $H \cap E^j$ is a perfect matching of H by induction.

If G is edge-maximal, then G_1 and G_2 are both edge-maximal. Thus every vertex v of G is incident to at least two edges in distinct sets (since the same property holds for v in G_1 or G_2). □

For a set E of edges in a graph G, a vertex v of G is *E-isolated* if v is incident to no edge in E.

Lemma 5.5. *Suppose that E is a set of edges in a K_5-minor-free graph G such that every triangle of G has exactly one edge in E, and if S is a subgraph of G isomorphic to V_8 then $E(S) \cap E$ is a perfect matching in S. Let V be the set of E-isolated vertices in G. Then G has a planar ω-decomposition D of width 2 with $V(D) = \{\{v\} : v \in V\} \cup \{\{v,w\} : vw \in E\}$ with no duplicate bags.*

Proof. By Theorem 5.3, we need only consider the following four cases.

Case (a). $G = K_4$: Say $V(G) = \{v, w, x, y\}$. Without loss of generality, $E = \{vw, xy\}$. Thus $V = \emptyset$. Then $D := K_2$, with bags $\{v, w\}$ and $\{x, y\}$, is the desired decomposition of G. Now assume that $G \neq K_4$.

Case (b). $G = V_8$: Thus E is a perfect matching of G. Then $D := K_4$, with one bag for each edge in E, is the desired decomposition of G.

Case (c). G is planar and has no separating triangle (see Figure 6): Fix a plane drawing of G. Thus every triangle of G is a face. Initially, let D be the planar decomposition of G with $V(D) := \{\{v\} : v \in V(G)\}$ and $E(D) := \{\{v\}\{w\} : vw \in E(G)\}$. For each edge $vw \in E$, introduce a new bag $\{v, w\}$ in D, and replace the edge $\{v\}\{w\}$ by the path $\{v\}\{v, w\}\{w\}$. Thus D is a planar subdivision of G. Now consider each triangle uvw of G. Without loss of generality, $vw \in E$. Replace the path $\{v\}\{u\}\{w\}$ in D by the edge $\{u\}\{v, w\}$. Since uvw is a face with only one edge in E, D remains planar. Moreover, $D(v)$ is a connected star for each vertex v of G. Since G has no separating triangle and $G \neq K_4$, each clique is a (≤ 3)-clique. Thus, by construction, each clique is contained in a bag of D, or is contained in the union of two adjacent bags of G. Therefore D is a planar ω-decomposition of G with width 2. The order is $|G| + |E|$. For each vertex $v \notin V$, there is an edge incident to v that is in E. Choose such an edge $vw \in E$. Thus $\{v\}\{v, w\}$ is an edge of D. Contract this edge and merge the bags. By Lemma 3.1, D remains a planar ω-decomposition. Now $V(D) = \{\{v\} : v \in V\} \cup \{\{v, w\} : vw \in E\}$.

Case (d). G is a (≤ 3)-sum of two smaller K_5-minor-free graphs G_1 and G_2: Let C be the join set. Let $E_1 := E \cap E(G_1)$ and $E_2 := E \cap E(G_2)$. Then every triangle of G_i has exactly one edge in E_i. Let V_i be the set of vertices of G_i that are E_i-isolated. By induction, each G_i has a planar ω-decomposition D_i of width 2 with $V(D_i) := E_i \cup V_i$. By

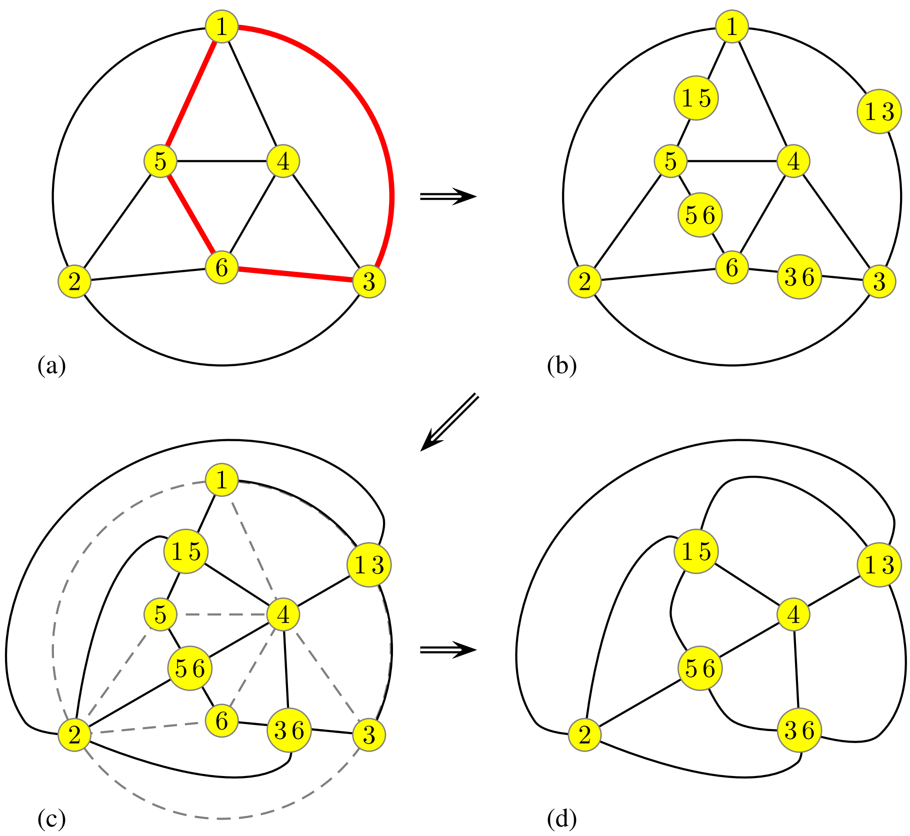

FIGURE 6. An example of the construction of a planar ω-decomposition in case (c) of Lemma 5.5, where $E = \{13, 15, 56, 36\}$ and $V = \{2, 4\}$.

Lemma 3.7 with $p = q = 4$ and $k_1 = k_2 = 2$, G has a planar ω-decomposition D of width 2 with $V(D) := V_1 \cup E_1 \cup V_2 \cup E_2$. Moreover, each bag of D_1 that intersects C is adjacent to each bag of D_2 that intersects C.

If there is a duplicate bag X in D, then one copy of X is from D_1 and the other copy is from D_2, and X intersects C. Thus the two copies are adjacent. Contract the edge XX in D into the bag X. By Lemma 3.1, D remains a planar ω-decomposition of G with width 2. Now D has no duplicate bags.

Every bag in D is either a vertex or an edge of G. If $\{v, w\}$ is a bag of D, then vw is in $E_1 \cup E_2 = E$. Conversely, if $vw \in E$ then $vw \in E_1$ or $vw \in E_2$. Thus the bags of cardinality 2 in D are in one-to-one correspondence with edges in E.

Suppose there is bag $\{v\}$ in D but $v \notin V$. Then $v \in C$, $v \in V_2$, and $v \notin V_1$ (or symmetrically, $v \in V_1$ and $v \notin V_2$). That is, v is incident to an edge $vw \in E_1$ but v is E_2-isolated. Now $\{v, w\}$ is a bag in D_1, and $\{v\}$ is a bag in D_2. These bags are adjacent in D. Contract the edge $\{v, w\}\{v\}$ in D into the bag $\{v, w\}$. By Lemma 3.1, D remains an ω-decomposition of G with width 2. An analogous argument applies if $v \in V_1$ and $v \notin V_2$. We have proved that if $\{v\}$ is a bag of D, then $v \in V$. Conversely, if $v \in V$ then $v \in V_1$

or $v \in V_2$ (possibly both), and there is a bag $\{v\}$ in D. Thus the singleton bags of D are in one-to-one correspondence with vertices in V.

Therefore $V(D) = \{\{v\} : v \in V\} \cup \{\{v, w\} : vw \in E\}$ with no duplicate bags. □

Proof of Theorem 5.2. Let H be an edge-maximal K_5-minor-free graph containing G as a spanning subgraph. By Lemma 5.4, there is a partition of $E(H)$ into three sets E^1, E^2, E^3 with the stated properties. We now construct a planar ω-decomposition of H of width 2 and order at most $\frac{4}{3}|H| - 2$, and at most $|H| - 2$ bags have cardinality 2. This is the desired decomposition of G (since $|G| = |H|$).

Let V^j be the set of E^j-isolated vertices in H. Consider an E^1-isolated vertex v. Then each edge incident to v is in $E^2 \cup E^3$. Since v is incident to (at least) two edges in distinct sets, v is not E^2-isolated and is not E^3-isolated. In general, $V^i \cap V^j = \emptyset$ for distinct i and j[14]. Thus $|V^i| \leq \frac{1}{3}|H|$ for some i. By Lemma 5.5, H has a planar ω-decomposition D of width 2 with order $|E^j| + |V^j| \leq |H| - 2 + \frac{1}{3}|H| = \frac{4}{3}|H| - 2$. □

The following two propositions, while not used to prove bounds on the crossing number, are of independent interest. First we consider strong planar 3-decompositions of K_5-minor-free graphs.

Proposition 5.6. *Every K_5-minor-free graph G with $|G| \geq 3$ has a strong planar 3-decomposition of width 3 and order $3|G| - 8$. Moreover, for all $n \geq 3$, there is a planar graph G, such that $|G| = n$ and every strong planar 3-decomposition of G with width 3 has order at least $3|G| - 8$.*

Proof. Add edges to G so that it is edge-maximal with no K_5-minor. This does not affect the claim. By Theorem 5.3, we need only consider the following three cases.

Case (a). G is a planar triangulation with no separating triangle: Let D be the dual graph of G. That is, $V(D) := F(G)$ and $E(D) := \{XY : X, Y \in F(G), |X \cap Y| = 2\}$. Then D is a planar graph. For each vertex v of G, $D(v)$ is the connected cycle consisting of the faces containing v. Every (≤ 3)-clique of G is contained in some face of G (since G has no separating triangle) and is thus in some bag of D. Thus D is a strong planar 3-decomposition of G. The order is $|F(G)| = 2|G| - 4$, which is at most $3|G| - 8$ unless $|G| = 3$, in which case one bag suffices.

Case (b). $G = V_8$: Then $\mathsf{cr}(V_8) = 1$, as illustrated in Figure 5(b). Thus by Lemma 4.2, G has a strong planar decomposition of width 2 and order $\lceil \frac{0}{2} \rceil + 12 + 1 = 13 < 3 \cdot |G| - 8$; see Figure 5(d). This decomposition is also a strong planar 3-decomposition since $K_3 \not\subseteq V_8$.

Case (c). G is a (≤ 3)-sum of two smaller K_5-minor-free graphs G_1 and G_2, each with $|G_i| \geq 3$. Let C be the join set. Thus C is a clique in G_1 and in G_2. By induction, each G_i has a strong planar 3-decomposition of width 3 and order $3|G_i| - 8$. By Lemma 3.7 with $p = q = k_1 = k_2 = 3$, G has a strong planar 3-decomposition D of width 3 and order $3|G_1| - 8 + 3|G_2| - 8 = 3(|G_1| + |G_2|) - 16$. If $|C| \leq 2$ then $|G_1| + |G_2| \leq |G| + 2$ and D has order at most $3|G| - 10$. Otherwise $|C| = 3$. Since the decompositions of G_1

[14]Note that it is possible for V_1, V_2, V_3 to partition $V(H)$; for example, when H is a planar Eulerian triangulation.

and G_2 are strong, C is a bag in both decompositions. Thus CC is an edge of D, which can be contracted by Lemma 3.1. Therefore D has order $3(|G_1| + |G_2|) - 16 - 1 = 3(|G| + 3) - 16 - 1 = 3|G| - 8$.

This completes the proof of the upper bound. It remains to prove the lower bound. Observe that in a strong planar 3-decomposition of width 3, every triangle is a distinct bag. The lower bound follows since there is a planar graph with $n \geq 3$ vertices and $3n - 8$ triangles [78]. □

It follows from Euler's Formula and Theorem 5.3 that every K_5-minor-free graph G has at most $3|G| - 6$ edges, and is thus 5-degenerate (also see Lemma 6.4 below). Thus by Lemma 3.5, G has a strong ω-decomposition isomorphic to G of width 5. Since $\omega(G) \leq 4$, it is natural to consider strong ω-decompositions of width 4.

Proposition 5.7. *Every K_5-minor-free graph G with $|G| \geq 4$ has a strong planar ω-decomposition of width 4 and order at most $\frac{4}{3}|G| - 4$. Moreover, for all $n \geq 1$, there is a planar graph G_n, such that $|G_n| = 3n$ and every strong ω-decomposition of G_n with width 4 has order at least $\frac{7}{6}|G_n| - 3$.*

Proof. Add edges to G so that it is edge-maximal with no K_5-minor. This does not affect the claim. By Theorem 5.3, we need only consider the following three cases.

Case (a). G is a planar triangulation with no separating triangle: If $G = K_4$ then the decomposition with one bag containing all four vertices satisfies the requirements. K_5 minus an edge is the only 5-vertex planar triangulation, and it has a separating triangle. Thus we can assume that $|G| \geq 6$. By Lemma 5.4[15], G has a set S of $|G| - 2$ edges such that every face of G has exactly one edge in S. Each edge $e = vw \in S$ is in two faces xvw and yvw; let $P(e) := \{v, w, x, y\}$. Let D be the graph with $V(D) := \{\{v\} : v \in V(G)\} \cup \{P(e) : e \in S\}$, where $\{v\}$ is adjacent to $P(e)$ if and only if $v \in P(e)$. Then D is a planar bipartite graph. For each vertex v of G, $D(v)$ is a (connected) star rooted at $\{v\}$. Every face (and thus every triangle) is in some bag of D. Thus D is a strong ω-decomposition of G with width 4 and order $2|G| - 2$. For each vertex v of G, select one bag P containing v, and contract the edge $\{v\}P$ in D. By Lemma 3.1, we obtain a strong ω-decomposition of G with width 4 and order $|G| - 2 \leq \frac{4}{3}|G| - 4$ (since $|G| \geq 6$).

Case (b). $G = V_8$: As illustrated in Figure 5(e), G has a strong planar ω-decomposition of width 4 and order $4 < \frac{4}{3}|G| - 4$.

Case (c). G is a (≤ 3)-sum of two smaller K_5-minor-free graphs G_1 and G_2, each with $|G_i| \geq 4$: Let C be the join set. By induction, each G_i has a strong planar ω-decomposition D_i of width 4 and order at most $\frac{4}{3}|G_i| - 4$. By Lemma 3.7 with $p \leq 3$ and $q = k_1 = k_2 = 4$, G has a strong planar ω-decomposition of width 4 and order $|D_1| + |D_2| \leq \frac{4}{3}|G_1| - 4 + \frac{4}{3}|G_2| - 4 = \frac{4}{3}(|G_1| + |G_2|) - 8 \leq \frac{4}{3}|G| - 4$.

[15]The full strength of the Four-Colour Theorem (used in the proof of Lemma 5.4) is not needed here. That G has (one) set S of $|G| - 2$ edges such that every face of G has exactly one edge in S quickly follows from Petersen's Matching Theorem applied to the dual; see [13].

This completes the proof of the upper bound. It remains to prove the lower bound. Let $G_1 := K_3$. As illustrated in Figure 7, construct G_{n+1} from G_n as follows. Insert a triangle inside some face of G_n, and triangulate so that each of the three new vertices have degree 4. This creates seven new triangles and no K_4. Thus $|G_i| = 3n$ and G_n has $7n - 6$ triangles. In a strong ω-decomposition of G_n, each triangle is in a bag. Since G_n contains no K_4, each bag of width 4 can accommodate at most two triangles. Thus the number of bags is at least half the number of triangles, which is $\frac{7}{2}n - 3 = \frac{7}{6}|G_i| - 3$. □

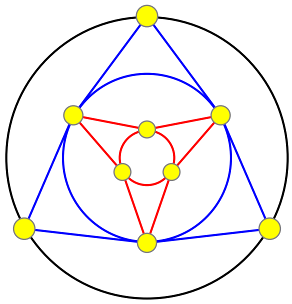

FIGURE 7. The graph G_3 in the lower bound of Proposition 5.7.

We conjecture that for $n \geq 2$, the graph G_n in Proposition 5.7 actually requires at least $\frac{4}{3}|G_n| - 4$ bags in every strong ω-decomposition of width 4.

6. GRAPHS EMBEDDED ON A SURFACE

Recall that $\mathbb{S}_\gamma$ is the orientable surface with γ handles. As illustrated in Figure 8, a *cycle* in $\mathbb{S}_\gamma$ is a closed curve in the surface. A cycle is *contractible* if it is contractible to a point in the surface. A noncontractible cycle is *separating* if it separates $\mathbb{S}_\gamma$ into two connected components.

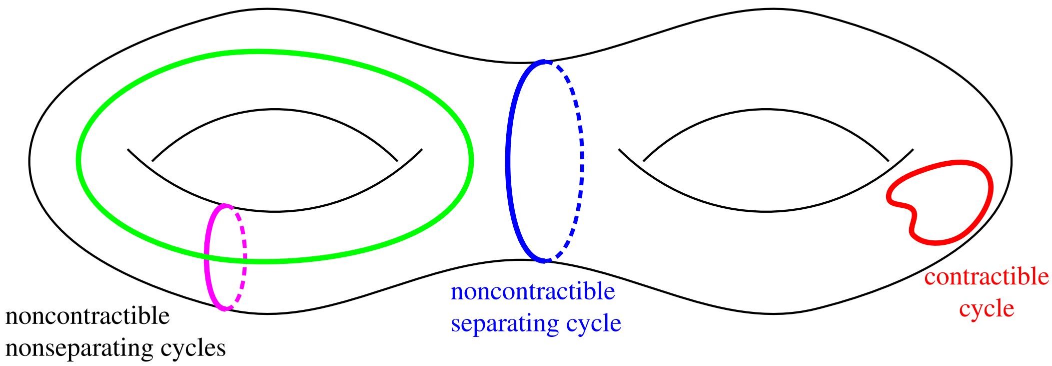

FIGURE 8. Cycles on the double torus.

Let G be a graph embedded in $\mathbb{S}_\gamma$. A *noose* of G is a cycle C in $\mathbb{S}_\gamma$ that does not intersect the interior of an edge of G. Let $V(C)$ be the set of vertices of G intersected by C. The *length* of C is $|V(C)|$.

Pach and Tóth [50] proved that, for some constant c_γ, the crossing number of every graph G of genus γ satisfies

$$\mathsf{cr}(G) \;\leq\; c_\gamma \sum_{v\in V(G)} \deg(v)^2 \;\leq\; 2c_\gamma\,\Delta(G)\,\|G\| \;. \tag{3}$$

The following lemma is probably well known.

Lemma 6.1. *Every graph with $n \geq 3$ vertices and genus γ has at most $(\sqrt{3\gamma}+3)n-6$ edges.*

Proof. Let G be an edge-maximal graph with n vertices and genus γ. Suppose that an embedding of G in $\mathbb{S}_\gamma$ has f faces. Euler's Formula states that $m-n-f=2\gamma-2$. Since G is edge-maximal, every face is bounded by three edges and each edge is in the boundary of two faces. Thus $m=3f/2$ and $f=2m/3$. Hence $m=3n+6\gamma-6$. If $\gamma=0$ then we are done. Now assume that $\gamma\geq 1$. We need to prove that $3n+6\gamma\leq(\sqrt{3\gamma}+3)n$. That is, $6\gamma\leq\sqrt{3\gamma}n$, or equivalently $\gamma\leq n^2/12$, which is true since K_n has genus $\lceil (n-3)(n-4)/12\rceil < n^2/12$; see [41, 58]. □

Equation (3) and Lemma 6.1 imply that

$$\mathsf{cr}(G) \;\leq\; c_\gamma\,\Delta(G)\,|G| \;. \tag{4}$$

By Lemma 4.2, G has a planar decomposition of width 2 and order $c_\gamma\,\Delta(G)\,|G|$. We now provide an analogous result without the dependence on $\Delta(G)$, but at the expense of an increased bound on the width.

Theorem 6.2. *Every graph G with genus γ has a planar decomposition of width 2^γ and order $3^\gamma|G|$.*

The key to the proof of Theorem 6.2 is the following lemma, whose proof is inspired by similar ideas of Pach and Tóth [50].

Lemma 6.3. *Let G be a graph with a 2-cell embedding in $\mathbb{S}_\gamma$ for some $\gamma\geq 1$. Then G has a decomposition of width 2, genus at most $\gamma-1$, and order $3|G|$.*

Proof. Since $\gamma\geq 1$, G has a noncontractible nonseparating noose. Let C be a noncontractible nonseparating noose of minimum length $k:=|V(C)|$. Orient C and let $V(C):=(v_1,v_2,\dots,v_k)$ in the order around C. For each vertex $v_i\in V(C)$, let $E^\ell(v_i)$ and $E^r(v_i)$ respectively be the set of edges incident to v_i that are on the left-hand side and right-hand side of C (with respect to the orientation). Cut the surface along C, and attach a disk to each side of the cut. Replace each vertex $v_i\in V(C)$ by two vertices v_i^ℓ and v_i^r respectively incident to the edges in $E^\ell(v_i)$ and $E^r(v_i)$. Embed v_i^ℓ on the left-hand side of the cut, and embed v^r on the right-hand side of the cut. We obtain a graph G' embedded in a surface of genus at most $\gamma-1$ (since C is nonseparating).

Let $L:=\{v_i^\ell : v\in V(C)\}$ and $R:=\{v_i^r : v\in V(C)\}$. By Menger's Theorem, the maximum number of disjoint paths between L and R in G' equals the minimum number of vertices that separate L from R in G'. Let Q be a minimum set of vertices that separate L

from R in G'. Then there is a noncontractible nonseparating noose in G that only intersects vertices in Q. (It is nonseparating in G since L and R are identified in G.) Thus $|Q| \geq k$ by the minimality of $|V(C)|$. Hence there exist k disjoint paths $P_1, P_2, \dots, P_k$ between L and R in G', where the endpoints of P_i are v_i^ℓ and $v_{\sigma(i)}^r$, for some permutation σ of $[1, k]$. In the disc with R on its boundary, draw an edge from each vertex $v_{\sigma(i)}^r$ to v_i^r such that no three edges cross at a single point and every pair of edge cross at most once. Add a new vertex $x_{i,j}$ on each crossing point between edges $v_{\sigma(i)}^r v_i^r$ and $v_{\sigma(j)}^r v_j^r$. Let G'' be the graph obtained. Then G'' is embedded in $S_{\gamma-1}$.

We now make G'' a decomposition of G. Replace v_i^ℓ by $\{v_i\}$ and replace v_i^r by $\{v_i\}$. Replace every other vertex v of G by $\{v\}$. Replace each 'crossing' vertex $x_{i,j}$ by $\{v_i, v_j\}$. Now for each vertex $v_i \in V(C)$, add v_i to each bag on the path P_i from v_i^ℓ to $v_{\sigma(i)}^r$. Thus $G''(v_i)$ is a (connected) path. Clearly $G''(v)$ and $G''(w)$ touch for each edge vw of G. Hence G'' is a decomposition of G with genus at most $\gamma - 1$. Since the paths $P_1, P_2, \dots, P_k$ are pairwise disjoint, the width of the decomposition is 2.

It remains to bound the order of G''. Let $n := |G|$. Observe that G'' has at most $n+k+\binom{k}{2}$ vertices. One of the paths P_i has at most $\frac{n+k}{k}$ vertices. For ease of counting, add a cycle to G' around R. Consider the path in G' that starts at v_i^ℓ, passes through each vertex in P_i, and then takes the shortest route from $v_{\sigma(i)}^r$ around R back to v_i^r. The distance between $v_{\sigma(i)}^r$ and v_i^r around R is at most $\frac{k}{2}$. This path in G' forms a noncontractible nonseparating noose in G (since if two cycles in a surface cross in exactly one point, then both are noncontractible).

The length of this noose in G is at most $\frac{n+k}{k} - 1 + \frac{k}{2}$ (since v_i^ℓ and v_i^r both appeared in the path). Hence $\frac{n+k}{k} - 1 + \frac{k}{2} \geq k$ by the minimality of $|V(C)|$. Thus $k \leq \sqrt{2n}$. Therefore G'' has at most $n + \sqrt{2n} + \binom{\sqrt{2n}}{2} \leq 3n$ vertices. □

Proof of Theorem 6.2. We proceed by induction on γ. If $\gamma = 0$ then G is planar, and G itself is a planar decomposition of width $1 = 2^0$ and order $n = 3^0 n$. Otherwise, by Lemma 6.3, G has a decomposition D of width 2, genus $\gamma - 1$, and order $3n$. By induction, D has a planar decomposition of width $2^{\gamma-1}$ and order $3^{\gamma-1}(3n) = 3^\gamma n$. By Lemma 3.4 with $p = k = 2$, and $\ell = 2^{\gamma-1}$, G has a planar decomposition of width $2 \cdot 2^{\gamma-1} = 2^\gamma$ and order $3^\gamma n$. □

Theorem 6.2 and Lemma 4.1 imply that every graph G with genus γ has crossing number $\mathsf{cr}(G) \leq 12^\gamma\, \Delta(G)^2\, |G|$, which for fixed γ, is weaker than the bound of Pach and Tóth [50] in (4). The advantage of our approach is that it generalises for graphs with an arbitrary excluded minor (and the dependence on γ is much smaller).

We now prove that a graph G embedded on a surface has an ω-decomposition with small width and linear order. To do so, we apply Lemma 3.6, which requires a bound on the degeneracy of G.

Lemma 6.4. *Every graph G of genus γ is $(2\sqrt{3\gamma} + 6)$-degenerate. If $\sqrt{3\gamma}$ is an integer then G is $(2\sqrt{3\gamma} + 5)$-degenerate.*

Proof. By Lemma 6.1, G has average degree $\frac{2\|G\|}{|G|} < 2(\sqrt{3\gamma} + 3)$. Thus G has a vertex of degree less than $2\sqrt{3\gamma} + 6$. Moreover, if $\sqrt{3\gamma}$ is an integer, then G has a vertex of degree at most $2\sqrt{3\gamma} + 5$. The result follows since every subgraph of G has genus at most γ. □

Theorem 6.5. *Every graph G of genus γ has a planar ω-decomposition of width $2^\gamma(2\sqrt{3\gamma}+7)$ and order $3^\gamma\,|G|$.*

Proof. By Theorem 6.2, G has a planar decomposition D of width at most 2^γ and order $3^\gamma n$. By Lemma 6.4, G is $(2\sqrt{3\gamma}+6)$-degenerate. Thus by Lemma 3.6, G has a planar ω-decomposition isomorphic to D with width $2^\gamma(2\sqrt{3\gamma}+7)$. □

7. H-MINOR-FREE GRAPHS

For integers $h \geq 1$ and $\gamma \geq 0$, Robertson and Seymour [62] defined a graph G to be *h-almost embeddable* in $\mathbb{S}_\gamma$ if G has a set X of at most h vertices such that $G \setminus X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$ such that:

- G_0 has an embedding in $\mathbb{S}_\gamma$,
- the graphs $G_1, G_2, \dots, G_h$ (called *vortices*) are pairwise disjoint,
- there are faces[16] $F_1, F_2, \dots, F_h$ of the embedding of G_0 in $\mathbb{S}_\gamma$, such that each $F_i = V(G_0) \cap V(G_i)$,
- if $F_i = (u_{i,1}, u_{i,2}, \dots, u_{i,|F_i|})$ in clockwise order about the face, then G_i has a strong $|F_i|$-cycle decomposition Q_i of width h, such that each vertex $u_{i,j}$ is in the j-th bag of Q_i.

The following 'characterisation' of H-minor-free graphs is a deep theorem by Robertson and Seymour [62].

Theorem 7.1 ([62])**.** *For every graph H there is a positive integer $h = h(H)$, such that every H-minor-free graph G can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in some surface in which H cannot be embedded.*

The following theorem is one of the main contributions of this paper.

Theorem 7.2. *For every graph H there is an integer $k = k(H)$, such that every H-minor-free graph G has a planar ω-decomposition of width k and order $|G|$.*

We prove Theorem 7.2 by a series of lemmas.

Lemma 7.3. *Every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ has a planar decomposition of width $h(2^\gamma+1)$ and order $3^\gamma\,|G|$.*

Proof. By Theorem 6.2, G_0 has a planar decomposition D of width at most 2^γ and order $3^\gamma\,|G_0| \leq 3^\gamma\,|G|$. We can assume that D is connected. For each vortex G_i, add each vertex in the j-th bag of Q_i to each bag of D that contains $u_{i,j}$. The bags of D now contain at most $2^\gamma h$ vertices. Now add X to every bag. The bags of D now contain at most $(2^\gamma+1)h$ vertices. For each vertex v that is not in a vortex, $D(v)$ is unchanged by the addition of the vortices, and is thus connected. For each vertex v in a vortex G_i, $D(v)$ is the subgraph of D induced by the bags (in the decomposition of G_0) that contain $u_{i,j}$, where v is in the j-th bag of Q_i. Now $Q_i(v)$ is a connected subgraph of the cycle Q_i, and for each vertex $u_{i,j}$, the subgraphs $G_0(u_{i,j})$ and $G_0(u_{i,j+1})$ touch. Thus $D(v)$ is connected. (This argument is

[16]Recall that we equate a face with the set of vertices on its boundary.

similar to that used in Lemma 3.4.) $D(v)$ is connected for each vertex $v \in X$ since D itself is connected. □

Lemma 7.4. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $d = d(h, \gamma)$, such that every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ is d-degenerate.*

Proof. If G is h-almost embeddable in $\mathbb{S}_\gamma$ then every subgraph of G is h-almost embeddable in $\mathbb{S}_\gamma$. Thus it suffices to prove that if G has n vertices and m edges, then its average degree $\frac{2m}{n} \leq d$. Say each G_i has m_i edges. G has at most hn edges incident to X. Thus $m \leq hn + \sum_{i=0}^{h} m_i$. By Lemma 6.1, $m_0 < (\sqrt{3\gamma}+3)n$. Now $m_i \leq \binom{h}{2}|F_i|$ by Equation (1) with D an $|F_i|$-cycle. Since $G_1, G_2, \dots, G_h$ are pairwise disjoint, $\sum_{i=1}^{h} m_i \leq \binom{h}{2}n$. Thus $m < (h + \sqrt{3\gamma} + 3 + \binom{h}{2})n$. Taking $d = h(h+1) + 2\sqrt{3\gamma} + 6$ we are done. □

Lemmas 3.6, 7.3 and 7.4 imply:

Corollary 7.5. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $k = k(h, \gamma) \geq h$, such that every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ has a planar ω-decomposition of width k and order $3^\gamma\,|G|$.* □

Now we bring in $(\leq h)$-sums.

Lemma 7.6. *For all integers $h \geq 1$ and $\gamma \geq 0$, every graph G that can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in $\mathbb{S}_\gamma$ has a planar ω-decomposition of width k and order $\max\{1, 3^\gamma(h+1)(|G|-h)\}$, where $k = k(h, \gamma)$ from Corollary 7.5.*

Proof. We proceed by induction on $|G|$. If $|G| \leq h$ then the decomposition of G with all its vertices in a single bag satisfies the claim (since $k \geq h$).

Now assume that $|G| \geq h+1$. If G is h-almost embeddable in $\mathbb{S}_\gamma$, then by Corollary 7.5, G has a planar ω-decomposition of width k and order $3^\gamma|G|$, which, since $|G| \geq h+1$, is at most $3^\gamma(h+1)(|G|-h)$, as desired.

Otherwise, G is a $(\leq h)$-sum of graphs G_1 and G_2, each of which, by induction, has a planar ω-decomposition of width k and order $\max\{1, 3^\gamma(h+1)(|G_i|-h)\}$. By Lemma 3.7, G has a planar ω-decomposition D of width k and order

$$|D| = \max\{1, 3^\gamma(h+1)(|G_1|-h)\} + \max\{1, 3^\gamma(h+1)(|G_2|-h)\} \ .$$

Without loss of generality, $|G_1| \leq |G_2|$. If $|G_2| \leq h$ then $|D| = 2 \leq 3^\gamma(h+1)(|G|-h)$, as desired. If $|G_1| \leq h$ and $|G_2| \geq h+1$, then $|D| = 1 + 3^\gamma(h+1)(|G_2|-h)$, which, since $|G| \geq |G_2|+1$, is at most $3^\gamma(h+1)(|G|-h)$, as desired. Otherwise, both $|G_1| \geq h+1$ and $|G_2| \geq h+1$. Thus the order of D is $3^\gamma(h+1)(|G_1|+|G_2|-2h) \leq 3^\gamma(h+1)(|G|-h)$, as desired. □

Proof of Theorem 7.2. Let $h = h(H)$ from Theorem 7.1. Let $\mathbb{S}_\gamma$ be the surface in Theorem 7.1 in which H cannot be embedded. If G has no H-minor then, by Theorem 7.1, G can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in $\mathbb{S}_\gamma$. By Lemma 7.6, G has a planar ω-decomposition of width k and order $3^\gamma(h+1)|G|$, where $k = k(h, \gamma)$ from Corollary 7.5. By Lemma 3.3, G has a planar ω-decomposition of width k' and order $|G|$, for some k' only depending on k, γ and h (all of which only depend on H). □

Theorem 7.2 and Lemma 4.1 imply the following quantitative version of Theorem 1.1.

Corollary 7.7. *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number at most $c\,\Delta(G)^2\,|G|$.* □

8. Graph Partitions

A *partition* of a graph is a proper partition of its vertex set. Each part of the partition is called a *bag*. The *width* of partition is the maximum number of vertices in a bag. The *pattern* (or *quotient graph*) of a partition is the graph obtained by identifying the vertices in each bag, deleting loops, and replacing parallel edges by a single edge. Observe that a graph G has a decomposition D of spread 1 if and only if G has a partition whose pattern is a subgraph of D.

A *tree-partition* is a partition whose pattern is a forest. The *tree-partition-width*[17] of a graph G is the minimum width of a tree-partition of G, and is denoted by $\mathsf{tpw}(G)$. Tree-partitions were independently introduced by Seese [65] and Halin [32], and have since been investigated by a number of authors [10, 11, 21, 22, 26, 27, 79].

A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [22]. In particular, Seese [65] proved the lower bound,

$$2 \cdot \mathsf{tpw}(G) \geq \mathsf{tw}(G) + 1\ ,$$

which is tight for even complete graphs. The best known upper bound is

$$\mathsf{tpw}(G) \leq 2\big(\mathsf{tw}(G) + 1\big)\big(9\,\Delta(G) - 1\big)\ , \tag{5}$$

which was obtained by the first author [77] using a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [21]. See [4, 17, 18, 23, 24, 45] for other results related to tree-width and graph partitions.

Here we consider more general types of partitions. A partition is *planar* if its pattern is planar. A relationship between planar partitions and rectilinear drawings is described in the following lemma[18].

Lemma 8.1. *Every graph G with a planar partition of width p has a rectilinear drawing in which each edge crosses at most $2\,\Delta(G)\,(p-1)$ other edges. Hence*

$$\overline{\mathsf{cr}}(G) \leq (p-1)\,\Delta(G)\,\|G\|.$$

Proof. Apply the construction from Lemma 4.1 with $s(v) = 1$ for every vertex v. We obtain a rectilinear drawing of G. Consider an edge vw of G. Say v is in bag X, and w is in bag Y. Then vw is drawn inside $D_\epsilon(XY)$. Thus, if two edges e_1 and e_2 of G cross, then an endpoint of e_1 and an endpoint of e_2 are in a common bag, and e_1 and e_2 have no endpoint in common. Thus each edge of G crosses at most $2\,\Delta(G)\,(p-1)$ other edges, and $\overline{\mathsf{cr}}(G) \leq \frac{1}{2}\sum_e 2\,\Delta(G)\,(p-1) = \Delta(G)\,(p-1)\|G\|$. □

[17] Tree-partition-width has also been called *strong tree-width* [11, 65].

[18] Note that Lemma 8.1 bounds the number of crossings per edge; see [48] for related results.

A graph is *outerplanar* if it has a plane drawing with all the vertices on one face. Obviously, $\mathsf{cr}^\star(G) = 0$ if and only if G is outerplanar. A partition is *outerplanar* if its pattern is outerplanar.

Lemma 8.2. *Every graph G with an outerplanar partition of width p has a convex drawing in which each edge crosses at most $2\,\Delta(G)\,(p-1)$ other edges. Hence*

$$\mathsf{cr}^\star(G) \leq (p-1)\,\Delta(G)\,\|G\| \enspace .$$

Proof. Apply the proof of Lemma 8.1 starting from a plane convex drawing of the pattern. □

Since every forest is outerplanar, every graph G has an outerplanar partition of width $\mathsf{tpw}(G)$. Thus Lemma 8.2 and Equation (5) imply the following quantitative version of Theorem 1.3.

Corollary 8.3. *Every graph G has a convex drawing in which each edge crosses less than*

$$4\,\Delta(G)\,\big(\mathsf{tw}(G)+1\big)\big(9\,\Delta(G)-1\big)$$

other edges. Hence

$$\mathsf{cr}^\star(G) < 18\big(\mathsf{tw}(G)+1\big)\,\Delta(G)^2\,\|G\| < 18\,\mathsf{tw}(G)\,\big(\mathsf{tw}(G)+1\big)\,\Delta(G)^2\,|G| \enspace .$$

□

Note the following converse result.

Proposition 8.4. *Suppose that a graph G has a convex drawing such that whenever two edges e and f cross, e or f crosses at most k edges. Then G has tree-width $\mathsf{tw}(G) \leq 3k+11$.*

Proof. First we construct a strong planar decomposition D of G (in a similar way to the proof of Lemma 4.2). Orient each edge of G. Replace each vertex v of G by the bag $\{v\}$ in D. Replace each crossing between oriented edges vw and xy by the bag $\{v,x\}$ in D. For each edge vw of G there is an edge $\{v,x\}\{w\}$ in D; replace it by the path $\{v,x\}\{v,w\}\{w\}$. Thus D is a strong planar decomposition of G with width 2. Observe that the distance between each bag in D and some bag $\{v\}$ on the outerface is at most $\lfloor\frac{k}{2}\rfloor+1$. Thus D is $(\lfloor\frac{k}{2}\rfloor+2)$-outerplanar[19]. Bodlaender [8] proved that every d-outerplanar graph has tree-width at most $3d-1$. Thus D has tree-width at most $3\lfloor\frac{k}{2}\rfloor+5$. That is, some tree T is a strong decomposition of D with width at most $3\lfloor\frac{k}{2}\rfloor+6$. By Lemma 3.4 with $p=2$, G has a strong decomposition isomorphic to T with width at most $6\lfloor\frac{k}{2}\rfloor+12$. That is, G has tree-width at most $6\lfloor\frac{k}{2}\rfloor+11$. □

[19] An outerplanar graph is called 1-*outerplanar*. A plane graph is k-*outerplanar* if the graph obtained by deleting the vertices on the outerface is $(k-1)$-outerplanar.

9. $K_{3,3}$-MINOR-FREE GRAPHS

In this section we prove Theorem 1.2, which gives an upper bound on the rectilinear crossing number of $K_{3,3}$-minor-free graphs. The proof employs the following characterisation by Wagner [76].

Lemma 9.1 ([76])**.** *A graph G is $K_{3,3}$-minor-free if and only if G can be obtained by (≤ 2)-sums from planar graphs and K_5.*

Lemma 9.2. *Let G be a $K_{3,3}$-minor-free graph. For every edge e of G, there is a matching M in G with the following properties:*

- $|M| \leq \frac{1}{3}(|G|-2)$,
- *each edge in M is disjoint from e,*
- *contracting M gives a planar graph.*

Proof. If G is planar, then the lemma is satisfied with $M = \emptyset$. Suppose that $G = K_5$. Let vw be an edge of G that is disjoint from e. Let $M := \{vw\}$. Then $|M| = 1 = \frac{1}{3}(|G|-2)$. The graph obtained by contracting vw is K_4, which is planar.

Now assume that G is not planar and not K_5. By Lemma 9.1, G is a (≤ 2)-sum of two smaller $K_{3,3}$-minor-free graphs G_1 and G_2. Then $e \in E(G_1)$ or $e \in E(G_2)$. Without loss of generality, $e \in E(G_1)$. If the join set of the (≤ 2)-sum is an edge, then let vw be this edge. Otherwise, let vw be any edge of G_2.

By induction, G_1 has a matching M_1 with the claimed properties (with respect to the edge e), and G_2 has a matching M_2 with the claimed properties (with respect to the edge vw). In particular, every edge in M_2 is disjoint from vw. Thus $M := M_1 \cup M_2$ is a matching of G (even if $vw \in M_1$). Moreover, every edge in M is disjoint from e. We have $|M| = |M_1| + |M_2| \leq \frac{1}{3}(|G_1|-2) + \frac{1}{3}(|G_2|-2) = \frac{1}{3}(|G_1|+|G_2|-4) \leq \frac{1}{3}(|G|-2)$.

Let H_i be the planar graph obtained by contracting M_i in G_i. Let H be the graph obtained by contracting M in G. Then H is a (≤ 2)-sum of H_1 and H_2. Thus H is planar. □

Corollary 9.3. *Every $K_{3,3}$-minor-free graph G has a planar partition with width 2 and at most $\frac{1}{3}(|G|-2)$ bags of cardinality 2.* □

It follows from Euler's Formula and Lemma 9.1 that every $K_{3,3}$-minor-free graph G has at most $3\,|G|-5$ edges. Thus Corollary 9.3 and Lemma 8.1 imply the following quantitative version of Theorem 1.2.

Corollary 9.4. *Every $K_{3,3}$-minor-free graph G has a rectilinear drawing in which each edge crosses at most $2\,\Delta(G)$ other edges. Hence*

$$\overline{\mathsf{cr}}(G) \leq \Delta(G)\,\|G\| \leq \Delta(G)\,(3\,|G|-5)\ .$$

□

10. OPEN PROBLEMS

We finish with some open problems. In the first two problems, H is a fixed graph; both problems remain open even with $H = K_5$.

(1) Can the dependence on $\Delta(G)$ in Corollary 7.7 be reduced from quadratic to linear? We conjecture that every H-minor-free graph G has crossing number

$$\mathsf{cr}(G) \leq c_H \sum_{v \in V(G)} \deg(v)^2 .$$

(2) Does every H-minor-free graph with bounded degree have linear rectilinear crossing number? This question has an affirmative answer if every H-minor-free graph has a planar partition with bounded width.

(3) The *Hadwiger number* of a graph G is the maximum integer n such that K_n is a minor of G. Is the Hadwiger number of G at most $c\mathsf{cr}(G)/|G|$ for some consyant c?

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