Generation of graphs with bounded branchwidth

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Abstract

Branchwidth is a connectivity parameter of graphs closely related to treewidth. Graphs of treewidth at most k can be generated algorithmically as the subgraphs of k-trees. In this paper, we investigate the family of edge-maximal graphs of branchwidth k, that we call k-branches. The k-branches are, just as the k-trees, a subclass of the chordal graphs where all minimal separators have size k. However, a striking difference arises when considering subgraph-minimal members of the family. Whereas K_{k+1} is the only subgraph-minimal k-tree, we show that for any $k \geq 7$ a minimal k-branch having q maximal cliques exists for any value of $q \notin \{3,5\}$, except for $k = 8, q = 2$. We characterize subgraph-minimal k-branches for all values of k. Our investigation leads to a generation algorithm, that adds one or two new maximal cliques in each step, producing exactly the k-branches.

1 Introduction

Branchwidth and treewidth are mutually related connectivity parameters of graphs: whenever one of these parameters is bounded by some fixed constant for a graph, then so is the other [16]. Since many graph problems that are NP-hard in general can be solved in linear time when restricted to such classes of graphs both treewidth and branchwidth have played a large role in many investigations in algorithmic graph theory. Tree-decompositions have traditionally been the choice when solving NP-hard graph problems by dynamic programming to give FPT algorithms when parameterized by treewidth, see e.g. [2, 15] for overviews. Recently it is the branchwidth parameter that has been in the focus of several algorithmic research results. For example, several papers [7, 5, 8, 9, 6] show that for graphs of bounded genus the base of the exponent in the running time of these FPT algorithms could be improved by the dynamic programming following instead a branch-decomposition of optimal branchwidth. Also, a strong heuristic algorithm for the travelling salesman problem [4] has been developed based on branch-decompositions and an exact (exponential-time) algorithm has been given to compute branchwidth [10]. Given these recent developments in favor of branchwidth one may wonder why treewidth has historically been preferred over branchwidth? Mainly, this is because of the equivalent definition of '*G has treewidth $\leq k$*' by '*G is a partial k-tree*'. This alternative definition is intuitively appealing since the k-trees are the graphs generated by the following very simple algorithm: '*Start with K_{k+1}; Repeatedly choose a k-clique C and add a new vertex adjacent to vertices in C*'. Can we define branchwidth in an analogous algorithmic way? This is the question that has inspired our research and in this paper we give an affirmative answer.

We start by investigating the family of edge-maximal graphs of branchwidth k, that we call <u>k-branches</u>. The k-branches are chordal, as can be easily deduced from earlier work on branchwidth [11, 10]. In Section 2 we report on related work [13] where we have given a characterization of k-branches. In Section 3 we consider subgraph-minimal k-branches. They form the starting graphs

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of our algorithm generating k-branches, just as the minimal k-tree K_{k+1} is the starting graph of the generation algorithm for k-trees. K_n has branchwidth $\lceil 2n/3 \rceil$ for any $n \geq 3$ and $K_{\lfloor 3(k-1)/2 \rfloor + 1}$ is one of the minimal k-branches. However, for $k \geq 7$ we find that there is a minimal k-branch on q maximal cliques for any $q \notin \{3, 5\}$, except for the pathological case $k = 8, q = 2$. We show that the minimal k-branches have clique trees that are caterpillars and give a characterization of the family of minimal k-branches for all values of k. Our investigation culminates in Section 4 with a non-deterministic generation algorithm, that adds one or two new maximal cliques in each step, yielding as output exactly the graphs that are k-branches, and whose spanning subgraphs (i.e., partial graphs) are exactly the graphs of branchwidth at most k. Our results lead to a better understanding of the branchwidth parameter by defining graphs of branchwidth k through the algorithmic concept of partial k-branches. The algorithm will generate a random graph of branchwidth k together its branch-decomposition and can be used to provide test instances for optimization codes based on branch-decomposition.

2 Definitions and earlier results

A *branch-decomposition* (T, μ) of a graph G is a tree T with nodes of degree one and three only, together with a bijection μ from the edge-set of G to the set of degree-one nodes (leaves) of T. For an edge e of T let T_1 and T_2 be the two subtrees resulting from $T \setminus \{e\}$, let G_1 and G_2 be the graphs induced by the edges of G mapped by μ to leaves of T_1 and T_2 respectively, and let $mid(e) = V(G_1) \cap V(G_2)$. The width of (T, μ) is the size of the largest $mid(e)$ thus defined. For a graph G its *branchwidth* $bw(G)$ is the smallest width of any branch-decomposition of G. [1]

A tree-decomposition $(T, \mathcal{X})$ of a graph G is an arrangement of the vertex subsets $\mathcal{X}$ of G, called bags, as nodes of the tree T such that for any two adjacent vertices in G there is some bag containing them both, and for each vertex of G the bags containing it induce a connected subtree. For a subtree T' of T the induced tree-decomposition $(T', \mathcal{X}')$ is the result of removing from $(T, \mathcal{X})$ all nodes of $V(T) \setminus V(T')$ and their corresponding bags.

Definition 1 *A k-*troika *(A, B, C) of a set X are 3 subsets of X such that $|A| \leq k$, $|B| \leq k$, $|C| \leq k$, and $A \cup B = A \cup C = C \cup B = X$. (A, B, C)* respects *$S_1, S_2, ..., S_q$ if any $S_i, 1 \leq i \leq q$ is contained in at least one of A, B or C.*

A necessary condition for a graph to be a k-branch is that it is a chordal graph where all minimal separators have size k, with the property that every maximal clique has a k-troika respecting the minimal separators contained in it [13]. This motivates the following definition.

Definition 2 *Let G be a chordal graph with C_G its set of maximal cliques and S_G its set of minimal separators. A tree-decomposition $(T, \mathcal{X})$ of G is called k-*full *if the following conditions hold: 1) The set of bags $\mathcal{X}$ is in 1-1 correspondence with $C_G \cup S_G$ (we call the nodes with bags in C_G the* maxclique nodes *and the nodes with bags in S_G the* minsep nodes*.) 2) The bags of the minsep nodes all have cardinality k. 3) There is an edge ij in the tree T iff $X_i \in S_G, X_j \in C_G$ and $X_i \subseteq X_j$. 4) Every maxclique bag X_j has a k-troika respecting its neighbor minsep bags.*

Note that if G has a k-full tree-decomposition then it is unique. We need additional constraints on k-full tree-decompositions to characterize exactly the k-branches.

Definition 3 *A* mergeable subtree *of a k-full tree-decomposition $(T, \mathcal{X})$ of a graph G is a subtree T' of T that: contains at least one edge, has leaves that are maxclique nodes, and satisfies:*

[1] The graphs of branchwidth 1 are the stars, and constitute a somewhat pathological case. To simplify certain statements we therefore restrict attention to graphs having branchwidth $k \geq 2$.

1. $|\{v : v \in X$ *where* X *a node in* $T'\}| \leq \lfloor 3k/2 \rfloor$
2. *Either the subtree* T' *has at most one node that in* T *has a neighbor in* $V(T) \setminus V(T')$ *or else* T' *is a path* X, B, Y *with* X, B, Y *and all their neighbors in* T *inducing a path* A, X, B, Y, C *satisfying* $B \setminus (A \cup C) = \emptyset$.

Lemma 1 *Let* $A - X - B - Y - C$ *be a path in* T *for some* k*-full tree-decomposition* $(T, \mathcal{X})$ *with* X *and* Y *maxclique nodes.* $X \cup Y$ *has a* k*-troika respecting* A, C *if and only if* $|X \cup Y| \leq \lfloor 3k/2 \rfloor$ *and* $B \setminus (A \cup C) = \emptyset$.

Proof If $|X \cup Y| > \lfloor 3k/2 \rfloor$ then $X \cup Y$ does not have a k-troika. Let $P = B \setminus (A \cup C)$. Note that we have $A \cap C \subseteq B$ and since $|A| = |C| = k$ we have $|A \cap C| = 2k - |(X \cup Y) \setminus P|$. But then $|X \cup Y| + |A \cap C| = 2k + |P|$ and this means that by Theorem 2 of [14] (also by results of [11]) $X \cup Y$ has a k-troika respecting A, C if and only if $P = \emptyset$. □

Lemma 1 is implicit in [13], and implies that for mergeable subtree T' we can add edges to G to make a clique of $\{v : v \in X$ where X a node in $T'\}$ without increasing branchwidth of G.

Definition 4 *A* k*-full tree-decomposition* $(T, \mathcal{X})$ *of a graph* G *is a* k-skeleton *of* G *if* G *has at least* $\lfloor 3(k-1)/2 \rfloor + 1$ *vertices and* T *does not have a mergeable subtree.*

Theorem 1 *[13]* G *is a* k*-branch* $\Leftrightarrow$ G *has a* k*-skeleton*

3 Minimal k-branches

We characterize the subgraph-minimal k-branches on q maximal cliques, by describing the structure of the minimal k-skeletons, as defined below. We divide the caracterization into two Theorems, one for the cases when $k \leq 6$ or $q \leq 5$ and the other for the cases $k \geq 7, q \geq 6$.

Definition 5 *A* k*-branch* G *is a* minimal k-branch *if no strict subgraph of* G *is a* k*-branch. Let the set of* minimal k-skeletons *be* $MS(k) = \{(T, \mathcal{X}) : (T, \mathcal{X})$ *is a* k*-skeleton but for no proper subtree* T' *of* T *is the induced tree-decomposition* $(T', \mathcal{X}')$ *a* k*-skeleton*$\}$. *Let* $MS(k, q)$ *be the set of minimal* k*-skeletons on* q *maxclique nodes.*

If G is a minimal k-branch then for its k-skeleton $(T_G, \mathcal{X})$ we have $(T_G, \mathcal{X}) \in MS(k)$. However, the graph represented by a minimal k-skeleton may have some cliques that are too big for it to be a minimal k-branch. For example, if $(T, \mathcal{X})$ is the tree T having a single maxclique node on 6 vertices then we have $(T, \mathcal{X}) \in MS(4)$ since it is a minimal 4-skeleton but the graph K_6 that it represents is not a minimal 4-branch since it contains the 4-branch K_5 as a subgraph. Since our algorithm in Section 4 builds k-skeletons, rather than graphs, we focus in the following on the minimal k-skeletons.

Lemma 2 *In a minimal* k*-skeleton* $(T, \mathcal{X})$*, the tree* T *does not contain a maxclique leaf* X *with path* $X - A - Y$ *and both* A *and* Y *having degree 2.*

Proof X and Y cannot be merged since otherwise G is not a k-branch, thus $|X \cup Y| > \lfloor \frac{3k}{2} \rfloor$. But then the subgraph induced by $X \cup Y$ is already a k-branch and G is not minimal: contradiction. □

See Figure 1 for an illustration of the following Theorem, which characterizes the minimal k-skeletons on q maximal cliques when $k \leq 6$ or $q \leq 5$.

Theorem 2 1. *For* $k \geqslant 2$, $MS(k, 1)$ *contains* $K_{\lfloor 3(k-1)/2 \rfloor + 1}$ *and if* k *even then also* $K_{\lfloor 3(k-1)/2 \rfloor + 2}$.

2. *For $k \leqslant 6$ and $k = 8$, $MS(k,2) = \emptyset$. For $k = 7$ and $k \geq 9$, $MS(k,2)$ is nonempty and consists of the trees with two maxclique nodes of size $x+k$ and $y+k$ (with k common vertices) for any $x \leq y$ satisfying*

$$3 - (k \bmod 2) \leq x \leq y \leq \lceil k/2 \rceil - 2 \qquad \textit{and} \qquad x + y \geq \lfloor k/2 \rfloor + 1 \tag{1}$$

3. *For any k, $MS(k,3) = \emptyset$.*
4. *For $k \leqslant 4$ and $k = 6$, $MS(k,4) = \emptyset$. For $k = 5$ and $k \geq 7$ $MS(k,4)$ is nonempty, and consists of the k-full tree-decompositions $(T,\mathcal{X})$ on $q = 4$ maxclique nodes X_1, X_2, X_3, Y with Y a node of degree 3 in T such that*
   (a) $|Y|, |X_i| \leqslant \lfloor \frac{3(k-1)}{2} \rfloor$ for any $i \in [1,3]$;
   (b) for any $i \in [1,3]$, $|X_i \cup Y| \leqslant \lfloor \frac{3k}{2} \rfloor$;
   (c) $\lfloor \frac{3k}{2} \rfloor + 1 \leqslant |X_i \cup Y \cup X_j|$ with $1 \leq i < j \leq 3$
5. *For any k, $MS(k,5) = \emptyset$*
6. *For any $k \leqslant 6$ and $q \geqslant 6$, $MS(k,q) = \emptyset$.*

Proof $q = 1$: This follows from the well-known fact that for $n \geq 3$, $bw(K_n) = \lceil 2n/3 \rceil$.

For the remaining cases observe that (*): if a minimal k-skeleton has $q \geq 2$ maxcliques then it has at least $\lfloor \frac{3k}{2} \rfloor + 1$ vertices and the number of vertices in each maxclique must be in the range $[k+1, \lfloor \frac{3(k-1)}{2} \rfloor]$.

$q = 2$: Let $G = (V,E)$ be a minimal k-skeleton with 2 maxcliques K_a and K_b with $a \leq b$. Since any minimal separator has size k we have $|V| = a + b - k$. From observation (*) above it follows that $\lfloor \frac{3k}{2} \rfloor + 1 \leqslant |V| \leqslant 2\lfloor \frac{3(k-1)}{2} \rfloor - k$. Resolving for k we get that if k is odd then $k \geqslant 7$, and if k is even then $k \geqslant 10$. Let $x = a - k, y = b - k$. It can be checked that the two Equations in (1) follow from observation (*) above. The first Equation of (1) ensures that each maxclique has by itself branchwidth less than k, and the second ensures that if we merge the two maxcliques then the result has branchwidth larger than k.

$q = 3$: By contradiction. By Lemma 2, the tree of the minimal k-skeleton cannot be a path. Therefore the maxcliques X, Y and Z would have to share a common minsep S. As no pair of maxcliques are mergeable, the size of the union of any pair of maxcliques is at least $\lfloor \frac{3k}{2} \rfloor + 1$. It follows that any pair of maxcliques is already a k-branch.

$q = 4$: By Lemma 2, the only possible topology for the tree T of the k-skeleton is the subdivided claw, i.e. with one maxclique having degree 3. Let X_1, X_2, X_3 be the three maxclique leaves and Y the degree 3 maxclique.

First notice that the subtree induced by X_i, X_j, Y can be merged iff condition *(c)* holds. Assume $(T,\mathcal{X})$ is a minimal k-skeleton. Condition *(a)* holds by observation (*) and condition *(b)* holds as by definition no pair of neighboring maxcliques induces a k-skeleton. Assume $(T,\mathcal{X})$ is a k-full tree decomposition satisfying the 3 conditions. By condition *(c)* it is a k-skeleton. Indeed no subtree can be merged as for any pair of neighboring maxcliques, say X_1 and Y, the set $X_1 \cup Y$ does not have any k-troika respecting $Y \cap X_2$ and $Y \cap X_3$. By conditions *(a)* and *(b)* and the fact that $MG(k,3) = \emptyset$, no subtree induces a k-skeleton. $(T,\mathcal{X})$ is thereby a minimal k-skeleton. It follows that:

$$\lfloor \frac{3k}{2} \rfloor + 1 \leqslant |X_i \cup Y \cup X_j| \leqslant 3\lfloor \frac{3(k-1)}{2} \rfloor - 2k \tag{2}$$

Resolving for k we find that if k is odd then $k \geqslant 5$, and if k is even then $k \geqslant 8$.

$q \geqslant 6$: For $k \leqslant 4$, as $K_{\lfloor \frac{3(k-1)}{2} \rfloor + 1} = k + 1$, there is no minimal k-skeleton on $q > 1$ maxclique. For $k = 5$ or 6, we show that the tree T of the k-skeleton cannot contain a path $X - A - Y$ with both X and Y maxclique leaves. Combined with Lemma 2, it implies that any maxclique leaf belongs to a subdivided claw. Assume such a path exists. We should have $|X| = |Y| = k + 1$ and thereby $|X \cup Y| = k + 2$: implying that the graph it represents is not a k-branch. If $k = 5$, such a subdivided claw contains only maxcliques of size 6, which is already a minimal 5-branch. Similarly, if $k = 6$, the subdivided claw only contains maxcliques of size 7. But then the subdivided claw only contains one maxclique leaf. Otherwise the two maxclique leaves and the degree 3 maxclique could have been merged without increasing the branchwidth (it would form a maxclique leaf of size 9). If any subdivided claw contains at most one maxclique leaf, the number of maxcliques cannot be finite: contradiction. Therefore for any $k \leqslant 6$ and $q \geqslant 6$, $MS(k, q) = \emptyset$

□

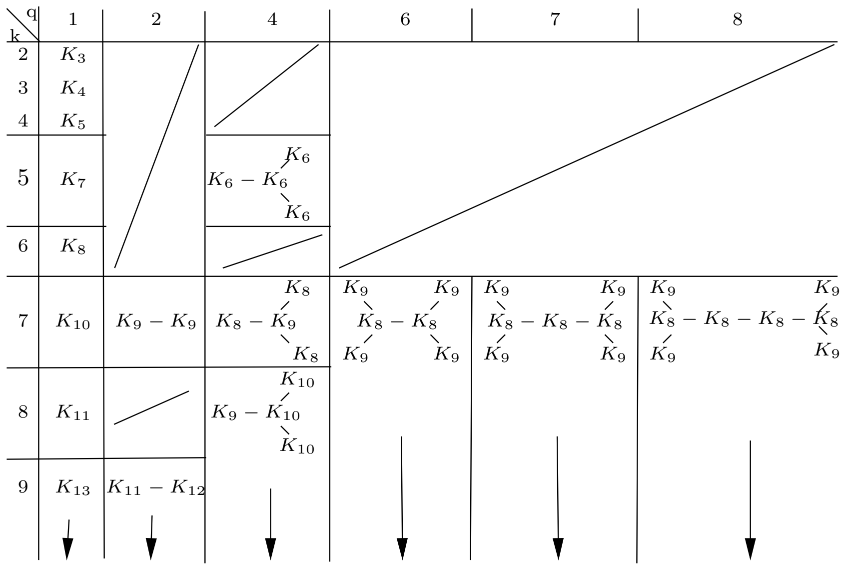

Figure 1: Examples of minimal k-skeletons $(T, \mathcal{X})$ on q maxclique nodes, for $k \leq 9, q \leq 8$. For $q = 4$ the following k-full tree-representations are minimal k-skeleton: the subdivided claw with $K_{\lfloor 3(k-1)/2 \rfloor}$ being the degree 3 maxclique, and K_{k+1} being the leaf cliques are minimal k-skeletons, for odd k; and for even k, two maxclique leaves should be replaced by K_{k+2}. Downward arrows indicate that minimal k-sketelons, with trees isomorphic to those depicted, exist also for larger k. Only the maxclique nodes are drawn. The minsep nodes have size k and appear on each edge of the trees. Only for the case $q = 8$ does the intersection of minsep nodes matter. For $q = 8$, if $A - X - B - Y - C$ the path with X and Y the maxclique nodes of degree two then minsep nodes A, B, C must satisfy $B \setminus (A \cup C) \neq \emptyset$.

From this characterization of minimal k-skeletons we can deduce the characterization of minimal k-branches. For lack of space we only sketch how to do this for the case $q = 2$. Note that if two distinct pairs $x \leq y$ and $x' \leq y'$ both satisfy Equations (1) then the graph associated with the first pair is a subgraph of the graph associated with the second pair if and only if $x \leq x'$ and $y \leq y'$. Thus, the minimal k-branches on $q = 2$ maximal cliques correspond with such smallest pairs $x \leq y$.

To describe the minimal k-skeletons for $k \geq 7$ having $q \geq 6$ the following definition of the adjacencies in a special caterpillar T will be useful (see also Figure 2).

Definition 6 *A tree T is a* special caterpillar *if T consists of a body which is a path $X_1, S_1, X_2, S_2, \ldots$ X_p, S_p, X_{p+1} alternating between maxclique and minsep nodes for some $p \geq 3$ with added hairs of length one or two (a hair of length one being a new maxclique node added as neighbor of a minsep node of the body, and a hair of length two being two new adjacent maxclique-minsep nodes with the minsep node added as neighbor of a maxclique node of the body) satisfying the following conditions:*

1. *at most one hair for each node of the body*

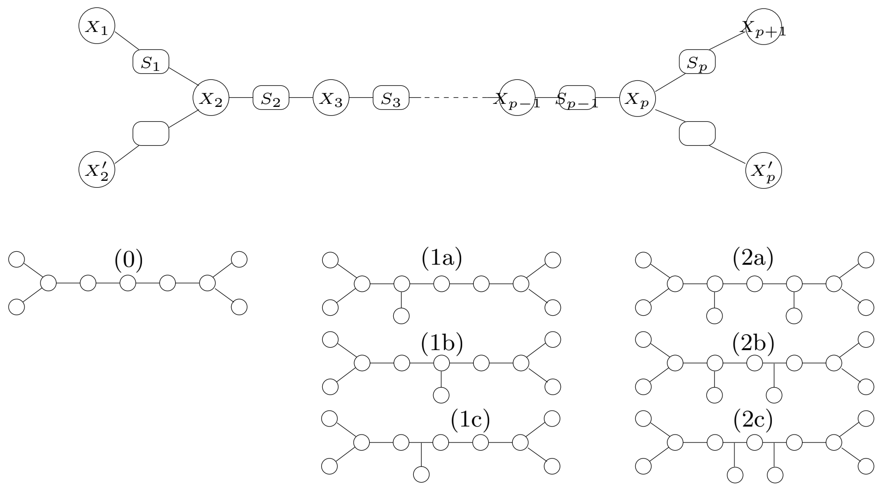

Figure 2: On top is a special caterpillar with names of nodes as in Definition 6, in Theorem 3 and in Algorithm Stage 1. Below are the 7 non-isomorphic special caterpillars with $p = 6$, with maxclique nodes drawn as circles and minsep nodes not drawn explicitly but present on any edge between two adjacent maxclique nodes. Thus, 1c, 2b, 2c have minsep nodes of degree 3.

2. *no hair on any of* $X_1, S_1, S_2, S_{p-1}, S_p, X_{p+1}$
3. *hair* X'_2 *on* X_2 *and hair* X'_p *on* X_p*, but no hair on minseps* $X'_2 \cap X_2, X'_p \cap X_p$
4. *if hair on* S_i *then no hair on* X_i *and no hair on* X_{i+1}
5. *if hair on* X_i *then not hairs on both of* X_{i-1} *and* X_{i+1}

Theorem 3 $(T, \mathcal{X})$ *is a minimal* k*-skeleton for some* $k \geq 7$ *on at least* $q \geq 6$ *maxclique nodes* $\Leftrightarrow$ $(T, \mathcal{X})$ *is a* k*-full tree-decomposition with* T *a special caterpillar whose bags satisfy (bag names as in Definition 6 and Figure 2):*

1. *either* $|X_1 \cup X_2 \cup X_3| \leq 3k/2$ *or* $|X'_2 \cup X_2 \cup X_3| \leq 3k/2$ *and also either* $|X_{p+1} \cup X_p \cup X_{p-1}| \leq 3k/2$ *or* $|X'_p \cup X_p \cup X_{p-1}| \leq 3k/2$
2. $|X_1 \cup X_2 \cup X'_2| > 3k/2$ *and* $|X_{p+1} \cup X_p \cup X'_p| > 3k/2$
3. *For maxcliques* X, Y *with a common neighbor,* $|X| \leq \lfloor 3(k-1)/2 \rfloor$ *and* $|X \cup Y| \leq 3k/2$
4. *If* S_i *has a hair then* $S_i \setminus (S_{i-1} \cup S_{i+1}) = \emptyset$
5. *If* X_i *has a hair then either i) no hair on* X_{i-1} *and* $S_{i-1} \setminus (S_{i-2} \cup S_i) = \emptyset$ *or ii) no hair on* X_{i+1} *and* $S_i \setminus (S_{i-1} \cup S_{i+1}) = \emptyset$
6. *If no hair on neither of* X_i, S_i, X_{i+1} *then* $S_i \setminus (S_{i-1} \cup S_{i+1}) \neq \emptyset$

Proof $\Leftarrow$: We first show that the k-full tree-decomposition $(T, \mathcal{X})$ is a k-skeleton, by showing that T does not have a mergeable subtree as in Definition 3. Any subtree T' having at most one node that in T has a neighbor in $V(T) \setminus V(T')$ is by condition 2 not mergeable since we would have $|\{v : v \in X \text{ and } X \text{ a maxclique node in } T'\}| > \lfloor 3k/2 \rfloor$. Any subtree T' which is a path X, B, Y with X, B, Y and all their neighbors in T inducing a path A, X, B, Y, C will by condition 6 satisfy $B \setminus (A \cup C) \neq \emptyset$ and is thus not mergeable. Thus $(T, \mathcal{X})$ is a k-skeleton and it remains to show that it is a minimal k-skeleton. We prove by contradiction, that for any proper subtree T' of T the induced tree-decomposition $(T', \mathcal{X}')$ is not a k-skeleton. Unless the graph G' that $(T', \mathcal{X}')$ represents has at least $\lfloor 3(k-1)/2 \rfloor + 1$ vertices, $(T', \mathcal{X}')$ is not a k-skeleton. By condition 3 this means that T' must contain at least 2 maxclique nodes. We show that in any such T' there is a mergeable subtree T''. There are 5 special cases of subtrees T' to consider:

1. Suppose maxclique bags of T' are X_1, X_2, X_2', X_3 or $X_{p-1}, X_p, X_p', X_{p+1}$. In both cases the 3 maxclique bags satisfying the size constraint in condition 1 form the mergeable subtree T".
2. T' contains a leaf X having a minsep neighbor S of degree 2 that itself has neighbor Y. By condition 3, X, S, Y makes up the mergeable subtree T''.
3. T' contains two maxclique leaves X, Y with a common minsep neighbor S. Again by condition 3 X, S, Y is the mergeable subtree.
4. Suppose in T there was a hair on minsep S_i and that T' does not contain this hair but does contain $S_{i-1}, X_i, S_i, X_{i+1}, S_{i+1}$. In this case the mergeable subtree T'' is X_i, S_i, X_{i+1} by condition 4 and Lemma 1.
5. Suppose T has a hair on maxclique X_i and that T' does not contain this hair but that T' does contain $X_{i-1}, S_{i-1}, X_i, S_i, X_{i+1}$. Since T is a special caterpillar, neither S_{i-1} nor S_i has a hair. Thus, condition 5 and Lemma 1 guarantee that either the subtree X_{i-1}, S_{i-1}, X_i or the subtree X_i, S_i, X_{i+1} is mergeable.

$\Rightarrow$: We establish three properties (A),(B),(C) of the nodes of any tree T of a minimal k-skeleton $(T, \mathcal{X})$. Recall that $MS(k)$ is the set of minimal k-skeletons.

(A) Any minsep node S of degree larger than 2 must have degree 3 with exactly one of its neighbors being a maxclique leaf and the other two having degree 2.

Assume first by contradiction S has two maxclique leaf neighbors X, Y. If $|X \cup Y| \leq \lfloor 3k/2 \rfloor$ then they could have been merged into a larger clique. If $|X \cup Y| > \lfloor 3k/2 \rfloor$ the subtree on the three nodes $X, X \cap Y, Y$ would induce a k-skeleton contradicting $(T, \mathcal{X}) \in MS(k)$. We next show that for any three components T_1, T_2, T_3 of $T \setminus S$ one of the three must be a single maxclique node, thereby establishing that S has degree 3 and exactly one maxclique leaf neighbor. Assume that none of T_1, T_2, T_3 is a single maxclique node. Let $X_2 \in V(T_2)$ be a neighbor of S. We claim that the maximal subtree T' of T containing X_2 as a leaf with parent S would already induce a k-skeleton. This since any subtree T'' of T' that is mergeable in T' would have to contain X_2 (otherwise it would be mergeable also in T) and it would have to contain either all of T_1 or all of T_3, say wlog T_1, (otherwise the new merged clique would contain two minimal separators A, B with $X_2 \neq (A \cup B)$.) But then the union of maxclique nodes in T_1 would have size less than $3k/2$, which means that T_1 would have been a mergeable subtree already in T (since the new merged clique also in T would have only the minsep neighbor S) contradicting $T \in MS(k)$. Thus S has degree 3 and one maxclique leaf neighbor X_1. Let us show that the neighbors X_2, X_3 have degree 2. Assume X_2 has 3 minsep neighbors A, B, S and consider $T' = T \setminus \{X_1\}$. Note that we cannot have T' representing a graph of branchwidth less than k since otherwise T_2 would already have been mergeable in T. As any mergeable subtree T'' of T' could not be mergeable in T, T'' would have to contain X_2 and X_3. It then would also have to contain either all of T_1 or T_2. Otherwise the new merged clique would contain two minimal separators A, B with $X_2 \cup X_3 \neq (A \cup B)$. This means that T_1 or T_2 was already mergeable in T: contradiction.

(B) Any maxclique node X of degree 3 has all 3 minsep neighbors A_1, A_2, A_3 of degree 2 and at least one of them has a maxclique leaf as neighbor.

By (A), the minseps A_i's all have degree 2. Let Y_i be the second neighbor of A_i and assume neither of Y_1, Y_2, Y_3 is a leaf. Consider the partition of T into the subdivided claw on maxcliques (A_1, A_2, A_3, X) and the three subtrees T_i rooted in Y_i $(i = 1, 2, 3)$. Let T' be the subtree of T consisting of nodes X, A_1, A_2 together with T_1, T_2. Note that T' cannot represent a graph of branchwidth less than k since then T_1 (and T_2) would have been mergeable in T. Moreover, any mergeable subtree T'' of T' could not be mergeable in T so T'' would have to contain Y_3 and X, but then it would have to contain either all of T_1 or T_2. This means that either the subtree T_1 or the subtree T_2 was already mergeable in T, a contradiction.

(C) T cannot contain a leaf X having a degree-2 parent B that itself has another degree-2 neighbor Y. (This is Lemma 2)

We are ready to describe all trees $T \in MS(k)$. There are two trees containing respectively 1 and 2 maxclique nodes except for $k = 8$ (see Theorem 2). For the remaining trees we note that (A), (B) and (C) together imply that for any maxclique leaf X in T with parent A we have either: A of degree 3 with the other two neighbors of A having degree 2 and not being leaves (call these leaves of type i); or A of degree 2 with parent Y of degree 3 having 3 neighbors of degree 2 with 1, 2 or 3, respectively, of these being neighbors of a leaf (leaves of type ii.1, ii.2, ii.3 respectively.) Moreover, all nodes of degree 3 in T (which is the maximum) have at least one neighbor that is a leaf or neighbor of a leaf. Thus we can use the 4 types (i, ii.1, ii.2, ii.3) as building-blocks for any tree $T \in MS(k)$. If we use a building-block of type ii.3) then there is only a unique tree possible, with 4 maxclique nodes, covered already in Theorem 2. Building blocks of type i) and ii.1) contain one leaf and two nodes needing new neighbors, while type ii.2) contains two leaves and one node needing a new neighbor. Thus, when using building-blocks of types i), ii.1) or ii.2) we must always have exactly two building-blocks of type ii.2), that will correspond to two ends of the body of a caterpillar having hairs of length 1 (type ii.1) or 2 (type i). A minsep node of degree 3 cannot be adjacent to a maxclique node of degree 3, because the maxclique hair of this minsep could then have been dropped and we would still have an induced k-skeleton. Likewise, no three consecutive maxclique nodes of the body all have a hair since then the middle hair could have been dropped and we would still have an induced k-skeleton. Thus, T is a special caterpillar.

To end the proof, it suffices to note that conditions 1-6 of the Theorem hold, since otherwise T would either have had a mergeable subtree or it would have been a k-skeleton but not minimal. For example, if condition 6 did not hold for some i then by Lemma 1 $X_i - S_i - X_{i+1}$ would have been a mergeable subtree. For space reasons we do not give the details of all cases. □

4 An algorithm that generates k-branches

In this section we give an algorithm generating each possible k-skeleton, which by Theorem 1 will correspond to generation of the k-branches. For space reasons proofs of correctness have been moved to the Appendix.

Definition 7 *We get an* extended *k-skeleton by taking a k-skeleton $(T, \mathcal{X})$ and adding zero or more minsep leaves with bag-size k as neighbors of maxclique nodes of T while ensuring that each maxclique node still has a k-troika respecting its minsep neighbors. When starting with a minimal k-skeleton $(T, \mathcal{X})$ the result is an* extended minimal *k-skeleton. We define $ES(k)$ to be the set of extended k-skeletons and $EMS(k)$ to be the set of extended minimal k-skeletons.*

Recall that $MS(k)$ are the minimal k-skeletons, and note that by definition $MS(k) \subseteq EMS(k) \subseteq ES(k)$. The algorithm is organised in 3 stages with the outputs of the previous stage forming the inputs to the next stage. Stage 1 generates $MS(k)$, Stage 2 generates $EMS(k)$ and Stage 3 generates $ES(k)$. Note that the extended k-skeletons $ES(k)$ have the dual property that we get a k-skeleton both if we remove all minsep leaves and also if we add a new legal maxclique leaf to each minsep leaf. For our generation algorithm this implies that generating k-branches is equivalent to generating extended k-skeletons where all leaves are maxclique nodes. The reason we generate extended k-skeletons, and not only the k-skeletons, is to be able to enforce that all eventual minsep neighbors of a maxclique node are added as soon as the maxclique node is added. This to easily satisfy the constraint that a maxclique node have a k-troika respecting its minsep neighbors.

Description of STAGE 1: Generation of the minimal k-skeletons $MS(k)$.
See Algorithm 1. The minimal k-skeletons on 1, 2, 4, or 6 maxclique nodes are generated by the

special rules `1clique, 2clique, 4clique` or `6clique` respectively. The special caterpillar T in a minimal k-skeleton $(T,\mathcal{X})$ on 6 maxclique nodes is unique, $p=6$ in Definition 6

For the minimal k-skeletons $(T,\mathcal{X})$ on more than 6 maxcliques we enter a Repeat-loop that will generate the special caterpillar T from left to right by adding in each iteration one or two new maxclique nodes to the current right end of its body. The Repeat-loop is prefixed and postfixed by special operations `Start` and `End` that add the building-blocks that in the proof of Theorem 3 are called type ii.2). Note that throughout the code the names of parameters denoting maxclique and minsep nodes are in accordance with Definition 6 and Figure 2, with certain exceptions. In particular, in the prefix operation `Start`$(X_1, X_2, X_2', X_3, Hair, S_3)$ the parameter $Hair$ is a maxclique leaf hair of length 2 added to maxclique X_3.

Algorithm 1: STAGE 1: Generate any $(T,\mathcal{X}) \in MS(k)$ by choosing 1,2,3,4 or 5

1: $(T,\mathcal{X}) := 1clique(X)$ s.t. Theorem 2, case $q=1$ holds ;
2: $(T,\mathcal{X}) := 2clique(X,Y)$ s.t. Theorem 2, case $q=2$ holds ;
3: $(T,\mathcal{X}) := 4clique(X,Y,Z,W)$ s.t. Theorem 2, case $q=4$ holds ;
4: $(T,\mathcal{X}) := 6clique(X_1,X_2,X_2',X_3,X_3',X_4)$ s.t. T is the unique special caterpillar with $p=3$ and $q=6$ and conditions 1,2,3 in Theorem 3 hold ;
5: **begin**
  first choose a or b while ensuring that condition 1 and 2 of Theorem 3 hold;
  a: `Start`(X_1,X_2,X_2',X_3,S_3), $i:=3$, `HasHair`$:=0$, `NeedsPair`$:=0$;
  b: `Start`$(X_1,X_2,X_2',X_3,Hair,S_3)$, $i:=3$, `HasHair`$:=1$, `NeedsPair`$:=1$;
  repeat
    if *`HasHair` and `NeedsPair`* **then** choose `rule I`;
    else if *`HasHair` and* **not** *`NeedsPair`* **then** choose `rule I,II` or `III`;
    else choose `rule II, III` or `IV`;
    `rule I`: `ADD`(S_i,X_{i+1},S_{i+1}) s.t. $S_i \setminus (S_{i-1}\cup S_{i+1})=\emptyset$;
    `rule II`: `ADD`(S_i,X_{i+1},S_{i+1}) s.t. $S_i \setminus (S_{i-1}\cup S_{i+1})\neq\emptyset$;
    `rule III`: `ADD`(S_i,X_{i+1},B,S_{i+1}) and `ADD`$(B,Hair)$;
    `rule IV`: `ADD`(S_i,X_{i+1},S_{i+1}) and `ADD`(S_i,W) s.t. $S_i \setminus (S_{i-1}\cup S_{i+1})=\emptyset$;
    if *`rule III` was chosen* **then** `HasHair`$:=1$ and `NeedsPair`$:=(S_i\setminus(S_{i-1}\cup S_{i+1})\neq\emptyset)$;
    else `HasHair`$:=0$ and `NeedsPair`$:=0$;
    $i:=i+1$;
  until *body of caterpillar is finished and `NeedsPair`*$=0$;
  `End`(S_i,X_i,X_i',X_{i+1}) s.t. Thm 3 (cond. 1 and 2) holds with $i=p$;
end

In the repeat-loop we maintain the loop invariant that S_i will be the minimal separator at the current right end of the body at which construction of the caterpillar will continue. Throughout STAGE 1 we make the assumption that when adding a new maxclique node X adjacent to some minsep node S then for any other neighbor Y of S the pair X,Y must satisfy condition 3 of Theorem 3. To not clutter the code we do not explicitly state these conditions. When adding new maxclique nodes, both here and in *Stage 3*, the syntax for the operation is `ADD`$(oldsep, newclique, newsep1, newsep2)$, where the two latter parameters may be missing. The *newclique* node is added as a neighbor of *oldsep* and the *newsep* nodes are added as neighbors of *newclique*. Thus, to extend the rightmost end of the body by a path S_i, X_{i+1}, S_{i+1} we use in `rule I` and `rule II` the operation `ADD`(S_i,X_{i+1},S_{i+1}). In `rule III` we are additionally adding a hair of length two consisting of minsep B and maxclique $Hair$ to the new maxclique node X_{i+1} and express this by the two operations `ADD`(S_i,X_{i+1},B,S_{i+1}) and `ADD`$(B,Hair)$. In `rule IV` we are additionally adding a hair W of length one to minsep node S_i and express this by the additional operation `ADD`(S_i,W). The boolean values `HasHair` and `NeedsPair` govern which of `rule I` to `rule IV` can be applied while ensuring that the conditions for minimal k-branches are fullfilled. `HasHair` is True iff the rightmost maxclique node X_i of the current body has a hair H attached to it. `NeedsPair` is True iff `HasHair` is True and the next-to-last maxclique node X_{i-1} would not be mergeable with X_i even if we had removed the hair H (in which case the next maxclique node X_{i+1} must satisfy that X_{i+1} and X_i would be mergeable if we had removed H.)

Description of STAGE 2: Generation of the set $EMS(k)$
For space reasons we do not show a separate algorithm in the text. The input to STAGE 2 is a minimal k-skeleton $(T, \mathcal{X}) \in MS(k)$ as generated by STAGE 1. STAGE 2 is a repeat-loop that can be exited at any time and which in each iteration adds one new minsep leaf S as neighbor of some maxclique node X of $(T, \mathcal{X})$, according to Definition 7. We must ensure that X will still have a k-troika respecting its minsep neighbors. If X already had one neighbor A then condition $\texttt{OK}(X, A, S)$ must hold, if it had two neighbors A, B then condition $\texttt{OK}(X, A, B, S)$ must hold, while if it had three neighbors then a new neighbor cannot be added. These conditions are used also in STAGE 3 and defined by: '$\texttt{OK}(X, A, B)$ is True iff $|X| + |A \cap B| \leq 2k$' and '$\texttt{OK}(X, A, B, C)$ is True iff $|A \cup B| = |A \cup C| = |B \cup C| = |X|$'.

Lemma 3 $EMS(k) = \{(T, \mathcal{X}) : \exists$ *sequence of choices in* STAGE 1 *and in* STAGE 2 *s.t.* STAGE 2 *gives as output* $(T, \mathcal{X})\}$

Description of STAGE 3: Generate the set $ES(k)$.
See Algorithm 2. As in STAGE 1 the rule adding a new maxclique node X adjacent to an existing minsep node A with new promise leaves B and C will have the syntax $\texttt{ADD}(A, X, B, C)$. In case we have one or zero promise leaves the syntax is $\texttt{ADD}(A, X, B)$ and $\texttt{ADD}(A, X)$. The shorthand $\texttt{ADD}(A, X, ...)$ can be replaced by any of the 3 rules. Similarly, the shorthand $\texttt{OK}(X, A, ...)$ appearing right after some $\texttt{ADD}(A, X, ...)$ has the intepretation that any third and fourth parameters B and C of the `ADD` also becomes a third and fourth parameter of the `OK`.

Algorithm 2: STAGE 3: Takes as input some $(T, X) \in EMS(k)$ produced by STAGE 2 and builds on this to produce as output an extended k-skeleton in $ES(k)$

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repeat
- Choose a minsep node A of T;
- **if** A *a leaf with parent* W *having a single other neighbor* S **then**
  - choose 1, 2, 3, 4 or 5;
  - 1: $\texttt{ADD}(A, New)$ s.t. $|W \cup New| > \lfloor 3k/2 \rfloor$;
  - 2: $\texttt{ADD}(A, New, B)$ s.t. $\texttt{OK}(New, A, B)$ and $|W \cup New| + |B \cap S| > 2k$;
  - 3: $\texttt{ADD}(A, New1)$ and $\texttt{ADD}(A, New2)$ s.t. $|New1 \cup New2| > \lfloor 3k/2 \rfloor$;
  - 4: $\texttt{ADD}(A, New1, B, ...)$ and $\texttt{ADD}(A, New2, ...)$ s.t. $\texttt{OK}(New1, A, B, ...)$ and $\texttt{OK}(New2, A, ...)$;
  - 5: $\texttt{ADD}(A, New, B, C)$ s.t. $\texttt{OK}(New, A, B, C)$;
- **else**
  - choose 6 or 7;
  - 6: $\texttt{ADD}(A, New, B, ...)$ s.t. `OK`;
  - 7: $\texttt{ADD}(A, New)$ s.t. $|Y \cup New| > \lfloor 3k/2 \rfloor$ for $\forall Y$ maxclique leaf with parent A;

until *done*;
Output extended k-skeleton $(T, \mathcal{X})$, which represents a k-branch iff it has no minsep leaves;

Theorem 4 $ES(k) = \{(T, \mathcal{X}) : \exists$ *sequence of choices of rules in the 3 stages s.t. output is* $(T, \mathcal{X})\}$

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A Appendix - Correctness proof for the algorithm

Lemma 3 *Proof* We first show that $MS(k) \supseteq \{(T, \mathcal{X}) : \exists$ sequence of choices in STAGE 1 that gives $(T, \mathcal{X})\}$. For T having 1, 2, 4 or 6 maxclique nodes this is clear by the rules for `1clique`, `2clique`, `4clique` and `6clique`. For 7 or more maxclique nodes we note that `rule I` and `rule II` does not add any hairs, while `rule III` adds a hair to the new maxclique node and `rule IV` adds a hair to the old minsep node. This means that conditions 1 and 4 of Definition 6 hold while conditions 2 and 3 hold by the `OK1` test on the `Start` and `End` additions. Condition 5 of Definition 6 holds since whenever the two last consecutive maxclique nodes of the body both have a hair then `HasHair` and `NeedsPair` are both True and none of `rule II, III, IV` or `End` addition could be applied, but only `rule I`. We have established that any T produced by the first stage of the algorithm is a special caterpillar and it remains to show that the 6 conditions of Theorem 3 hold for T. Conditions 1 and 2 hold by the `OK1` test of the `Start` and `End` additions. Condition 3 is enforced but as mentioned we did not include it in this extended abstract not to clutter the Algorithm. Condition 4 holds since only `Rule III` adds such a hair and the condition is explicitly mentioned in the rule. Finally, if the last maxclique node of the body has a hair and part i) of condition 5 fails, then `HasPair` and `NeedsPair` are both True and only `rule I` can be applied, which enforces part ii) of condition 5. Condition 6 holds since `rule II` does enforce $S_i \setminus (S_{i-1} \cup S_{i+1}) \neq \emptyset$ while `rule III, IV` enforce addition of a hair, and `rule I` is applied only when `HasHair` is already True.

We next show that $MS(k) \subseteq \{(T, \mathcal{X}) : \exists$ sequence of choices in STAGE 1 that gives $(T, \mathcal{X})\}$: For T having 1, 2, 4 or 6 maxclique nodes this is clear by the rules for `1clique`, `2clique`, `4clique` and `6clique`. For 7 or more maxclique nodes we need to show that any $(T, \mathcal{X}) \in MS(k)$ could have been generated by STAGE 1 of the Algorithm. First note that the `OK1` tests at the `Start` and `End` additions will allow either of the inequalities of condition 1 and condition 2 of Theorem 3 to hold. Thus we know that the ends of the special caterpillar T can be generated correctly. For the rest of T note that the boolean values `HasHair` and `NeedsPair` ensure that the Rules that can be applied will allow any $(T, \mathcal{X})$ satisfying conditions 3, 4, 5 and 6 of Theorem 3.

Thus we know that at the start of STAGE 2 we have any $(T, \mathcal{X})$ in $MS(k)$. To prove the lemma it thus suffices to note that STAGE 2 enforces exactly the conditions imposed on extended minimal k-skeletons in $EMS(k)$ as given by Definition 7. □

Theorem 4 *Proof* $\supseteq$: By induction on the number of iterations of the repeat loop in the STAGE 3. For the base case, by Lemma 3 we know that the second stage gives $(T, \mathcal{X}) \in EMS(k)$ which by definition is a member of $ES(k)$. For the inductive case, each of the 7 addition rules preserves membership in $ES(k)$, since we can easily check that the condition for having a mergeable subtree given in Definition 3 will never be met.

$\subseteq$: Since $(T, \mathcal{X}) \in ES(k)$ we know that there exists at least one subtree T' of T with the induced tree-decomposition $(T', \mathcal{X}') \in MS(k)$. The proof will be by structural induction on the tree T. *Base case:* If the subtree T' mentioned above is the subtree of T that we get by removing all minsep leaves from T then we have $(T, \mathcal{X}) \in EMS(k)$ and are done since by Lemma 3 there is a sequence of choices such that the STAGE 2 gives T.

Inductive case: Assuming the base case does not hold we show that T contains a smaller subtree T' with the induced tree-decomposition $(T', \mathcal{X}') \in ES(k)$ and such that application of some rule 1-7 of the Algorithm to the tree T' would give the tree T. We call a maxclique node *pendant* if it has at most one neighbor that is not a leaf. We call a pendant node *prunable* if its minsep parent itself has at most one non-pendant neighbor. We call a prunable node *good* if for the subtree T' resulting from removing it and any of its leaf neighbors we have the induced tree-decomposition in $ES(k)$, meaning that in particular T' itself contains a subtree whose induced tree-decomposition is a minimal k-skeleton. In the inductive case we will consider a good prunable maxclique node X with parent A, which exists by the Claim below.

Claim: A tree T with $(T, \mathcal{X}) \in ES(k)$ contains a good prunable node X iff $(T, \mathcal{X}) \notin EMS(k)$.
Proof: Consider the subtree of T where we have removed all minsep leaves and then removed all maxclique leaves. Any minsep leaf in this subtree is the parent of a prunable node. If none of these prunable nodes are good in T, then the subtree of T that we get after removing all minsep leaves must induce a tree-decomposition which is a minimal k-skeleton and thus $(T, \mathcal{X}) \in EMS(k)$. On the other hand, if one of these prunable nodes are good, then by definition $(T, \mathcal{X}) \notin EMS(k)$. ♢

Let us first assume that A does not have any non-pendant neighbors. Then the intersection of any two maximal cliques in T is equal to A. Let T' be the subtree resulting from removing X and any of its leaf neighbors from T. By induction T' could be generated by a sequence of choices of the algorithm. If X has no leaf neighbors we apply rule 7 $\texttt{ADD}(A, X)$ to T' to get T and if X has some leaf neighbors $B, ...$ we apply rule 6 $\texttt{ADD}(A, X, B, ...)$ to T' to get T. Note that the conditions allowing application of these rules in the Algorithm must be True, otherwise $(T, \mathcal{X})$ would not be an extended k-skeleton.

For the remaining cases, we have the good prunable maxclique node X with parent A and with the unique non-pendant neighbor of A being called W. The possible arrangements of these nodes can be described by 3 numbers (x, y, z) where

- $x \in \{1, 2\}$ describes the number of neighbors W has apart from A,
- $y \in \{1, 2, 3\}$ describes the number of pendant maxclique neighbors A has, with 3 denoting any number larger than 2
- $z \in \{0, 1, 2\}$ describes the number of children leaves X has

The rest of the proof considers in turn each of these cases, showing that one of rules 1-7 could have been applied to a subtree T' of T with $(T', \mathcal{X}') \in ES(k)$, to add the prunable maxclique node X and possibly some of the other nodes as well, to yield the tree $(T, \mathcal{X})$. Note that rules 3 and 4 actually add 2 new maxclique nodes as neighbors of the minsep A, whereas the other rules add only a single neighbor.

Let us start with a full argument for the case $(x = 1, y = 1, z = 1)$. We then have in the tree T a minsep leaf B with its parent X being a pendant maxclique node with a path $B - X - A - W - S$ in the tree T such that nodes X, A, W all have degree 2. Consider the tree $T' = T \setminus \{B, X\}$ and note that the assumption $(T, \mathcal{X}) \in ES(k)$ implies $(T', \mathcal{X}') \in ES(k)$ since in particular no cliques could be merged without increasing branchwidth in T' because then they could also be merged in T. The crucial point here is that T' contains the *promise* leaf A and any clique that is merged with W in T' would still need to have a k-troika respecting also A. By the induction hypothesis there is a derivation sequence giving $(T', \mathcal{X}')$ and we argue that this derivation sequence followed by application of rule 2 with parameters $\texttt{ADD}(A, X, B)$ will yield $(T, \mathcal{X})$. The fact that applying this rule to $(T', \mathcal{X}')$ would yield $(T, \mathcal{X})$ is obvious so all we need to check is that the conditions of the Algorithm allow application of rule 2. Rule 2 is prefaced by the condition *'If A a leaf with parent W having a single other neighbor S choose 1,2,3,4 or 5'*, and we first note that minsep leaf A in the tree T' does indeed satisfy this condition. Rule 2 has the further condition *'if $\texttt{OK}(X, A, B)$ and $|W \cup X| + |B \cap S| > 2k$'* that holds for the following reason: Since T is a k-skeleton we must have $\texttt{OK}(X, A, B)$ since otherwise X could not have a k-troika respecting A, B, while condition $|W \cup X| + |B \cap S| > 2k$ must be True since otherwise we could have merged the two cliques X and W in T without increasing branchwidth contradicting $(T, \mathcal{X}) \in ES(k)$.

We now argue for the remaining cases. By inspecting the Algorithm, we note that rules 1-5 are used only if A is a leaf and $x = 1$. Thus when $y = 1$, meaning that A has in T a single maxclique pendant neighbor X, we take the subtree resulting from removing X and its leaf neighbors and apply to it rule 1 if $z = 0$, rule 2 if $z = 1$ and rule 5 if $z = 2$. In each case this will add X to A. This covers all cases of $x = 1, y = 1$. If $x = 1, y = 2$ then the minsep node A has two pendant maxclique neighbors X_1, X_2. Let X_i have σ_i minsep leaf neighbors and assume that $\sigma_1 \geq \sigma_2$. We let $z = \sigma_1$ and apply rule 3 in case $z = 0$ and we apply rule 4 in case $z \in \{1, 2\}$, adding both maxcliques X_1 and X_2. Note that no minimal k-skeleton has a minsep node A of degree 3 with two leaves. Thus, both X_1 and X_2 are good prunable nodes. Moreover, since no minimal k-skeleton has a minsep node A of degree 2 with one neighbor being X a leaf and the other neighbor W having degree 2, then in fact we can remove both X_1 and X_2 and still be guaranteed to have a subtree T' of T with $(T', \mathcal{X}') \in ES(k)$. We have thus argued all cases of $x = 1, y \in \{1, 2\}, z \in \{0, 1, 2\}$.

It remains to argue for $x = 2$ and also all cases where $y = 3$. For all these cases we use rule 7 if $z = 0$ and rule 6 otherwise, adding a single maxclique neighbor X to minsep A. Consider the subtree T' of T resulting from removing X and its leaf neighbors. We note that the 'Else' pre-condition for rules 6 and 7 hold, that the pre-condition $\texttt{OK}(New, A, B, ...)$ for rule 6 holds, and that the pre-condition $|Y \cup X| > \lfloor 3k/2 \rfloor$ for rule 7 holds since if X had no leaf neighbor and Y was another maxclique leaf neighbor of A then we would have $|Y \cup X| > \lfloor 3k/2 \rfloor$, as otherwise $(T, \mathcal{X})$ would not be a k-skeleton. We need to argue that $(T', \mathcal{X}') \in ES(k)$, in particular that W could not in T' be merged with some maxclique Y into a larger new clique without increasing branchwidth (the reason we cannot merge W with two other maxcliques Y_1, Y_2 in T' is because then Y_1, Y_2 could have been merged already in T). In all cases the argument is that in T' the new clique would have two minsep neighbors S_1, S_2 and the newly merged clique would have to respect both S_1 and S_2 which is not possible as $S_1 \cup S_2 \neq W \cup Y$. In particular, in case $x = 2$ we take for S_1, S_2 the 2 minsep neighbors of W different from A and in case $x = 1, y = 3$ we can use $S_1 = A$ since the minsep A would have remained a minimal separator also in T'. □