# FPT algorithms for domination in biclique-free graphs

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**Abstract.** A class of graphs is said to be biclique-free if there is an integer $t$ such that no graph in the class contains $K_{t,t}$ as a subgraph. Large families of graph classes, such as any nowhere dense class of graphs or $d$-degenerate graphs, are biclique-free. We show that various domination problems are fixed-parameter tractable on biclique-free classes of graphs, when parameterizing by both solution size and $t$. In particular, the problems $k$-Dominating Set, Connected $k$-Dominating Set, Independent $k$-Dominating Set and Minimum Weight $k$-Dominating Set are shown to be FPT, when parameterized by $t + k$, on graphs not containing $K_{t,t}$ as a subgraph. With the exception of Connected $k$-Dominating Set all described algorithms are trivially linear in the size of the input graph.

## 1 Introduction

The $k$-dominating set problem is one of the most well-studied NP-complete problems in algorithmic graph theory. Given a graph $G$ and an integer $k$, we ask if $G$ contains a set $S$ of at most $k$ vertices such that every vertex of $G$ is either in $S$ or adjacent to a vertex of $S$. To cope with the intractability of this problem it has been studied both in terms of approximability [15] (relaxing the optimality) and fixed-parameter tractability (relaxing the runtime). In this paper we consider also weighted $k$-domination and the variants asking for a connected or independent $k$-dominating set.

The $k$-dominating set problem is notorious in the theory of fixed-parameter tractability (see [10, 22, 12] for an introduction to parameterized complexity). It was the first problem to be shown $W[2]$-complete [10], and it is hence unlikely to be FPT, i.e. unlikely to have an algorithm with runtime $f(k)n^c$ for $f$ a computable function, $c$ a constant and $n$ the number of vertices of the input graph. However, by restricting the class of input graphs, say to planar graphs, we can obtain FPT algorithms [2], even if the problem remains NP-complete on planar graphs [13]. In the race to find the boundary between FPT and $W$-hardness one typically wants the weakest possible restriction when proving FPT and the strongest possible restriction when proving $W$-hardness. In this paper, we push the tractability frontier forward by considering the above variants of $k$-dominating set on $t$-biclique free graphs and showing that they are FPT when parameterized by $k + t$. The $t$-biclique free graphs are those that do not contain

$K_{t,t}$ as a subgraph, and to the best of our knowledge, they form the largest class of graphs for which FPT algorithms are known for $k$-dominating set. Our algorithms are simple and rely on results from extremal graph theory that bound the number of edges in a $t$-biclique free graph, see the Bollobás book [4].

The parameterized complexity of the dominating set problem has been heavily studied with the tractability frontier steadily pushed forward by enlarging the class of graphs under consideration. One such line of improvements for $k$-dominating set consists of the series of FPT algorithms starting with planar graphs by Alber et al. [2], followed by bounded genus graphs by Ellis et al. [11], $H$-minor free graphs by Demaine et al. [8], bounded expansion graphs by Nesetril and Ossona de Mendez [19], and culminating in the FPT algorithm for nowhere dense classes of graphs by Dawar and Kreutzer [7]. See Figure 1. Alon and Gutner [3] have shown that $k$-dominating set on $d$-degenerate graphs parameterized by $k + d$ is FPT. Nowhere dense classes and $d$-degenerate graph classes are incomparable and Dawar and Kreutzer [7] mention that: 'it would be interesting to compare nowhere dense classes of graphs to graph classes of bounded degeneracy'. In this paper we base such a comparison on the fact that any nowhere dense class of graphs or $d$-degenerate class of graphs is a $t$-biclique-free class of graphs, for some $t$. See Section 2. Let us remark that also for a nowhere dense class $\mathcal{C}$ of graphs there is a parameter analogous to the $d$ in $d$-degenerate graphs and the $t$ in $t$-biclique-free graphs, with Dawar and Kreutzer [7] mentioning that: '...the exact parameter dependence of the algorithms depends on the function $h$, which is determined by the class of structures $\mathcal{C}$'. A relation between the different graph class properties can be seen in Figure 1.

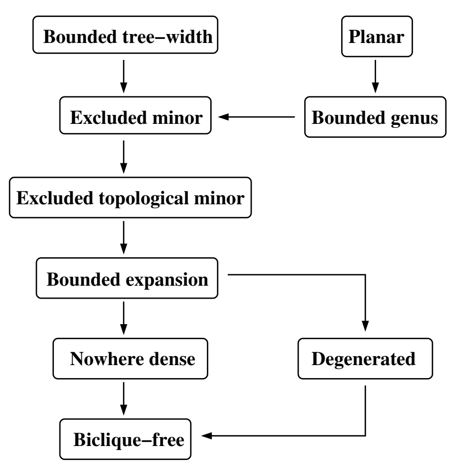


**Fig. 1.** Inclusion relations between some of the mentioned graph class properties. We refer to Nesetril and Ossona de Mendez [20] for a more refined view.

Raman and Saurabh [24] have shown that $k$-dominating set is $W[2]$-hard on $K_3$-free graphs and FPT on graphs not containing $K_{2,2}$ as a subgraph (i.e. 2-biclique-free graphs). Philip et al. [23] have shown that $k$-dominating set on graphs not containing $K_{i,j}$ as a subgraph (which they call $K_{i,j}$-free) has a polynomial kernel, which is stronger than simply saying that it is FPT. However, their algorithm is parameterized by $k$ only and considers $i + j$ to be a constant. They mention explicitly that: 'Another challenge is to...get a running time of the form $O(n^c)$ for $K_{i,j}$-free graphs where $c$ is independent of $i$ and $j$.' In this paper we do not directly meet this challenge but instead do something related. By showing that $k$-dominating set on $t$-biclique free graphs is FPT when parameterized by $k + t$, we generalize all FPT results for $k$-domination on restricted graph classes that we have found in the literature. Note that we could not expect to meet the challenge of a polynomial kernel when parameterizing by $k + t$, as Dom et al. [9] have shown that the $k$-Dominating Set problem on $d$-degenerate graphs (a subclass of $(d + 1)$-biclique-free graphs) does not have a kernel of size polynomial in both $d$ and $k$ unless the polynomial hierarchy collapses to the third level.

Our result extends to showing that connected $k$-domination and independent $k$-domination on $t$-biclique-free graphs are both FPT when parameterized by $k+t$. Note that Cygan et al. [6] have shown that connected $k$-domination has no polynomial kernel on graphs of degeneracy 2 (a subclass of 3-biclique-free graphs) unless the polynomial hierarchy collapses to the third level. For connected $k$-domination we use a subroutine developed by Misra et al. [18] for the Group Steiner Tree problem. The FPT borderline for connected $k$-domination and independent $k$-domination prior to our work resembled the one for $k$-domination, with Dawar and Kreutzer [7] showing that both problems are FPT on nowhere dense classes of graphs and Golovach and Villanger [14] showing that both problems are FPT on $d$-degenerate graphs when parameterized by $k + d$. Our algorithm generalizes these results.

Our result extends also to weighted $k$-domination. Alon and Gutner [3] show that weighted $k$-domination on $d$-degenerate graphs parameterized by $d + k$ is FPT but fixed-parameter tractability was not known for nowhere dense classes of graphs prior to our result for $t$-biclique free graphs.

A famous open problem in parameterized complexity is the question if there is an FPT algorithm deciding if a graph $G$ is $k$-biclique-free and in the Conclusion section we briefly mention this open problem in light of our algorithms.

## 2 Graph classes and problems

We use standard graph theory terminology. For a graph $G = (V, E)$ and $S \subseteq V$ we denote by $N[S]$ the vertices that are either in $S$ or adjacent to a vertex of $S$, and we denote by $G[S]$ the subgraph of $G$ induced by $S$. We denote always $|V| = n$ and $|E| = m$. The distance between two vertices is the number of edges in a shortest path linking them. Let us consider some classes of graphs.

**Definition 1 (Degenerate classes).** *A class of graphs $\mathcal{C}$ is said to be degenerate if there is an integer $d$ such that every induced subgraph of any $G \in \mathcal{C}$ has a vertex of degree at most $d$.*

Many interesting families of graphs are degenerate. For example, graphs embeddable on some fixed surface, degree-bounded graphs and non-trivial minor-closed families of graphs. Another broad property of graph classes, recently introduced by Nesetril and Ossona de Mendez [21], is the property of being nowhere dense. There are several equivalent definitions, we use the following based on the concept of a shallow minor. The radius of a connected graph $G$ is the minimum over all vertices $v$ of $G$ of the maximum distance between $v$ and another vertex. For non-negative integer $r$ a graph $H$ is a shallow minor at depth $r$ of a graph $G$ if there exists a subgraph $X$ of $G$ whose connected components have radius at most $r$, such that $H$ is a simple graph obtained from $G$ by contracting each component of $X$ into a single vertex and then taking a subgraph.

**Definition 2 (Nowhere dense classes).** *A class of graphs $\mathcal{C}$ is said to be nowhere dense if there is a function $f$ such that for every $r \geq 0$ the graph $K_{f(r)}$ is not a shallow minor at depth $r$ of any $G \in \mathcal{C}$.*

Many interesting families of graphs are nowhere dense, like graphs of bounded expansion and graphs locally excluding a minor. We now consider a class of graphs which was shown by Philip et al. [23] to strictly contain the degenerate classes of graphs and show that it also contains the nowhere dense classes. We denote by $K_{t,t}$ the complete bipartite graph with $t$ vertices on each side of the bipartition.

**Definition 3 (Biclique-free classes).** *A class of graphs $\mathcal{C}$ is said to be $t$-biclique-free, for some $t > 0$, if $K_{t,t}$ is not a subgraph of any $G \in \mathcal{C}$, and it is said to be biclique-free if it is $t$-biclique-free for some $t$.*

**Fact 1** *Any degenerate or nowhere dense class of graphs is biclique-free, but not vice-versa.*

*Proof.* For completeness we give a full proof. We first show that if a class of graphs $\mathcal{C}$ is degenerate then it is biclique-free. Assume that every induced subgraph of any $G \in \mathcal{C}$ has a vertex of degree at most $d$. Then $K_{d+1,d+1}$ cannot be a subgraph of any $G \in \mathcal{C}$ since its vertices would induce a subgraph where every vertex has degree larger than $d$.

We next show that if a class of graphs $\mathcal{C}$ is nowhere dense then it is biclique-free. Assume a function $f$ such that for every $r \geq 0$ the graph $K_{f(r)}$ is not a shallow minor at depth $r$ of any $G \in \mathcal{C}$. Then $K_{f(1)-1,f(1)-1}$ cannot be a subgraph of any $G \in \mathcal{C}$ since we in such a subgraph could contract a matching of $f(1) - 2$ edges crossing the bipartition and get $K_{f(1)}$ as a shallow minor at depth 1.

Finally, we show a biclique-free class of graphs $\mathcal{C}$ that is neither degenerate nor nowhere dense. For any value of $k \geq 2$ there exists a $k$-regular graph of girth 5, call it $R_k$, see e.g. [1]. Since $K_{2,2}$ is a 4-cycle the class $\{R_k : k \geq 2\}$ is biclique-free but not degenerate. Let $S_k$ be the graph obtained by subdividing once each edge of $K_k$. Since $S_k$ contains $K_k$ as a shallow minor at depth 1 the class $\{S_k : k \geq 2\}$ is biclique-free but not nowhere dense. Let the class $\mathcal{C}$ contain, for each value of $k \geq 2$, the graph we get by taking one copy of $S_k$ and one copy of $R_k$ and adding a single edge between some vertex of $S_k$ and some vertex of $R_k$ to make it a connected graph. The class $\mathcal{C}$ is biclique-free but it is neither degenerate nor nowhere dense.

A $k$-dominating set of a graph $G = (V, E)$ is a set $S \subseteq V$ with $|S| = k$ and $N[S] = V$. We will be considering parameterized versions of several domination-type problems in biclique-free classes of graphs. In each case we ask for a dominating set of size exactly $k$ but note that an algorithm for this problem can also be used to find the smallest $k$.

$k$-Dominating Set
Input: Integers $k, t$ and a $t$-biclique-free graph $G$.
Parameter: $k + t$
Question: Is there a $k$-dominating set in $G$?

Connected $k$-Dominating Set
Input: Integers $k, t$ and a $t$-biclique -free graph $G$.
Parameter: $k + t$
Question: Is there a $k$-dominating set $S$ in $G$ with $G[S]$ connected?

Independent $k$-Dominating Set
Input: Integers $k, t$ and a $t$-biclique -free graph $G$.
Parameter: $k + t$
Question: Is there a $k$-dominating set $S$ in $G$ with $G[S]$ having no edges?

Weighted $k$-Dominating Set
Input: Integers $k, t$ and a $t$-biclique -free graph $G$ with positive vertex weights.
Parameter: $k + t$
Output: A $k$-dominating set $S$ in $G$ with minimum sum of weights, if it exists.

## 3 Simple algorithms for domination in biclique-free graph classes

When studying the boundary between $W$-hardness and fixed parameter tractability one tries to find the strongest restriction such that the problem remains hard and the weakest restriction such that the problem becomes fixed parameter tractable. In most cases the arguments on both sides become more and more involved as one approaches the boundary. In this section we give fairly simple algorithms for variants of the $k$-dominating set problem on the class of biclique-free graphs.

### 3.1 Extremal combinatorics and high degree vertices

Before starting to describe the algorithm we need some tools from extremal combinatorics.

Bollobás in his book "Extremal Graph Theory"[4] discusses the so called Zarankiewicz Problem of giving an upper bound for the number of edges in graphs where $K_{s,t}$ is forbidden as a subgraph for integers $s, t$. It is worthwhile to point out that there is a significant difference between forbidding a $K_t$ and a $K_{s,t}$, as a graph without $K_t$ may contain $\Omega(n^2)$ edges while a graph without $K_{s,t}$ contains $O(n^{2-1/t})$ edges. Another difference that we have already mentioned is that the $k$-dominating set problem is $W[2]$-hard on $K_3$ free graphs [24] while it is fixed parameter tractable on $t$-biclique-free graphs. The proposition below turns out to be very useful when studying graphs without $K_{s,t}$ as a subgraph.

**Proposition 1 (Bollobás [4] VI.2).** *For integers $s, t$ let $G = (V_1, V_2, E)$ be a bipartite graph not containing $K_{s,t}$ as a subgraph where $|V_1| = n_1$ and $|V_2| = n_2$. Then for $2 \leq s \leq n_2$ and $2 \leq t \leq n_1$ we have that $|E| < (s-1)^{\frac{1}{t}}(n_2 - t + 1)n_1^{1-\frac{1}{t}} + (t-1)n_1$.*

A convenient consequence of Proposition 1 is that we can use it to say something about the number of high degree vertices in graphs that are $t$-biclique-free. For ease of notation let $f(k,t) = 2k(t + 1 + (4k)^t)$.

**Lemma 1.** *Let $k$ and $t$ be positive integers and let $G$ be a $t$-biclique-free graph on $n$ vertices where $f(k,t) \leq n$. Then there are less than $(4k)^t$ vertices of $G$ with degree at least $\frac{n-k}{k}$.*

*Proof.* On the contrary let us assume that there exists a vertex set $X \subset V$ where $(4k)^t = |X|$ and each vertex $v \in X$ has degree at least $\frac{n-k}{k}$ in $G$. Clearly such a vertex set also exists if there are more then $(4k)^t$ vertices of degree at least $\frac{n-k}{k}$. Let $Y = V \setminus X$ and define $x = |X|$ and thus $|Y| = n - x$. There are now at least $x(\frac{n-k}{k} - x)$ edges in $G$ between vertex sets $X$ and $Y$.

As $G$ is a $t$-biclique-free graph we know by Proposition 1 that the number of edges between $X$ and $Y$ in $G$ is less than $(t-1)^{\frac{1}{t}}(n-x-t+1)x^{1-\frac{1}{t}} + (t-1)x$ which is trivially at most $2(n-x)x^{1-\frac{1}{t}} + tx$. As $x = (4k)^t$ we aim for a contradiction by starting from the observation that,

$$\begin{aligned}
x(\tfrac{n-k}{k} - x) &< 2(n-x)x^{1-\frac{1}{t}} + tx \\
(\tfrac{n-k}{k} - x) &< 2(n-x)x^{-\frac{1}{t}} + t \\
n/k &< 2nx^{-\frac{1}{t}} - 2x^{1-\frac{1}{t}} + t + 1 + x \\
1 &< \tfrac{2k}{x^{1/t}} - (2kx^{1-\frac{1}{t}})/n + k(t+1+x)/n \\
1 &< \tfrac{2k}{(4k)^{t/t}} - (2k(4k)^{t(1-\frac{1}{t})})/n + k(t+1+(4k)^t)/n \\
1 &< \tfrac{1}{2} + k(t+1+(4k)^t)/n \\
\tfrac{1}{2} &< k(t+1+(4k)^t)/n
\end{aligned}$$

The assumption was that $f(k,t) = 2k(t + 1 + (4k)^t) \leq n$ which means that $k(t + 1 + (4k)^t)/n \leq \frac{1}{2}$ and we get the contradiction.

### 3.2 Enumeration of partial dominating sets

A simple way to decide if a graph has a dominating set of size $k$ is to enumerate all inclusion minimal dominating sets and check if one of them is of size at most $k$. If the goal is an FPT algorithm this approach fails already for planar graphs as they may contain $O(n^k)$ minimal dominating sets of size $k+1$.[1] Our way around this obstruction is to build the dominating sets in stages by enumerating only some subsets of each dominating set of size at most $k$ in such a way that all remaining vertices can be classified into a "small" number of equivalence classes. This provides a framework where several variants of domination can be discussed and compared.

Like before let $f(k,t) = 2k(t+1+(4k)^t)$.

**Lemma 2.** *For positive integers $k$ and $t$ let $G$ be a $t$-biclique-free graph on $n$ vertices where $f(k,t) \leq n$. Then there exists an algorithm that in time $O((n+m)k \cdot (4k)^{tk})$ outputs a family of vertex subsets $\mathcal{F}$ such that $|\mathcal{F}| \leq (4k)^{tk}$ and for any vertex set $S$ where $|S| \leq k$ and $V = N[S]$ there is $X \in \mathcal{F}$ with $X \subseteq S$ such that $X$ dominates at least $n - f(k,t)$ vertices of $G$, i.e. $|N[X]| \geq n - f(k,t)$.*

*Proof.* For a graph $G$ let us say that a family $\mathcal{F}$ of vertex subsets satisfies invariant $\mathcal{D}$ (for Domination) if for every $S \subseteq V$ such that $|S| \leq k$ and $N[S] = V$ there exists $X \in \mathcal{F}$ such that $X \subseteq S$. A family $\mathcal{F}$ of vertex subsets is defined to be of branch depth $i$ if it holds for every $X \in \mathcal{F}$ that if $|V \setminus N[X]| > f(k,t)$ then $|X| \geq i$. For short we will denote a family $\mathcal{F}$ of vertex subsets satisfying invariant $\mathcal{D}$ of branch depth $i$ as $\mathcal{F}_{\mathcal{D},i}$. Note that to prove the lemma it suffices to find in $O((n+m)k \cdot (4k)^{tk})$ time a family $\mathcal{F}_{\mathcal{D},k}$ since by the Domination invariant there is for every $S$ with $|S| \leq k$ and $N[S] = V$ some $X \in \mathcal{F}_{\mathcal{D},k}$ with $X \subseteq S$, and since this is at branch depth $k$ we know that if $|X| < k$ then $|N[X]| \geq n - f(k,t)$ and if $|X| \geq k$ then necessarily $X = S$ so that $|N[X]| = n$.

We will prove the lemma by induction on $i$, where the induction hypothesis is that a family $\mathcal{F}_{\mathcal{D},i}$ of cardinality at most $(4k)^{ti}$ can be computed in $O((n+m)i \cdot (4k)^{ti})$ time. The base case is obtained by simply observing that $\{\emptyset\}$ satisfies invariant $\mathcal{D}$ and is of branch depth 0, i.e. we can take $\mathcal{F}_{\mathcal{D},0} = \{\emptyset\}$.

Now for the induction step. Let us assume that a family $\mathcal{F}_{\mathcal{D},i-1}$ of cardinality $(4k)^{t(i-1)}$ is provided and let us argue how $\mathcal{F}_{\mathcal{D},i}$ can be obtained in $\mathcal{O}((n+m) \cdot (4k)^t)$ time. Let $X$ be one of the at most $(4k)^{t(i-1)}$ elements in $\mathcal{F}_{\mathcal{D},i-1}$ of cardinality $i-1$ where $|V \setminus N[X]| > f(k,t)$. Every dominating set $S \supset X$ with $|S| \leq k$ has to dominate all vertices of $V \setminus N[X]$. Thus, at least one of the vertices of $S \setminus X$ has to dominate at least $\frac{|V \setminus N[X]|}{|S \setminus X|}$ of these vertices. Let $Z$ be the set of vertices in $V \setminus X$ that dominates at least $\frac{|V \setminus N[X]|}{|S \setminus X|}$ of the vertices in $V \setminus N[X]$.

[1] Consider an independent set of size $n$ that is partitioned into $k$ colours of size $n/k$. For each colour add one vertex that is adjacent to all vertices of this colour and add one additional vertex $u$ adjacent to all $n$ vertices in the independent set. We obtain $(n/k)^k$ minimal dominating sets by selecting $u$ and exactly one vertex from each colour class.

By Lemma 1 $|Z| \leq (4k)^t$ and it is a trivial task to obtain $Z$ in $O(n+m)$ time provided $G$ and $X$.

Now update $\mathcal{F}_{\mathcal{D},i-1}$ by removing $X$ and adding set $X \cup \{w\}$ for every $w \in Z$. As every dominating set $S \supset X$ with $|S| \leq k$ contains at least one vertex of $Z$ the thus updated set $\mathcal{F}_{\mathcal{D},i-1}$ satisfies the invariant $\mathcal{D}$. Repeat this procedure for every set $X \in \mathcal{F}_{\mathcal{D},i-1}$ such that $|X| = i-1$ and $|V \setminus N[X]| > f(k,t)$. As there are at most $(4k)^{t(i-1)}$ such sets of size $i-1$ and each of these are replaced by at most $(4k)^t$ sets the resulting family of sets is of size at most $(4k)^t \cdot |\mathcal{F}_{\mathcal{D},i-1}|$ and is of branching depth $i$. This completes the proof.

Each element $X \in \mathcal{F}_{\mathcal{D},k}$ will now be used to define an equivalence relation on $V \setminus X$ based on the possible neighbors among the undominated vertices $V \setminus N[X]$.

**Definition 4.** *For a graph $G = (V,E)$ and a vertex set $X \subset V$ let $W = V \setminus N[X]$. Let $\equiv_X$ be the binary relation on $V \setminus X$ with $u \equiv_X v$ if $N[u] \cap W = N[v] \cap W$. This is an equivalence relation and we say that $v \in V \setminus X$ belongs to the equivalence class $\mathcal{E}$ corresponding to $N[v] \cap W$.*

**Definition 5.** *For a graph $G = (V,E)$ and a vertex set $X \subset V$ a set $A = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$ of equivalence classes of $\equiv_X$ is called dominating if mapping each element $\mathcal{E}_i \in A$ to an arbitrary vertex $v_i \in \mathcal{E}_i$ (it does not matter which since all vertices in $\mathcal{E}_i$ have the same closed neighborhoods in $W$) we have $V = N[X] \cup \bigcup_{i=1}^{r} N[v_i]$. Let $\mathcal{A}_{X,r}$ be defined as the set of all equivalence classes of cardinality $r$ that are dominating.*

**Lemma 3.** *For positive integers $k$ and $t$ let $G$ be a $t$-biclique-free graph on $n$ vertices where $f(k,t) \leq n$. Let $X$ be an element of the family $\mathcal{F}_{\mathcal{D},k}$ and let $r \leq k - |X|$. Then $|\mathcal{A}_{X,r}| \leq 2^{r \cdot f(k,t)}$ and $\mathcal{A}_{X,r}$ can be computed in $O((n+m) \cdot 2^{r \cdot f(k,t)})$ time.*

*Proof.* Let $W = V \setminus N[X]$. Note $|W| \leq f(k,t)$ and hence there are at most $2^{f(k,t)}$ subsets of $W$. For each vertex $v \in V \setminus X$ compute $N[v] \cap W$ and add $v$ to its equivalence class. Note there are at most $2^{f(k,t)}$ equivalence classes. For each of the $(2^{f(k,t)})^r$ possible subsets $A = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$ of equivalence classes add $A$ to $\mathcal{A}_{X,r}$ if $A$ is dominating. The running time for this is $O((n+m) \cdot 2^{r \cdot f(k,t)})$.

### 3.3 Various domination problems on biclique-free graph classes

This subsection combines the previous combinatorial results into algorithms for different domination problems.

**Theorem 1.** *Given a $t$-biclique-free graph $G = (V,E)$ the following problems, as defined in Section 2, are fixed parameter tractable when parameterizing by $k+t$:*

1. *$k$-Dominating set, $O((n+m) \cdot 2^{O(tk^2(4k)^t)})$ time,*
2. *Connected $k$-Dominating set,*
3. *Independent $k$-Dominating set,*

*4.* Weighted $k$-Dominating set.

*Proof.* In all cases, if $|V| < f(k,t) = 2k(t+1+(4k)^t)$ we can simply enumerate all $\binom{f(k,t)}{k}$ vertex subsets of size $k$ and test in $O((n+m)\cdot k^2)$ time if the specific properties for the problem is satisfied. Otherwise we first enumerate the family $\mathcal{F}_{\mathcal{D},k}$ containing at most $(4k)^{tk}$ elements in $O((n+m)k\cdot(4k)^{tk})$ time using Lemma 2. For each element $X \in \mathcal{F}_{\mathcal{D},k}$ we apply Lemma 3 and compute $\mathcal{A}_{X,r}$ which is of size at most $2^{r\cdot f(k,t)}$ in time $O((n+m)\cdot 2^{r\cdot f(k,t)})$. The value $r \leq k-|X|$ for which this is computed will depend on the problem to be solved in the following way:

1. By definition, there is a $k$-Dominating set in $G$ if and only if there is some $X \in \mathcal{F}_{\mathcal{D},k}$ for which $\mathcal{A}_{X,k-|X|} \neq \emptyset$. Applying Lemma 3 $(4k)^{tk}$ times give a total running time of $O((n+m)\cdot 2^{O(tk^2(4k)^t)})$.
2. In Misra et al. [18] an FPT algorithm is given for the following Group Steiner Tree problem:
   - Given a graph $H$, subsets of vertices $T_1, T_2, ..., T_l$, and an integer $p$, with parameter $l \leq p$, does there exist a subgraph of $H$ on $p$ vertices that is a tree $T$ and includes at least one vertex from each $T_i, 1 \leq i \leq l$?
   We claim that our input graph $G$ has a connected $k$-dominating set if and only if there exists some $X \in \mathcal{F}_{\mathcal{D},k}$ and some $r \in \{0,1,...,k-|X|\}$ and some $A \in \mathcal{A}_{X,r}$ such that the following Group Steiner Tree problem has a Yes-answer:
   - Let $X = \{v_1, v_2, ..., v_{|X|}\}$ and $A = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$. Set $H = G$ and set the vertices of the equivalence class $\mathcal{E}_i$ to form a subset $T_i$, for each $1 \leq i \leq r$ and additionally $T_i = \{v_i\}$ for $r+1 \leq i \leq r+|X|$. We thus have $l = r+|X|$ and set $p = k$.
   Let us argue for the claim. For one direction assume that $S$ is a connected $k$-dominating set of $G$. We then know there exists $X \subseteq S$ with $X \in \mathcal{F}_{\mathcal{D},k}$. Let $A = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$ be the equivalence classes of $\equiv_X$ containing at least one vertex of $S$. Note that $0 \leq r \leq k-|X|$ and $A \in \mathcal{A}_{X,r}$. The Group Steiner Tree problem we formed from this $X$ and $A$ has a Yes-answer, for example by taking any subgraph of $G[S]$ inducing a tree. For the other direction, if the Group Steiner Tree problem formed by some $X$ and $A$ has a Yes-answer by some tree $T$ then the set of vertices of this tree $T$ will necessarily be a connected $k$-dominating set of $G$.
   The total running time will be FPT in $k+t$ as the possible choices of $X$ and $A$ are a function of $k+t$ only and the Group Steiner Tree problem is FPT.
3. For every $X \in \mathcal{F}_{\mathcal{D},k}$ such that $G[X]$ is an independent set, check by brute force in $O((n+m)\cdot 2^{f(k,t)})$ if there exists an independent dominating set of size $k-|X|$ in the graph $G[V \setminus N[X]]$. The runtime will be linear FPT.
4. For each vertex $v \in V$ let $w(v)$ be the weight of the vertex. The weight $w(A)$ of a set $A = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$ of equivalence classes is defined as $\sum_{i=1}^{r} w(v_i)$ where class $\mathcal{E}_i$ is mapped to the minimum weight vertex $v_i \in \mathcal{E}_i$. The $k$-dominating set of minimum weight is then obtained by minimizing over all $X \in \mathcal{F}_{\mathcal{D},k}$ such that $\mathcal{A}_{X,k-|X|} \neq \emptyset$ the value of $\min_{A \in \mathcal{A}_{X,k-|X|}} w(A) + \sum_{v \in X} w(v)$. The runtime will be linear FPT.

## 4 Conclusion

In this paper we have pushed forward the FPT boundary for the $k$-dominating set problem, and some of its variants, to the case of $t$-biclique-free graphs parameterized by $k + t$. This generalizes all FPT algorithms for $k$-dominating set on special graph classes we have found in the literature, in particular the algorithms for the uncomparable classes $d$-degenerate and nowhere dense. By the result of Dom et al. [9] this problem does not have a polynomial kernel unless the polynomial hierarchy collapses to the third level.

The basic idea for our algorithm is to branch on vertices of sufficiently high degree until the remaining vertices can be partitioned into a small number of equivalence classes. Unsurprisingly, the applied techniques have some similarities with the algorithms for the $d$-degenerate classes and nowhere dense classes. Usually the algorithms become more complicated as they apply to more general classes of graphs. In this paper our generalized algorithm is still fairly simple with a running time that can trivially be summed up to be $O((n + m) \cdot 2^{O(tk^2(4k)^t)})$.

The described algorithm resolves the $k$-dominating set problem on all $t$-biclique-free graphs. As the algorithm only uses extremal combinatorial properties of the graph class there is no need to verify that the input graph indeed is $t$-biclique-free. Consider the following somewhat strange parameterized problem:

$k$-Dominating Set or $k$-Biclique
Input: Graph $G$ and integer $k$.
Parameter: $k$
Output: Either '$G$ is not $k$-biclique-free' or '$G$ has a $k$-dominating set' or '$G$ does not have a $k$-dominating set'

By using our algorithm for $k$-dominating set on an arbitrary graph $G$, with $k = t$, we actually get an FPT algorithm for $k$-Dominating Set or $k$-Biclique, since we either conclude by Lemma 1 that $G$ is not $k$-biclique-free or we have few high degree vertices and are able to decide if the graph has a dominating set of size $k$. This gives us a somewhat strange result, resembling the situation for the Ramsey problem asking if a graph has either a clique of size $k$ or an independent set of size $k$ [17, 16]. The $k$-dominating set problem is $W[2]$-hard on general graphs and Bulatov and Marx [5] gives reason to believe that deciding if a graph $G$ is $k$-biclique-free is not FPT on general graphs. Nevertheless, one of the two problems can always be resolved in FPT time.

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