Edge-maximal graphs of branchwidth k: the k-branches

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Abstract

Treewidth and branchwidth are two closely related connectivity parameters of graphs. Graphs of treewidth at most k have well-known alternative characterizations as subgraphs of chordal graphs and as partial k-trees. In this paper we give analogous alternative characterizations for graphs of branchwidth at most k. We first show that they are the subgraphs of chordal graphs where every maximal clique X has three subsets of size at most k each such that any two subsets have union X, with the property that every minimal separator contained in X is contained in one of the three subsets. Secondly, we give a characterization of the edge-maximal graphs of branchwidth k, that we call k-branches.

Key words: Graph decomposition, chordal, branchwidth, edge maximal

1 Introduction

Branchwidth and treewidth are connectivity parameters of graphs introduced in the proof of the Graph Minors Theorem by Robertson and Seymour [RS91]. The two parameters are related by the following inequalities $branchwidth(G) \leqslant treewidthG+1 \leqslant \lfloor 3/2\, branchwidth(G) \rfloor$ illustrated in Figure 1. The graphs of treewidth at most k have also been studied under the name of partial k-trees [AP85]. The k-trees are in fact the edge-maximal graphs of treewidth k, definable as chordal graphs having all minimal separators of size k and all maximal cliques of size $k+1$ [Ros74]. In this paper we give a similar characterization

⋆ Part of these results have appeared in the extended abstracts [PT05b, PT05a]

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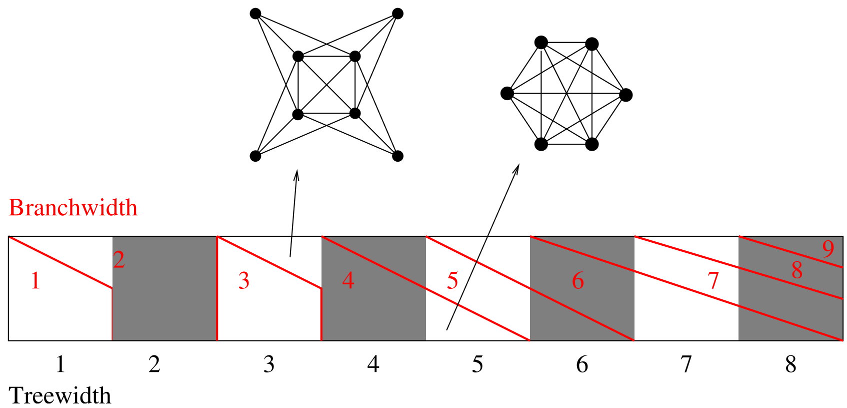

Fig. 1. Venn diagram showing branchwidth of graphs of treewidth at most 8. Two chordal graphs of branchwidth 4 are shown: the 5-tree K_6 and a 3-tree on 8 vertices.

of the edge-maximal graphs of branchwidth k, that we call k-branches. It was previously known that the 1-branches are the stars, that the 2-branches are the 2-trees, and it could be deduced from work of Bodlaender and Thilikos [BT99] that the 3-branches are the 3-trees having no three-dimensional cube as a minor. Also, it could be deduced from work of Kloks, Kratochvil and Muller [**?**] that k-branches are chordal. Apart from this, little was known previously about the edge-maximal graphs of branchwidth k.

Chordal graphs are the node intersection graphs of subtrees of a tree and a central tool in many investigations of tree-like properties of graphs. A recent result by Mazoit [Maz04] established links between chordal graphs and branchwidth by showing that the branchwidth of a graph is equal to the minimum over the branchwidth of all its *efficient triangulations*, see his paper for exact definitions. Graphs of treewidth at most k are precisely the subgraphs of chordal graphs where every maximal clique X has size $k+1$. In this paper we characterize graphs of branchwidth at most k as subgraphs of chordal graphs where every maximal clique X has three subsets of size at most k each such that any two subsets have union X, with the property that every minimal separator contained in X is contained in one of the three subsets. In fact the following succinct definition of both treewidth and branchwidth follows from these characterizations, by replacing the underlined words by the words in parenthesis:

For any $k \geq 2$ a graph G on vertices $v_1, v_2, ..., v_n$ has branchwidth at most k (treewidth at most $k-1$) if and only if there is a ternary tree T with subtrees $T_1, T_2, ..., T_n$ such that if v_i and v_j adjacent then subtrees T_i and T_j share at least one edge (node) of T, and each edge (node) of T is shared by at most k of the subtrees.

Finally, let us mention that the results in this paper have also been applied to obtain an algorithm generating k-branches [PPT06].

2 Standard definitions

We consider simple undirected and connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote G subgraph of H by $G \subseteq H$ which means that $V(G) = V(H)$ and $E(G) \subseteq E(H)$, and we also say that H is a supergraph of G. For a set $A \subseteq V(G)$, $G(A)$ denotes the subgraph of G induced by the vertices in A. A is called a *clique* if $G(A)$ is complete. The set of neighbors of a vertex v in G is $N(v) = \{u \mid uv \in E(G)\}$. A vertex set $S \subset V(G)$ is a *separator* if $G(V(G) \setminus S)$ is disconnected. Given two vertices u and v, S is a *u, v-separator* if u and v belong to different connected components of $G(V(G) \setminus S)$. A u, v-separator S is *minimal* if no proper subset of S separates u and v. S is a *minimal separator* of G if there exist two vertices u and v in G such that S is a minimal u, v-separator. A graph is *chordal* if it contains no induced cycle of length ≥ 4. A *triangulation* of a graph G is a chordal supergraph of G. In a *clique tree* of a chordal graph G the nodes are in 1-1 correspondence with the maximal cliques of G and the set of nodes whose maximal cliques contain a given vertex form a subtree. We usually refer to nodes of a tree and vertices of a graph.

A *tree-decomposition* $(T, \mathcal{X})$ of a graph G is a tree T with nodes mapped to a set $\mathcal{X}$ of vertex subsets of $V(G)$, also called *bags*, such that 1) the set of bags covers the vertices of $V(G)$; 2) for any edge xy of $E(G)$ there exists a bag containing both x and y; and 3) the bags containing any given vertex induce a subtree of T. The *width* of a tree-decomposition is the size of its largest bag minus one and the *treewidth* of a graph is the smallest width of any of its tree-decompositions (see e.g. [Bod97]).

A *branch-decomposition* (T, μ) of a graph G is a tree T with nodes of degree one and three only, together with a bijection μ from the edge-set of G to the set of degree-one nodes (leaves) of T. For an edge e of T let T_1 and T_2 be the two subtrees resulting from $T \setminus \{e\}$, let G_1 and G_2 be the graphs induced by the edges of G mapped by μ to leaves of T_1 and T_2 respectively, and let $mid(e) = V(G_1) \cap V(G_2)$. The width of (T, μ) is the size of the largest $mid(e)$ thus defined. For a graph G its *branchwidth* $bw(G)$ is the smallest width of any branch-decomposition of G [1] (see e.g. [RS91]).

[1] The connected graphs of branchwidth 1 are the stars, and constitute a somewhat pathological case. To simplify we therefore restrict attention to graphs having branchwidth $k \geq 2$, in other words our statements are correct only for *graphs having at least two vertices of degree more than one.*

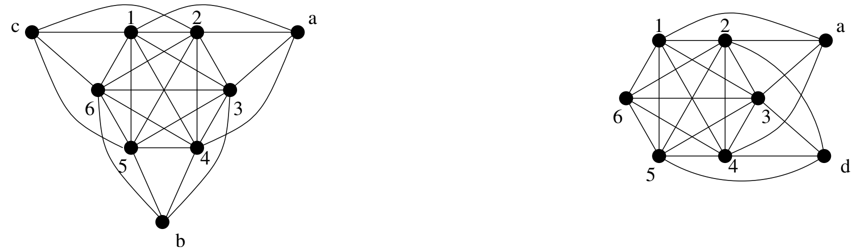

Fig. 2. The chordal graph on the left has branchwidth 4 (the maximal clique 123456 has the three subsets 1234, 3456, 1256 of size 4 that satisfy the conditions of Theorem 2) whereas the graph on the right has branchwidth 5.

3 Subgraphs of chordal graphs

Here is a well-known alternative characterization of treewidth, see e.g. [vL90].

Theorem 1 *A graph has treewidth at most k iff it is a subgraph of a chordal graph H where every maximal clique X of H satisfies $|X| \leq k+1$.*

In this section we give a similar alternative characterization of branchwidth.

Theorem 2 *A graph has branchwidth at most k iff it is a subgraph of a chordal graph H where every maximal clique X of H has three subsets of size at most k each such that any two subsets have union X, with the property that every minimal separator of H contained in X is contained in one of the three subsets.*

See Figure 3 for an example. We prove the theorem in two steps, first showing a characterization in terms of subtree-representations of a ternary tree. For sake of completeness, we give a proof from basic principles, not relying on previous work.

Definition 1 *A* ternary subtree-representation *$R = (T, \{T_1, T_2, ..., T_n\})$ is a pair where T is a tree with vertices of degree at most three and $T_1, T_2, ..., T_n$ are subtrees of T.*

- *Its* edge intersection graph *$EI(R)$ has vertex set $\{v_1, v_2, ..., v_n\}$ and edge set $\{v_iv_j : T_i$ and T_j share an edge of $T\}$;*
- *Its* node intersection graph *$NI(R)$ has the same vertex set but edge set $\{v_iv_j : T_i$ and T_j share a node of $T\}$.*

For a node u of T, we call the set of vertices $X_u = \{v_i : T_i$ contains $u\}$ the bag *of u, and $\{X_u : u \in V(T)\}$ the bags of R.*

With the above terminology we can easily move between the view of a ternary subtree-representation R as a tree T with a set of subtrees $\{T_1, T_2, ..., T_n\}$ or as a tree T with a set of bags $\{X_u : u \in V(T)\}$. When manipulating the latter we must simply ensure that for any vertex in $EI(R)$ the set of bags containing that vertex corresponds to a set of nodes of T inducing a subtree, i.e. a connected subgraph.

Definition 2 *Let $R = (T, \{T_1, ..., T_n\})$ be a ternary subtree-representation. The* edge-weight *of R is the maximum, over all edges uv of T, of the number of subtrees in $\{T_1, ..., T_n\}$ that contain edge uv.*

We are in this paper only interested in the edge intersection graphs of ternary subtree-representations having bounded edge-weight k. We start by showing that we can restrict ourselves to ternary subtree-representations for which the edge intersection and node intersection graphs are the same.

Lemma 1 *For any ternary subtree-representation $R = (T, \{T_1, \dots T_n\})$ of edge-weight k there exists a ternary subtree-representation $R' = (T', \{T'_1, \dots T'_n\})$ of edge-weight k with $EI(R) = EI(R') = NI(R')$.*

Proof: Clearly $EI(R) \subseteq NI(R)$. Assume $ab \in E(NI(R)) \setminus E(EI(R))$, i.e. some bag X_i of R contains two vertices a, b with no neighbor j of i having $\{a, b\} \subseteq X_j$. Since i has at most three neighbors, this means that one of a or b, say a, is in the bag of at most one of these neighbors. We can wlog assume that $EI(R)$ is connected and thus let j be the neighbor of i to whose bag a belongs. We subdivide the edge ij in the tree T with the new node having bag $X_i \cap X_j$, and remove a from X_i. Repeat this procedure until the resulting ternary subtree-representation R' satisfies $EI(R') = NI(R')$. When we are done R' has edge-weight k with $EI(R) = EI(R') = NI(R')$. □

Lemma 2 *A graph G has branchwidth at most $k \Leftrightarrow$ there is a ternary subtree-representation R of edge-weight at most k with $G \subseteq EI(R) = NI(R)$.*

Proof: $\Rightarrow$: Take a branch-decomposition (T, μ) of G of width k, i.e. $|mid(e)| \leq k$ for each $e \in E(T)$. We construct a ternary subtree-representation $R = (T', S)$ of edge-weight k with $G \subseteq EI(R)$. T' is constructed from T by for each leaf l of T adding a new leaf l' and making it adjacent to l. For vertex $a \in V(G)$ consider the smallest spanning subtree of T containing all leaves of T that are mapped by μ to an edge incident with a. The subtree T_a will be this subtree augmented by leaf l' for each leaf l of T that it contains. This completes the description of $R = (T', \{T_a : a \in V(G)\}$. For any two adjacent vertices $\{a, b\}$ of G we have $\mu^{-1}(l) = \{a, b\}$ for some leaf l of T, and thus the subtrees corresponding to a and b share the edge ll' of T' which implies that $G \subseteq EI(R)$. If R does not satisfy $EI(R) = NI(R)$ we apply Lemma 1. If vertex a has subtree T_a containing edge e of T, then there are edges incident with a mapped to leaves in both subtrees of T arising from deleting the edge e, and thus $a \in mid(e)$. But this means that the edge-weight of R is at most k.

$\Leftarrow$: Let $R = (T, \mathcal{S})$ be a ternary subtree-representation R of edge-weight at most k with $G \subseteq EI(R) = NI(R)$. We construct a branch-decomposition (T', μ) of G with width k. Associate each edge ab of G with an edge e of T such that the subtrees T_a and T_b corresponding to a and b both contain e. Subdivide the tree edge e by as many new nodes as there are edges of G associated to e, thus creating for each edge ab associated to e a new tree node e_{ab}. Furthermore, add a new leaf node l_{ab}, make it adjacent to e_{ab} and set $\mu(ab) = l_{ab}$. Let T''' be the tree we have constructed so far. It contains T as a minor. Consider the smallest spanning subtree T'' of T''' having the set of leaves $\{l_{ab} : ab \in E(G)\}$. Iteratively contract edges of T'' incident to a vertex of degree two until all inner vertices have degree three. The resulting tree is T'. Note that as we constructed T' from T in stages we could at each stage have updated the subtree T_a corresponding to vertex a to a new subtree $T_a^{'}$ so that we would still have a ternary subtree-representation $R' = (T', \mathcal{S}')$ with $G \subseteq EI(R')$. For example, $T_a^{'}$ should contain every 'subdivision node' on a tree edge f if T_a contained f, it should contain l_{ab} for any edge ab incident with a, and it should naturally shrink if it contained a removed leaf or contracted edge. Moreover, $(T', \mathcal{S}')$ has edge-weight at most k since never during this process did we increase the edge-weight beyond what it was. T' has nodes of degree one and three only and μ is a bijection between its leaves and the edges of G so (T', μ) is a branch-decomposition of G. It remains to show that it has width k, i.e. that for any edge e of T' we have $|mid(e)| \leq k$. We claim that $mid(e) \subseteq \{a : T_a^{'} \text{ contains edge } e\}$. Consider $a \in mid(e)$. There must exist two leaves l_{ab}, l_{ac} of T', one in each of the two subtrees of $T' \setminus e$, such that $a \in \mu^{-1}(l_{ab})$ and $a \in \mu^{-1}(l_{ac})$. Since the subtree $T_a^{'}$ of a contains both l_{ab} and l_{ac} it must also contain e. □

The notion of k-troikas [2] in the following definition will simplify several statements.

Definition 3 *A k-troika (A, B, C) of a set X are 3 subsets of X such that $|A| \leq k$, $|B| \leq k$, $|C| \leq k$, and $A \cup B = A \cup C = C \cup B = X$. (A, B, C)* respects *$S_1, S_2, ..., S_q$ if any $S_i, 1 \leq i \leq q$ is contained in at least one of A, B or C.*

For example, $(\{1,2,3,4\}, \{3,4,5,6\}, \{1,2,5,6\})$ is a 4-troika of $\{1,2,3,4,5,6\}$. We are ready to prove Theorem 2 which can now be rephrased as follows:

Theorem 2 *A graph G has branchwidth at most $k \Leftrightarrow G$ is subgraph of a chordal graph H and every maximal clique X of H has a k-troika respecting the minimal separators of H contained in X.*

[2] A troika is a horse-cart drawn by three horses, and when the need arises any two of them should also be able to pull the cart

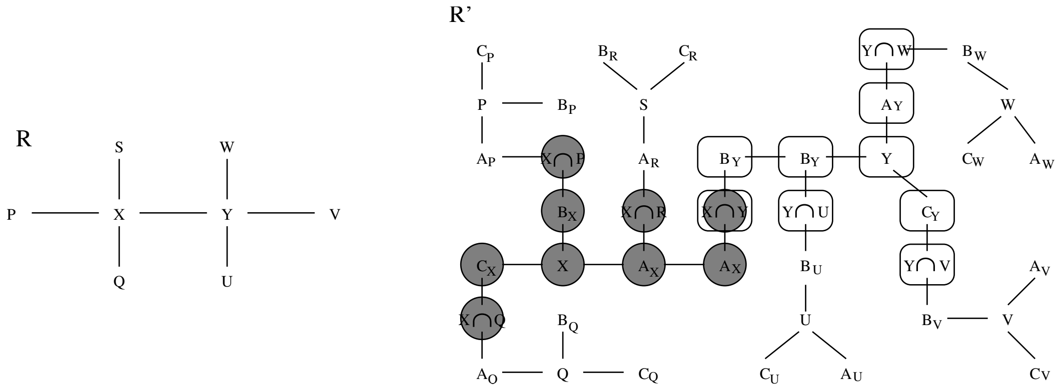

Fig. 3. Illustration of how the local conditions, i.e. existence of a k-troika (A_M, B_M, C_M) at each maximal clique M, yields a global condition, i.e. branchwidth at most k. R is a clique tree of chordal graph H and R' the ternary subtree-representation of edge-weight k with $H \subseteq EI(R')$. Rectangular nodes form the subtree of clique Y and circular nodes the subtree of clique X, sharing the node $X \cap Y$.

Proof: $\Rightarrow$: By Lemma 2 there exists a ternary subtree-representation R of edge-weight k with $G \subseteq EI(R) = NI(R)$. Since $NI(R)$ is a node intersection graph of subtrees of a tree it is a chordal graph [Gav74], and $H = EI(R) = NI(R)$ will indeed be our chordal graph H having G as a subgraph. By the Helly property of (vertex) intersection of subtrees of a tree, every maximal clique of H is a bag X_u for some node u of the tree. If $|X_u| \leq k$ then it clearly has a k-troika respecting any subset, so let us assume $|X_u| > k$. Since any pair a, b of nodes from X_u is adjacent in H, we must have $\{a, b\}$ contained also in one of the neighboring bags. Let the pairwise intersection of X_u and the three bags of its three neighbors be A, B and C. This means that any two of A, B, C must have union X_u since if for example $a \in X_u$ but $a \notin A \cup B$ then we would be forced to have $C = X_u$, since C would have to contain a and all its neighbors in X_u contradicting the fact that R has edge-weight k. Any minimal separator S of the chordal graph H is the intersection of two maximal cliques corresponding to two bags X_u, X_v. If we assume $A = X_u \cap X_w$, for w the neighbor of u on the path from u to v in T, then we have $S = X_u \cap X_v \subseteq A$ since otherwise the subtree corresponding to a vertex $a \in (X_u \cap X_v) \setminus A$ would be disconnected.

$\Leftarrow$: Consider any clique tree of H. Note that this clique tree can be viewed as a pair $R = (T, \mathcal{S})$ just as our ternary subtree-representations with $H = NI(R)$ and every bag inducing a maximal clique of H, except that nodes of T can have degree larger than 3. We construct from this a ternary subtree-representation $R' = (T', \mathcal{S}')$ of edge-weight k with $G \subseteq H \subseteq EI(R')$ which by Lemma 2 will imply that G has branchwidth at most k. Let X be a maximal clique whose node in T has q neighbors corresponding to maximal cliques $Z_1, Z_2, ..., Z_q$, and let (A, B, C) be the k-troika of X respecting minimal separators $X \cap Z_1, ..., X \cap Z_q$. This means there exists a partition P_A, P_B, P_C of $\{1, 2, ..., q\}$

such that $X \cap Z_i \subseteq A$ for $i \in P_A$, $X \cap Z_i \subseteq B$ for $i \in P_B$, $X \cap Z_i \subseteq C$ for $i \in P_C$. For maximal clique X we construct a ternary subtree as follows: we have a central node with bag X adjacent to three paths: one path with $\max\{1, |P_A|\}$ bags A, one path with $\max\{1, |P_B|\}$ bags B and one with $\max\{1, |P_C|\}$ bags C. For each $i \in \{1, 2, ..., q\}$ we have a leaf-node with bag $X \cap Z_i$ as neighbor of a node on these paths, e.g. if $i \in P_A$ the leaf-node should be the neighbor of a node with bag A, if $i \in P_B$ then B, and if $i \in P_C$ then C, such that q of the nodes on the 3 paths get one leaf each. (see Figure 3). Construct such a ternary subtree for each maximal clique X, i.e. for each node of T. Then, for each pair of maximal cliques X, Y that are bags of two neighboring nodes in T we identify the following two leaves into a single node: $X \cap Y$ in the subtree constructed for X and $Y \cap X$ in the subtree constructed for Y. The resulting tree T' has no node of degree more than three and together with bags as indicated it forms the ternary subtree-representation $R' = (T', \mathcal{S}')$. R' has edge-weight at most k since any subset of a k-troika has size at most k. We show that $H \subseteq EI(R')$. For any edge $ab \in E(H)$ we have $\{a, b\} \subseteq X$ for some maximal clique X. The k-troika (A, B, C) of X has the property that any vertex $a \in X$ must be in two out of A, B, C, so that we must have $\{a, b\}$ contained in one of A, B or C. Thus the edge ab is in $EI(R')$ and $H \subseteq EI(R')$. $\Box$

4 Characterization of edge-maximal graphs

Here is a well-known characterization of edge-maximal graphs of treewidth k, the so-called k-trees [Ros74].

Theorem 3 *[Ros74] A graph G is a k-tree iff*

(1) G is chordal
(2) Every minimal separator of G has size k
(3) Every maximal clique of G has size $k+1$

The edge-maximal graphs of branchwidth k have not been studied previously.

Definition 4 *A graph G of branchwidth k is called a k-branch if adding any edge to G will increase its branchwidth.*

In this section we characterize k-branches by five conditions. The first two conditions are common with k-trees, i.e. being chordal and having minimal separators of size k only. The third condition comes from Theorem 2, i.e. that maximal cliques have a k-troika respecting minimal separators, and this condition can be compared to the third condition for k-trees above. The fourth condition is a size constraint and could be compared to the trivial size con-

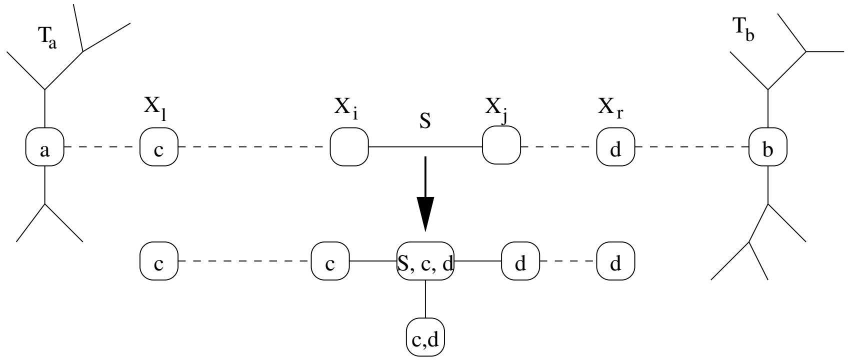

Fig. 4. The tree representation $R = (T, \mathcal{S})$ of a graph G of branchwidth k having a minimal a, b-separator S of size strictly less than k. Then G can be augmented by adding an edge cd without increasing the branchwidth.

straint $|V(G)| \geq k + 1$ for k-trees (strictly speaking Theorem 3 needs such a size constraint to avoid the empty graph.) We start by proving these four conditions on k-branches.

Lemma 3 *Any k-branch G satisfies the four conditions:*

(1) G is chordal
(2) Every minimal separator of G has size k
(3) Every maximal clique of G has a k-troika respecting minimal separators
(4) G has at least $\lfloor 3(k-1)/2 \rfloor + 1$ vertices

Proof: By Theorem 2, we know that any k-branch must be a chordal graph in which every maximal clique has a k-troika respecting its minimal separators. If $|V(G)| < \lfloor 3(k-1)/2 \rfloor + 1$ then, by Theorem 2, G would have branchwidth less than k since the clique on $|V(G)|$ vertices would have a $(k-1)$-troika. Thus conditions 1, 3 and 4 hold.

It remains to show condition 2, *i.e.* that all minimal separators of G must have size k. Let S be a minimal (a, b)-separator of G and consider a ternary subtree-representation $R = (T, \mathcal{S})$ of edge-weight k with $G = EI(R) = NI(R)$, which is guaranteed to exist by Lemma 2. There is a unique path P in T between the subtrees corresponding to a and b. For every node i on this path its bag X_i contains S and there must exist two adjacent nodes i, j for whom $X_i \cap X_j = S$, otherwise S would not be a minimal a, b-separator. But then we must have $|S| \leq k$ since otherwise the edge-weight of R would be more than k. We now show that if $|S| < k$ then we can add an edge to G without increasing its branchwidth. Assume that moving from left to right on path P we first hit i and then its neighbor j. Move left from node i and right from node j until encountering the first nodes l and r with bags not contained in S, say $c \in X_l \setminus S$ and $d \in X_r \setminus S$. We now add vertex c to every bag corresponding to a node on the path from l to i and vertex d to every bag on the path from r to j. Note that the intersection of any two bags corresponding to adjacent nodes on the

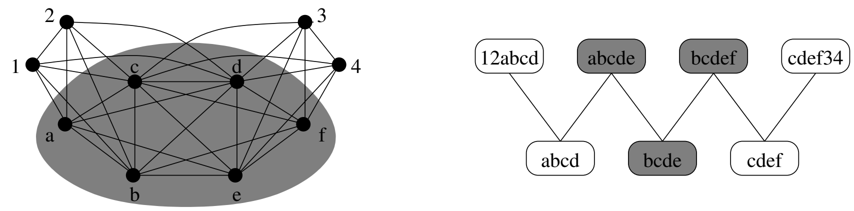

Fig. 5. A graph of branchwidth 4 satisfying all four conditions of Lemma 3 and yet edge af can be added without increasing its branchwidth. Its maxclique-minsep graph is also given and the mergeable subtree is shaded.

l to r path now has size $|S|+1$. Now subdivide the edge ij with the new node having bag $S \cup \{c, d\}$ and also hang a new leaf attached to it with bag $\{c, d\}$ (see Figure 4). If $|S| < k$ we would now have a ternary subtree-representation R' of edge-weight k with $EI(R') = NI(R')$. By Lemma 2 this would mean that the graph $EI(R')$ which is G with added edge cd has branchwidth k. So G could not have been a k-branch. $\square$

Note that any graph satisfying the four conditions in Lemma 3 will have branchwidth k, by Theorem 2 and the fact that no $k-1$-troika can respect a minimal separator of size k. However, Figure 4 shows a graph satisfying all four conditions for $k = 4$ and yet it is not a 4-branch. To state the fifth condition needed to characterize k-branches, the following auxiliary structure will be useful.

Definition 5 *Let C_G be the set of maximal cliques and S_G the set of minimal separators of a graph. The maxclique-minsep graph of G is a bipartite graph with vertex set $C_G \cup S_G$ and edge set $\{SX : S \in S_G \wedge X \in C_G \wedge S \subset X\}$.*

See Figure 4 for an example. We first show that the maxclique-minsep graph defines a tree-decomposition of the graph whenever conditions 1 and 2 of Lemma 3 hold.

Observation 1 *If G is a chordal graph where every minimal separator has size k then its maxclique-minsep graph is a tree-decomposition of G where every bag induces a clique, and we call it* the maxclique-minsep tree-decomposition *of G.*

Proof: We first show that the maxclique-minsep graph is a tree. Let S be a minimal separator of G that is the intersection of two maximal cliques, say X and Y. Then S is a minimal x, y-separator for any $x \in X \backslash S$ and $y \in Y \backslash S$, since otherwise there would exist a minimal separator of size larger than k. Assume by way of contradiction that the maxclique-minsep graph has an X, Y-path $X, S_1, X_1 \ldots S_k, Y$ avoiding S. As the minimal separators of G all have size k, for any minimal separator S_i ($1 \leqslant i \leqslant k$), we have $S_i \setminus S$ non-empty, which implies that the induced subgraph $G((X \cup X_1 \cup \cdots \cup Y) \setminus S)$ is connected. This contradicts the fact that S is an x, y-separator and thus the maxclique-minsep graph is a tree. It is easy to check that it is also a tree-decomposition where every bag induces a clique. □

The fifth condition ensures that the maxclique-minsep tree-decomposition is in some sense maximal, and uses the following definition of a mergeable subtree.

Definition 6 *Let G be a chordal graph where every minimal separator has size k, thus having a maxclique-minsep tree-decomposition $(T, \mathcal{X})$. A* mergeable subtree *of $(T, \mathcal{X})$ is a non-trivial subtree T' of T all of whose leaves are maxclique nodes, satisfying $|\{v : v \in X \text{ where } X \text{ a node in } T'\}| \leq \lfloor 3k/2 \rfloor$ and either*

(1) T' has at most one node that in T has a neighbor in $V(T) \setminus V(T')$, or
(2) T' is a path X, B, Y with X, B, Y and all their neighbors in T inducing a path A, X, B, Y, C satisfying $B \setminus (A \cup C) = \emptyset$

See Figure 4 for an example of a mergeable subtree for $k = 4$.

Theorem 4 *A graph G is a k-branch iff*

(1) G is chordal
(2) Every minimal separator of G has size k
(3) Every maximal clique of G has a k-troika respecting minimal separators
(4) G has at least $\lfloor 3(k-1)/2 \rfloor + 1$ vertices
(5) The maxclique-minsep tree-decomposition of G has no mergeable subtree

Proof: $\Rightarrow$: By Lemma 3 we already know that the first four conditions hold. If the fifth did not hold then let T' be a mergeable subtree of its maxclique-minsep tree-decomposition $(T, \mathcal{X})$. In that case the graph obtained from G by adding edges to make the set of vertices $X' = \{v : v \in X$ where X a node in $T'\}$ into a clique (i.e. merging the maximal cliques in T') would still have branchwidth k. Its maxclique-minsep tree-decomposition is obtained by merging the subtree T' into a single node X' of T. We need to show that X' has a k-troika respecting its minsep neighbors. We have $|\{v : v \in X$ where X a node in $T'\}| \leq \lfloor 3k/2 \rfloor$, so in case there exists at most one node A in $V(T) \setminus V(T')$ that has a neighbor in T', then we can trivially find a k-troika of X' satisfying A. Let us therefore assume T' is a path X, B, Y with X, B, Y and all their neighbors in T inducing a path A, X, B, Y, C satisfying $B \setminus (A \cup C) = \emptyset$. We show that in this case (A, C, S) is a k-troika of $X \cup Y$ respecting A, C, where $S = A \setminus C \cup C \setminus A$. Firstly, since by assumption X has a k-troika respecting A, B, and Y has a k-troika respecting B, C, and $|A| = |B| = |C| = k$, we must have $X = A \cup B$ and $Y = B \cup C$ and since $B \setminus (A \cup C) = \emptyset$ we must therefore have $X \cup Y = A \cup C$. Also, $A \cup S = C \cup S = A \cup C = X \cup Y$, and obviously (A, C, S) respects A, C. It remains to show that $|S| \leq k$. Note that $|S| = |A \setminus C \cup C \setminus A| = |A \cup C| - |A \cap C|$. Since $A \cup C = X \cup Y$ we have by assumption that $|A \cup C| \leq \lfloor 3k/2 \rfloor$. Also, we must have $|A \cap C| \geq \lfloor k/2 \rfloor$ since otherwise $|A \cup C| = |A| + |C| - |A \cap C| = k + k - |A \cap C| > \lfloor 3k/2 \rfloor$. Thus $|S| = |A \cup C| - |A \cap C| \leq \lfloor 3k/2 \rfloor - \lfloor k/2 \rfloor \leq k$.

$\Leftarrow$: Conditions 1 and 3 imply that G has branchwidth at most k by Theorem 2. If G has only one maximal clique then it has branchwidth k by condition 4. If G has more than one maximal clique then it has branchwidth k by conditions 4 and 2 and the fact that no $k-1$-troika can respect a minimal separator of size k. To prove that G is a k-branch we assume for sake of contradiction that some strict supergraph H of G is a k-branch and that it has a maxclique-minsep tree-decomposition T_H. Note first that since every minimal separator of both G and H is of size k then H cannot contain a minimal separator that is not also a minimal separator of G. Thus the minsep nodes of T_H are a subset of the minsep nodes of T_G. Consider the connected subtrees that result from removing the minsep nodes of T_H from T_G. Note that the maximal cliques of H must be in 1-1 correspondence with these subtrees. As H is a strict supergraph of G, there is at least one such subtree T' of T_G, corresponding to a maximal clique $X' = \{v : v \in X$ where X a node in $T'\}$ of H, containing at least two maxclique nodes of T_G. We show that either T' is a mergeable subtree of T_G or else it contains a mergeable subtree.

The rest of the proof is a case analysis on the number d of neighbors that X' has in T_H. Firstly, we must have $|X'| \leq \lfloor 3k/2 \rfloor$ since otherwise X' would not have a k-troika. If $d = 1$ then T' is a mergeable subtree of T_G by case 1 of Definition 6. We cannot have $d \geq 4$ since a k-troika can respect at most three distinct subsets of size k. If $d = 3$ then X' has three distinct minsep neighbors A, B, C and the only possible k-troika respecting all three is (A, B, C) and thus $A \cup B = B \cup C = A \cup C = X'$. A, B, C are three nodes of the tree T_G and thus wlog we can assume that the path from A to B in T_G does not pass through C. But this means that C is not contained in $A \cup B$ and thus $A \cup B \neq A \cup C$ and thus X' does not have a k-troika respecting A, B, C. Thus we have $d = 2$ so that X' has two minsep neighbors A, B. We show that T' is a mergeable subtree or contains a mergeable subtree. Since X' has a k-troika respecting A, B we have $A \cup B = X'$ which means that the path from A to B in T_G must pass through all maxclique nodes of T'. Thus T' is a path $X_1, S_1, X_2, ..., S_{t-1}, X_t$, with all these nodes and all their neighbors in T_G inducing a path $A, X_1, S_1, X_2, ..., S_{t-1}, X_t, B$ in T_G. We claim that X_1, S_1, X_2 would already be a mergeable subtree of T_G. Since X' has a k-troika respecting A, B we must have $A \cup B = X_1 \cup X_2 \cup ... \cup X_t$. But then by the interval structure of these maximal cliques we have $S_1 \setminus (A \cup S_2) = \emptyset$ and thus X_1, S_1, X_2 is a mergeable subtree of $(T, \mathcal{X})$ of G by case 2 of Definition 6. Thus, we have shown that if some strict supergraph of G is a k-branch then G satisfies condition 5. $\square$

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