Chordal digraphs*

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Abstract

We re-consider perfect elimination digraphs, that were introduced by Haskins and Rose in 1973, and view these graphs as directed analogues of chordal graphs. Several structural properties of chordal graphs that are crucial for algorithmic applications carry over to the directed setting, including notions like simplicial vertices, perfect elimination orderings, and vertex layouts. We show that semi-complete perfect elimination digraphs are also characterised by a set of forbidden induced subgraphs resemblant of chordless cycles. Moreover, just as the chordal graphs are related to treewidth, the perfect elimination digraphs are related to Kelly-width.

1 Introduction

In a paper from 2008, Hunter and Kreutzer [10] generalised a graph searching game, in which a robber attempts to avoid capture by a number of cops, to a searching game on directed graphs. The numbers of cops needed to guarantee capture in these games correspond to the treewidth of the graph and the Kelly-width of the digraph[1]. Just as the graphs of treewidth at most k are the partial k-trees, the digraphs of Kelly-width at most k are the partial k-DAGs. If in the iterative construction of k-trees any value of k is allowed then we construct the chordal graphs. In this paper, we study the class of digraphs constructed by likewise allowing any value of k in the iterative construction of k-DAGs. These digraphs can be seen as a generalisation of chordal graphs to digraphs, and are related to Kelly-width in the same way that chordal graphs are related to treewidth.

Chordal graphs have many applications in algorithmic graph theory and also in practical computing. Chordal graphs appear in a paper by Rose from 1970 [14] studying Gaussian elimination on sparse systems of linear equations, but then under the name of perfect elimination graphs. This study was generalised in a 1973 paper by Haskins and Rose [9], to the case of a non-symmetric system, and gave rise to the definition of a perfect elimination digraph. The class of digraphs studied in our paper is precisely the class of perfect elimination digraphs, giving a historical precedence for viewing them as the directed analogue of chordal graphs.

We will see that perfect elimination digraphs share many properties with chordal graphs, e.g. having a perfect elimination ordering, being definable by the existence of a linear vertex layout, or by an iterative construction process. However, the chordal graphs get their name from being exactly those graphs where every cycle of length at least 4 has a chord. Our main aim in this paper is to investigate whether the perfect elimination digraphs share a similar property.

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[1]Ned Kelly was an infamous Australian bushranger adept at hiding from cops.

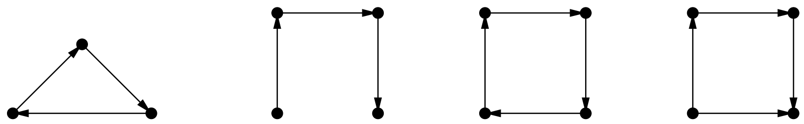

Figure 1: If G is a perfect elimination digraph then $\mathrm{uni}(G)$ does not contain a copy of any of the four depicted digraphs as an induced subgraph.

In Section 2 of the paper, we give formal definitions. Unlike the perfect elimination graphs (chordal graphs) the interest in perfect elimination digraphs seems to have lain dormant since the publication in 1978 of a paper by Rose and Tarjan [15]. In Section 3, we restate and reprove some of the implicit and explicit results on perfect elimination digraphs from the 1970s, bringing the terminology in line with more recent investigations of chordal graphs. We then focus on a characterisation of perfect elimination digraphs by forbidden induced subgraphs.

Let us describe our results informally. Partition the arcs of a digraph G to define two digraphs $\mathrm{uni}(G)$ and $\mathrm{bi}(G)$, with $\mathrm{uni}(G)$ containing the arcs (u, v) for which (v, u) is not an arc, and $\mathrm{bi}(G)$ containing the arcs (u, v) for which (v, u) is also an arc. Our first result states that if $\mathrm{bi}(G)$ is empty (equivalently, if G is an orientation) then G is a perfect elimination digraph if and only if G is acyclic (equivalently, contains no chordless directed cycle of length 3 or larger). On the other hand, if $\mathrm{uni}(G)$ is empty then G is a perfect elimination digraph if and only if G contains no chordless directed cycle of length 4 or larger. Note that G could be viewed as an undirected graph precisely when $\mathrm{uni}(G)$ is empty, and hence the latter result shows that indeed the perfect elimination digraphs are a generalisation of chordal graphs. We also show that if G is a perfect elimination digraph then $\mathrm{bi}(G)$ contains no chordless directed cycle of length 4 or larger. These results are in Section 4. An important and large class of digraphs are the *semi-complete* digraphs, where every pair of vertices has at least one arc. In Section 5, we show that a semi-complete digraph G is a perfect elimination digraph if and only if the underlying graph of $\mathrm{bi}(G)$ is chordal and $\mathrm{uni}(G)$ does not contain an induced subgraph isomorphic to a digraph in Figure 1.

In Section 6, we summarise our results and pose several open problems.

2 Preliminaries

We consider directed and undirected graphs. All graphs in this paper are simple and finite. In particular, our graphs have no loops. Directed graphs are called *digraphs*. A "graph" may be directed or undirected. Let F be a graph. A *vertex layout* for F is a linear ordering $\beta = \langle x_1, \ldots, x_n \rangle$ of the vertices of F. For an ordered vertex pair u, v of F, we write $u \prec_\beta v$ if $u = x_i$ and $v = x_j$ for some indices i, j with $1 \leq i, j \leq k$ and $i < j$.

Let G be a digraph. The vertex set of G is denoted as $V(G)$, and the arc set of G is denoted as $A(G)$. The arcs of G are denoted as (u, v). For a vertex pair u, v of G, if (u, v) or (v, u) is an arc of G then u and v are *adjacent* in G; otherwise, if neither (u, v) nor (v, u) is an arc of G, u and v are *non-adjacent* in G. If (u, v) is an arc of G then u is an *in-neighbour* of v and v is an *out-neighbour* of u in G. The *in-neighbourhood* of u in G, denoted as $N_G^{\mathrm{in}}(u)$, is the set of the in-neighbours of u in G, and the *out-neighbourhood* of u in G, denoted as $N_G^{\mathrm{out}}(u)$, is the set of the out-neighbours of u in G. For a set X of vertices of G, $G[X]$ denotes the *subgraph* of G *induced* by X, which is the digraph on vertex set X, and for every ordered vertex pair u, v of G, $(u, v) \in A(G[X])$ if and only if $u, v \in X$ and $(u, v) \in A(G)$. A digraph G' is an induced subgraph of G if there is $X \subseteq V(G)$ such that $G' = G[X]$. For a digraph G', we say that G *contains a copy of* G' *as an induced subgraph* if there is $X \subseteq V(G)$ such that G' and $G[X]$ are isomorphic, i.e., G' is obtained from $G[X]$ by renaming the vertices. For x a vertex of G, $G-x$ is the subgraph of G induced by $V(G) \setminus \{x\}$. For u and v not necessarily different vertices of G and k an integer with $k \geq 0$, a *directed* u,v*-path of* G *of length* k is a sequence $(x_0, \ldots, x_k)$ of pairwise different vertices of G where $x_0 = u$ and $x_k = v$ and $(x_i, x_{i+1}) \in A(G)$ for every $0 \leq i < k$. A *directed cycle of length* k of G is a sequence $C = (x_1, \ldots, x_k)$ of pairwise different vertices of G such that C is a directed x_1, x_k-path of G (of length $k-1$) and $(x_k, x_1) \in A(G)$. Note that a directed cycle can also be seen as a directed x, x-path for some vertex x of G. Since G has no loops, every directed cycle of G has length at least 2.

Let H be an undirected graph. The vertex set of H is denoted as $V(H)$, the edge set of H is denoted as $E(H)$, and the edges of H are denoted as $\{u, v\}$. For a vertex pair u, v of H, if $\{u, v\}$ is an edge of H then u and v are *adjacent* in H; otherwise, u and v are *non-adjacent* in H. For a vertex u of H, the *neighbourhood* of u in H, $N_H(u)$, is the set of the vertices of H that are adjacent to u in H, and $N_H[u] =_{\text{def}} N_H(u) \cup \{u\}$. A set X of vertices of H is a *clique* of H if the vertices from X are pairwise adjacent in H. H is called *complete* if every vertex pair of H is adjacent. Let k be an integer with $k \geq 3$. A *cycle of length* k of H is a sequence $C = (x_1, \ldots, x_k)$ of k pairwise different vertices of H such that $\{x_i, x_{i+1}\} \in E(H)$ for every $1 \leq i < k$ and $\{x_1, x_k\} \in E(H)$. An edge $\{u, v\}$ of H is a *chord* of C if $u = x_i$ and $v = x_j$ for some indices i, j with $1 \leq i, j \leq k$ and $1 < |j - i| < k - 1$. A cycle without chords is called *chordless.*

An undirected graph without chordless cycles of length at least 4 is called *chordal.* Chordal undirected graphs have a large number of different characterisations, such as by properties of minimal separators [4] or as intersection graphs [3, 6, 17]. Another characterisation is the following, through vertex layouts.

Theorem 2.1 ([14])**.** *An undirected graph* H *is chordal if and only if* H *has a vertex layout* β *such that for every vertex triple* u, v, w *of* H *with* $u \prec_\beta v \prec_\beta w$*, if* $\{v, u\}$ *and* $\{u, w\}$ *are edges of* H *then* $\{v, w\}$ *is an edge of* H*.*

A digraph is called *acyclic* if it contains no directed cycle. It particularly holds for every vertex pair u, v of an acyclic digraph G that (u, v) is not an arc of G or (v, u) is not an arc of G. The following characterisation is folklore: A digraph G is acyclic if and only if G has a vertex layout β such that $u \prec_\beta v$ for every arc (u, v) of G. Such a vertex layout for G is called a *topological ordering.* The *underlying graph* of a digraph G is the undirected graph H on vertex set $V(G)$ such that for every vertex pair u, v of G, $\{u, v\} \in E(H)$ if and only if u and v are adjacent in G.

3 Perfect elimination digraphs and simple results

Haskins and Rose introduced the class of perfect elimination digraphs as a directed analogue of undirected graphs having a perfect elimination scheme [9]. It is implicit in their work that perfect elimination digraphs can be equivalently defined through vertex layouts, and this was also mentioned by Kleitman [11]. We state the two definitions using terminology appropriate to our study and give a proof of their equivalence.

Definition 3.1. *Let* G *be a digraph with vertex layout* β*. We say that* β *is* directed transitive *if for every ordered vertex triple* u, v, w *of* G *with* $u \prec_\beta v$ *and* $u \prec_\beta w$*,* $(v, u) \in A(G)$ *and* $(u, w) \in A(G)$ *implies* $(v, w) \in A(G)$*.*

Note the similarity to the vertex layout characterising chordal graphs in Theorem 2.1.

The definition of chordal graphs in terms of perfect elimination schemes is intimately related to their alternative definition in terms of an inductive construction process, and it is easy to move from one definition to the other. So also for perfect elimination digraphs. We prefer to give the definition in terms of the inductive construction process. The definition is based on a digraph notion that resembles the notion of a clique in undirected graphs; we call this a *d-clique*.

Definition 3.2. *Let G be a digraph. Let A and B be sets of vertices of G, where $A \cap B$ may be non-empty. We call (A, B) a* d-clique *of G if for every ordered vertex pair a, b of G with $a \in A$ and $b \in B$ and $a \neq b$, (a, b) is an arc of G.*

We can say that (A, B) is a d-clique of G if G contains all arcs from A to B. Note that $A \cap B$ induces a complete digraph in G.

Definition 3.3. *The class of* perfect elimination digraphs *is inductively defined as follows:*

1) a digraph on a single vertex is a perfect elimination digraph

2) let G be a perfect elimination digraph, let u be a vertex that does not appear in G, let (A, B) be a d-clique of G; the digraph that is obtained from G by adding u and the arcs from the set $\{(a, u) : a \in A\} \cup \{(u, b) : b \in B\}$ is a perfect elimination digraph.

We can say that a perfect elimination digraph is built from a single vertex by repeatedly adding vertices and joining them to d-cliques. This construction process defines sequences of vertices. Let G be a perfect elimination digraph. If G is a digraph on a single vertex x then $\langle x \rangle$ is a *construction sequence* for G, and if G is obtained from a perfect elimination digraph G' by adding a new vertex x then $\langle x, y_1, \ldots, y_n \rangle$ is a *construction sequence* for G whenever $\langle y_1, \ldots, y_n \rangle$ is a construction sequence for G'. It is important to observe that every perfect elimination digraph has a construction sequence. Let us mention that the reversal of a construction sequence will correspond to a perfect elimination scheme, with the next vertex x to be eliminated from the digraph having the property that in the remaining graph the pair consisting of its set of out-neighbors A and its set of in-neighbors B form a d-clique. We return to this in Proposition 3.6. Let us first consider the alternative definition in terms of vertex layouts.

Proposition 3.4. *Let G be a digraph with vertex layout $\beta = \langle x_1, \ldots, x_n \rangle$. Then, G is a perfect elimination digraph with construction sequence β if and only if β is directed transitive.*

Proof. Let G have at least two vertices, let $G' =_{\text{def}} G - x_1$ and $\beta' =_{\text{def}} \langle x_2, \ldots, x_n \rangle$. Assume that G' is a perfect elimination digraph with construction sequence β' if and only if β' is directed transitive.

Assume that β' is directed transitive for G'. Let u, v, w be an ordered vertex triple of G with $u \prec_\beta v$ and $u \prec_\beta w$ and $(v, u) \in A(G)$ and $(u, w) \in A(G)$. If $x_1 \prec_\beta u$ then u, v, w are vertices of G', and $(v, w) \in A(G)$. If $x_1 = u$ then $(N_G^{\text{in}}(u), N_G^{\text{out}}(u))$ is a d-clique of G, and $(v, w) \in A(G)$. Thus, β is directed transitive for G.

Assume that G' is a perfect elimination digraph with construction sequence β'. Let $A =_{\text{def}} N_G^{\text{in}}(x_1)$ and $B =_{\text{def}} N_G^{\text{out}}(x_1)$. Note that $x_1 \prec_\beta y$ for every vertex y from $A \cup B$. The definition of directed transitive vertex layouts implies $(a, b) \in A(G)$ for every ordered vertex pair a, b of G with $a \in A$ and $b \in B$ and $a \neq b$, and so, (A, B) is a d-clique of G. Thus, G is a perfect elimination digraph with construction sequence β. ∎

Since every sublayout of a directed transitive vertex layout is also directed transitive, it is an easy corollary of Proposition 3.4 that the class of perfect elimination digraphs is closed under taking induced subgraphs. This also means that every digraph with an induced subgraph that is not a perfect elimination digraph cannot be a perfect elimination digraph itself. Therefore, perfect elimination digraphs admit a characterisation via forbidden induced subgraphs.

A digraph G is *weakly connected* if for every vertex pair u, v of G, there is a directed u, v-path or a directed v, u-path in G. The *weakly connected components* of a digraph are the maximal induced subgraphs that are weakly connected. Appending construction sequences of weakly connected perfect elimination digraphs yields a construction sequence for the disjoint union of the digraphs. Thus, a digraph is a perfect elimination digraph if and only if each of its weakly connected components is a perfect elimination digraph.

We consider another alternative characterisation of perfect elimination digraphs. Let G be a digraph and let x be a vertex of G. We say that x is *di-simplicial* in G if $(N_G^{\text{in}}(x), N_G^{\text{out}}(x))$ is a d-clique of G. Di-simplicial vertices admit a characterisation through directed paths. For G a digraph, x a vertex of G and $P = (x_0, \ldots, x_k)$ some directed path in G, $P{-}x$ is the vertex sequence that emerges from $(x_0, \ldots, x_k)$ by deleting vertex x, if it appears in P. Thus, if $x = x_i$ for some index x with $0 \le i \le k$ then $P{-}x = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$.

Lemma 3.5. *Let G be a digraph, and let x be a vertex of G. Then, x is di-simplicial in G if and only if for every directed path P of G of length at least 1, $P{-}x$ is a directed path of G.*

Proof. Assume that x is di-simplicial in G. Let $P = (x_0, \ldots, x_k)$ be a directed path of G of length at least 1. If x does not appear on P then $P = P{-}x$, and $P{-}x$ is a directed path of G. If $x = x_0$ or $x = x_k$ then $P{-}x$ is a directed path of G. So, assume that $x = x_i$ for some index i with $0 < i < k$. Since $x_{i-1} \in N_G^{\text{in}}(x)$ and $x_{i+1} \in N_G^{\text{out}}(x)$, it holds that $(x_{i-1}, x_{i+1}) \in A(G)$, and thus, $P{-}x$ is a directed path of G.

For the converse, assume that x is not di-simplicial. Then, there is a vertex pair a, b of G with $a \in N_G^{\text{in}}(x)$ and $b \in N_G^{\text{out}}(x)$ and $(a, b) \notin A(G)$. Thus, $P =_{\text{def}} (a, x, b)$ is a directed path of G of length at least 1 and $P{-}x$ is not a directed path of G. ∎

We use di-simplicial vertices to characterise perfect elimination digraphs. Note that the proof of Proposition 3.4 also shows that the first (leftmost) vertex of a construction sequence for a perfect elimination digraph is a di-simplicial vertex, which implies that every perfect elimination digraph has a di-simplicial vertex. The following result can be seen as a different formulation of the following fact: A digraph is a perfect elimination digraph if and only if each of its induced subgraphs has a di-simplicial vertex.

Proposition 3.6. *Let G be a digraph. Then, G is a perfect elimination digraph if and only if G can be reduced to a digraph on a single vertex by repeatedly deleting an arbitrary di-simplicial vertex.*

Proof. If G is a digraph on a single vertex then the statement easily holds. So, let G have at least two vertices. Assume that G is a perfect elimination digraph. Then, G has a di-simplicial vertex x, and $G{-}x$ can be reduced to a digraph on a single vertex by repeatedly deleting an arbitrary di-simplicial vertex.

For the converse, assume that G can be reduced to a digraph on a single vertex by repeatedly deleting an arbitrary di-simplicial vertex. If G has exactly one vertex then G is a perfect elimination digraph. Assume that G has at least two vertices, and let x be the vertex that is picked first. Then, x is a di-simplicial vertex of G, and since $G{-}x$ can be reduced to a digraph on a single vertex by repeatedly deleting an arbitrary di-simplicial vertex, $G{-}x$ is a perfect elimination digraph. Then, G is obtained from $G{-}x$ by adding x in the sense of Definition 3.3, and G is a perfect elimination digraph. ∎

Figure 2: The left side digraph is a perfect elimination digraph, that has exactly one di-simplicial vertex. The right side digraph has two strongly connected components on two vertices each, that are perfect elimination digraphs, but the whole digraph is not a perfect elimination digraph, particularly since it has no di-simplicial vertex.

To conclude this section, we review the results of this section and relate them to known results for undirected graphs. Perfect elimination digraphs have strong similarities to chordal undirected graphs. A vertex of an undirected graph is *simplicial* if its neighbourhood is a clique. It can be verified easily that simplicial vertices of undirected graphs admit a characterisation that is analogous to the one of Lemma 3.5. This particularly means that the restriction of being di-simplicial to undirected graphs coincides with the notion of being simplicial, and this shows a correspondence between perfect elimination digraphs and chordal undirected graphs. Nevertheless, perfect elimination digraphs are more complex in structure than chordal undirected graphs and therefore do not have so strong properties as chordal undirected graphs. For example, every chordal undirected graph that is not complete has a pair of non-adjacent simplicial vertices [4]. A similar result does not hold for perfect elimination digraphs. The left side digraph of Figure 2 is a perfect elimination digraph on four vertices, that has non-adjacent vertices, and the digraph has exactly one di-simplicial vertex, namely the bottom vertex.

Let G be a digraph on n vertices and m arcs. It can be tested in $\mathcal{O}(n+m)$ time whether a specific vertex is di-simplicial. Thus, a di-simplicial vertex of a digraph can be found in $\mathcal{O}(nm)$ time, or it can be output that no such vertex exists. The result of Proposition 3.6 therefore implies an easy $\mathcal{O}(n^2m)$-time algorithm for recognising perfect elimination digraphs. Rose and Tarjan gave a more efficient implementation of the recognition algorithm, that has running time $\mathcal{O}(nm)$ [15]. The main idea is to keep a list of missing arcs, that separate a vertex from being di-simplicial, and to update this list with every deleted vertex. Rose and Tarjan also asked the question of how good the running time is and whether it can be improved. They showed that verifying whether an acyclic digraph is transitive is linear-time reducible to recognising perfect elimination digraphs [15], thus, recognising perfect elimination digraphs is bounded from below by Boolean matrix multiplication.

We have learnt that a digraph is a perfect elimination digraph if and only if all its weakly connected components are perfect elimination digraphs. Can this characterisation be strengthened to hold for strongly connected components? This is not the case, since there are digraphs that are not perfect elimination digraphs but each of their strongly connected components is a perfect elimination digraph, even if we require the strongly connected components to have more than one vertex. Such an example is the right side digraph of Figure 2.

4 Two classes of perfect elimination digraphs

We aim at a characterisation of the perfect elimination digraphs through forbidden induced subgraphs. We will not give such a characterisation for the whole class of perfect elimination digraphs. But, we will give characterisations for large subclasses of perfect elimination digraphs. In this section, we consider two classes of digraphs, that are defined by the number of arcs between vertex pairs.

A pair of adjacent vertices of a digraph can be connected by exactly one arc or by two arcs. In this section, we consider digraphs for which either all pairs of adjacent vertices fall into the one class or all pairs of adjacent vertices fall into the other class. Let G be a digraph. By $\mathrm{uni}(G)$, we denote the digraph on vertex set $V(G)$ such that for every ordered vertex pair u, v of G, (u, v) is an arc of $\mathrm{uni}(G)$ if and only if $(u, v) \in A(G)$ and $(v, u) \notin A(G)$. We can say that $\mathrm{uni}(G)$ is the restriction of G to the arcs that connect two vertices in a unique way. We call $\mathrm{uni}(G)$ the uni-*restriction* of G. If $\mathrm{uni}(G) = G$ then G is an orientation of an undirected graph.

Proposition 4.1. *Let G be a digraph with $\mathrm{uni}(G) = G$. Then, G is a perfect elimination digraph if and only if G is acyclic.*

Proof. If G is acyclic then G has a vertex layout that is a topological ordering. Topological orderings are directed transitive vertex layouts, and thus, G is a perfect elimination digraph due to Proposition 3.4.

For the converse, assume that G is a perfect elimination digraph. Due to Proposition 3.4, G has a directed transitive vertex layout β. For a contradiction, suppose that G is not acyclic. Then, there is a smallest integer k with $k \geq 3$ such that G has a directed cycle $(x_1, \ldots, x_k)$ of length k. Note that G cannot have a directed cycle of length at most 2 due to our assumptions about loops and the restriction of G to $\mathrm{uni}(G) = G$. Without loss of generality, we may assume that $x_1 \prec_\beta x_2$ and $x_1 \prec_\beta x_k$. Then, $(x_k, x_1) \in A(G)$ and $(x_1, x_2) \in A(G)$, so that $(x_k, x_2) \in A(G)$, and $(x_2, \ldots, x_k)$ is a directed cycle of G of length $k - 1$, contradicting the choice of k as smallest possible integer. Thus, G is acyclic. ∎

Proposition 4.1 completely characterises the perfect elimination digraphs that are orientations of undirected graphs. These are exactly the acyclic digraphs. Let G be a digraph, and assume that G has a directed cycle $C = (x_1, \ldots, x_k)$. An arc (u, v) of G is a *chord* of C in G if $u = x_i$ and $v = x_j$ for some i, j with $1 \leq i, j \leq k$ and $1 < |i - j| < k$. Note that, like for undirected graphs, a chord of a directed cycle C shortcuts the cycle and makes a subpath of C into a directed cycle. However, unlike for undirected graphs, a chord of C can make exactly one subpath of C into a directed cycle. A directed cycle is *chordless* if it has no chord.

The acyclic digraphs are the digraphs without directed cycles. Since directed cycles of arbitrary length are forbidden, we can strengthen: The acyclic digraphs are the digraphs without chordless directed cycles. In case of digraphs that are orientations of undirected graphs, we can strengthen even further: A digraph G with $\mathrm{uni}(G) = G$ is a perfect elimination digraph if and only if G has no chordless directed cycle of length at least 3. Since every undirected graph has an acyclic orientation, every undirected graph is the underlying graph of some perfect elimination digraph, so that the underlying graph of a digraph does not provide a necessary or a sufficient condition on whether a digraph is a perfect elimination digraph.

As the second class of digraphs that we consider in this section, we study the perfect elimination digraphs for which all pairs of adjacent vertices are connected by two arcs. Let G be a digraph. By $\mathrm{bi}(G)$, we denote the digraph on vertex set $V(G)$ such that for every ordered vertex pair u, v of G, (u, v) is an arc of G if and only if $(u, v) \in A(G)$ and $(v, u) \in A(G)$. We call $\mathrm{bi}(G)$ the bi-*restriction* of G. Observe for every digraph G that an arc of G is an arc of either $\mathrm{uni}(G)$ or $\mathrm{bi}(G)$. In particular, $A(G) = A(\mathrm{uni}(G)) \cup A(\mathrm{bi}(G))$. If $\mathrm{bi}(G) = G$ then G is obtained from an undirected graph by replacing every edge by the two possible arcs. Unlike for $\mathrm{uni}(G)$, the structure of $\mathrm{bi}(G)$ provides a necessary condition for a digraph G to be a perfect elimination digraph.

Lemma 4.2. *Let G be a digraph. If G is a perfect elimination digraph then $\mathrm{bi}(G)$ contains no chordless directed cycle of length at least 4.*

Proof. Assume that G is a perfect elimination digraph. If $\text{bi}(G)$ contains no directed cycle of length at least 4 then the claim of the lemma holds. So, as the other case, assume that $\text{bi}(G)$ contains a directed cycle of length at least 4. Let β be a directed transitive vertex layout for G, that exists due to Proposition 3.4. Let k be an integer with $k \geq 4$, and let $C = (x_1, \ldots, x_k)$ be a directed cycle of $\text{bi}(G)$ of length k. Observe that $(x_1, x_k, x_{k-1}, \ldots, x_2)$ is also a directed cycle of G. Without loss of generality, we may assume that $x_1 \prec_\beta x_2$ and $x_1 \prec_\beta x_k$. Thus, $(x_2, x_1), (x_k, x_1) \in A(G)$ and $(x_1, x_2), (x_1, x_k) \in A(G)$ and $(x_2, x_k), (x_k, x_2) \in A(G)$. This means that x_2 and x_k are adjacent in $\text{bi}(G)$, and C has a chord in G. It follows that no directed cycle of $\text{bi}(G)$ of length at least 4 is chordless. ∎

Lemma 4.2 shows that the underlying graph of $\text{bi}(G)$ for G a perfect elimination digraph must be chordal. Or, as a negative condition, if the underlying graph of $\text{bi}(G)$ is not chordal then G cannot be a perfect elimination digraph. The converse is true for a class of digraphs.

Proposition 4.3. *Let G be a digraph with $\text{bi}(G) = G$. Then, G is a perfect elimination digraph if and only if G contains no chordless directed cycle of length at least 4.*

Proof. If G is a perfect elimination digraph then the claim follows from the assumption that $\text{bi}(G) = G$ and from Lemma 4.2.

For the converse, assume that G contains no chordless directed cycle of length at least 4. Then, the underlying graph H of G is chordal. Due to Theorem 2.1, H has a vertex layout β such that for every vertex triple u, v, w of H with $u \prec_\beta v \prec_\beta w$, $\{v, u\} \in E(H)$ and $\{u, w\} \in E(H)$ implies $\{v, w\} \in E(H)$. We show that β is a directed transitive vertex layout for G. Let u, v, w be an ordered vertex triple of G with $u \prec_\beta v$ and $u \prec_\beta w$ and $(v, u), (u, w) \in A(G)$. Then, $\{v, u\}, \{u, w\} \in E(H)$, and thus $\{v, w\} \in E(H)$, which particularly means that $(v, w) \in A(G)$. Applying Proposition 3.4, we conclude that G is a perfect elimination digraph. ∎

As a corollary of Proposition 4.3, we conclude that a digraph G with $\text{bi}(G) = G$ is a perfect elimination digraph if and only if the underlying graph of G is chordal. It is important to note here that every chordal undirected graph is the underlying graph of some digraph G with $\text{bi}(G) = G$.

In this section, we have studied two classes of perfect elimination digraphs and given characterisations by forbidden induced subgraphs. The perfect elimination digraphs that are equal to their uni-restrictions are characterised by the class of directed cycles as the set of forbidden induced subgraphs. However, each such directed cycle can be an induced subgraph of $\text{uni}(G)$ of some perfect elimination digraph G, by having a chord in $\text{bi}(G)$.

5 Semi-complete perfect elimination digraphs

In this section, we consider digraphs whose underlying undirected graph is complete and characterise the perfect elimination digraphs of this type by forbidden subgraphs. Before we concentrate on the main objective of this section, we present a simple and important closure property for perfect elimination digraphs. Let G be a digraph. The digraph $\text{rev}(G)$ is the digraph on vertex set $V(G)$, and for every ordered vertex pair u, v of G, (u, v) is an arc of $\text{rev}(G)$ if and only if (v, u) is an arc of G. We call $\text{rev}(G)$ the *reverse digraph* of G. Note that the reverse digraph of the reverse digraph of G is G itself, i.e., $\text{rev}(\text{rev}(G)) = G$. The following result is verified straightforward.

Lemma 5.1. *Let G be a digraph. Then, G is a perfect elimination digraph if and only if $\text{rev}(G)$ is a perfect elimination digraph.*

Lemma 5.1 shows that the class of perfect elimination digraphs is closed under taking reverse digraphs. Thus, the set of forbidden induced subgraphs for perfect elimiation digraphs is also closed under taking reverse digraphs. Therefore, studying classes of perfect elimination digraphs with respect to forbidden induced subgraphs is preferable for reverse digraph closed classes. The studied classes in Section 4 are closed under taking reverse digraphs.

In this section, we consider semi-complete perfect elimination digraphs. A digraph G is *semi-complete* if every vertex pair of G is adjacent. Equivalently said, the underlying graph of a semi-complete digraph is complete. Note that the vertex pairs of semi-complete digraphs may be connected by one or two arcs. We give a complete characterisation of semi-complete perfect elimination digraphs by forbidden induced subgraphs. We obtain this result by studying the vertices of $\mathrm{uni}(G)$ that are di-simplicial in $\mathrm{bi}(G)$. Our approach to the forbidden induced subgraphs characterisation is by giving a characterisation of semi-complete digraphs without di-simplicial vertices. Remember from Proposition 3.6 that every perfect elimination digraph has a di-simplicial vertex.

Let F be a semi-complete digraph and let u, v, w be an ordered vertex triple of F of pairwise different vertices. We call (u, v, w) a *witness triple for u in F* if one of the following three conditions is satisfied:

- (u, v, w) is a witness triple of the first type:
  u and v are non-adjacent in $\mathrm{uni}(F)$ and u and w are non-adjacent in $\mathrm{uni}(F)$ and (v, w) is an arc of $\mathrm{uni}(F)$

- (u, v, w) is a witness triple of the second type:
  u and w are non-adjacent in $\mathrm{uni}(F)$ and either (u, v) and (v, w) are arcs of $\mathrm{uni}(F)$ or (w, v) and (v, u) are arcs of $\mathrm{uni}(F)$

- (u, v, w) is a witness triple of the third type:
  (v, u), (u, w) and (w, v) are arcs of $\mathrm{uni}(F)$.

Witness triples witness that a vertex is not di-simplicial.

Lemma 5.2. *Let F be a semi-complete digraph, and let u be a vertex of F. Then, u is a di-simplicial vertex of F if and only if there is no witness triple for u in F.*

Proof. Assume that F has a witness triple (u, v, w) for u. We show that u is not di-simplicial in F. We distinguish between the three cases:

- *(u, v, w) is a witness triple of the first type*
  since v and w are adjacent to u in $\mathrm{bi}(F)$, it holds that $w \in N_F^{\mathrm{in}}(u)$ and $v \in N_F^{\mathrm{out}}(u)$, and since $(w, v) \notin A(F)$, it follows that $(N_F^{\mathrm{in}}(u), N_F^{\mathrm{out}}(u))$ is not a d-clique of F

- *(u, v, w) is a witness triple of the second type*
  since (w, u) is an arc of F, if (u, v) and (v, w) are arcs of $\mathrm{uni}(F)$ then (w, v) is not an arc of F, and $(N_F^{\mathrm{in}}(u), N_F^{\mathrm{out}}(u))$ is not a d-clique of F, and since (u, w) is an arc of F, if (w, v) and (v, u) are arcs of $\mathrm{uni}(F)$ then (v, w) is not an arc of F, and $(N_F^{\mathrm{in}}(u), N_F^{\mathrm{out}}(u))$ is not a d-clique of F

- *(u, v, w) is a witness triple of the third type*
  (v, u), (u, w) and (w, v) are arcs of $\mathrm{uni}(F)$, (v, w) is not an arc of F, and $(N_F^{\mathrm{in}}(u), N_F^{\mathrm{out}}(u))$ is not a d-clique of F.

In each of the three cases, $(N_F^{\text{in}}(u), N_F^{\text{out}}(u))$ is not a d-clique of F, and thus, u is not di-simplicial in F.

For the converse, let u not be di-simplicial in F. This means that $(N_F^{\text{in}}(u), N_F^{\text{out}}(u))$ is not a d-clique of F, so that there are vertices v, w with $v \in N_F^{\text{in}}(u)$ and $w \in N_F^{\text{out}}(u)$ and $v \neq w$ such that (v, w) is not an arc of F. Since v and w are adjacent in F, it follows that $(w, v) \in A(F)$. If v and w are adjacent to u in $\text{bi}(F)$ then (u, w, v) is a witness triple of the first type for u in F. If v and w are non-adjacent to u in $\text{bi}(F)$ then (u, v, w) is a witness triple of the third type for u in F. If v is adjacent to u in $\text{bi}(F)$ and w is adjacent to u in $\text{uni}(F)$ then (u, w, v) is a witness triple of the second type, and if w is adjacent to u in $\text{bi}(F)$ and v is adjacent to u in $\text{uni}(F)$ then (u, v, w) is a witness triple of the second type for u in F. Thus, in each of the possible cases, u has a witness triple in F. ∎

We are interested in minimal semi-complete digraphs without di-simplicial vertices. We show next that a di-simplicial vertex is a restricted type of simplicial vertex. This gives an easy but powerful tool for the main result in this section.

Lemma 5.3. *Let G be a digraph, and let u be a vertex of G. If u is di-simplicial in G then u is simplicial in the underlying graph of* $\text{bi}(G)$.

Proof. Let H be the underlying graph of $\text{bi}(G)$. If H is a complete undirected graph then every vertex of H is simplicial. Assume that H is not complete. Assume that u is not a simplicial vertex of H. Due to the definition of simplicial vertices, there are vertices v, w of H such that v and w are adjacent to u in H and v and w are non-adjacent in H. It follows that $(u, v), (v, u), (u, w), (w, u)$ are arcs of G and either (v, w) is an arc of G or (w, v) is an arc of G. This means that $v, w \in N_G^{\text{in}}(u)$ and $v, w \in N_G^{\text{out}}(u)$, and $(N_G^{\text{in}}(u), N_G^{\text{out}}(u))$ is not a d-clique of G, so that u is not di-simplicial in G. ∎

For the proof of our main result, we need two properties of chordal undirected graphs. We repeat some definitions for undirected graphs. Let H be an undirected graph. For a set X of vertices of H, $H \setminus X$ is the undirected graph on vertex set $V(H) \setminus X$, and a vertex pair u, v of $H \setminus X$ is adjacent in $H \setminus X$ if and only if u and v are adjacent in H. Let $k \geq 0$ be an integer, and let u, v be a vertex pair of H. A vertex sequence $(x_0, \ldots, x_k)$ of pairwise different vertices of H is a *u, v-path of H of length k* if $x_0 = u$ and $x_k = v$ and $\{x_i, x_{i+1}\} \in E(H)$ for every $0 \leq i < k$. H is called *connected* if H has a u, v-path for every vertex pair u, v of H. If there is a vertex pair u, v of H such that H has no u, v-path then H is called *disconnected*. A *connected component* of a disconnected undirected graph H is the connected undirected graph $H \setminus Y$ for some inclusion-minimal set $Y \subseteq V(H)$.

Let H be an undirected graph. A *maximal clique* of H is a clique of H that is not properly contained in another clique of H. A *clique tree* for H is an ordered pair $(T, \mathcal{B})$ where T is a tree and $\mathcal{B}$ is the set of the maximal cliques of H, and there is a 1-to-1 correspondence between the maximal cliques in $\mathcal{B}$ and the nodes of T such that for every ordered node triple a, b, c of T, let B_a, B_b, B_c denote the maximal cliques from $\mathcal{B}$ that correspond to respectively a, b, c, if b is a node on the a, c-path of T then $B_a \cap B_c \subseteq B_b$. An undirected graph is chordal if and only if it has a clique tree [3, 6, 17], also [2].

Lemma 5.4. *Let H be a chordal undirected graph.*

1) *Let x and y be vertices of H, and assume that x and y are adjacent and simplicial in H. Then, $N_H[x] = N_H[y]$.*

2) *Let x be a vertex of H, and assume that $N_H[x] \subset V(H)$. Every connected component of $H \setminus N_H[x]$ contains a vertex that is simplicial in H.*

Proof. We prove the first statement. Since x is simplicial in H and x and y are adjacent, it holds that $N_H[x] \subseteq N_H[y]$, and since y is simplicial in H and x and y are adjacent, it holds that $N_H[y] \subseteq N_H[x]$.

We prove the second statement. Let $(T, \mathcal{B})$ be a clique tree for H. For every node v of T, denote by B_v the maximal clique from $\mathcal{B}$ that corresponds to v. Let R be a node of T with $x \in B_R$; since $\{x\}$ is a clique of H, such a node must exist. Since B_R is a clique of H, it directly follows that $B_R \subseteq N_H[x]$. We assign a rank to every node of T: R has rank 0, and for every node u of T, the rank of u is the smallest integer k such that T has an u, R-path of length k.

Let C be a connected component of $H \setminus N_H[x]$. Let a be a node of T of highest rank such that B_a contains a vertex from C. Observe that $a \neq R$. Let b be the node of T of lowest rank such that a and b are adjacent in T. Since T is a tree, b is uniquely defined. Since B_a and B_b are different maximal cliques of H, it holds that $B_a \setminus B_b$ is non-empty. Since B_a contains a vertex from C, every vertex in B_a is adjacent to some vertex of C in H. So, every vertex from B_a that is not contained in $N_H[x]$ is a vertex of C. If $x \in B_a$ then every vertex from $B_a \setminus \{x\}$ is adjacent to x in H, i.e., $B_a \subseteq N_H[x]$, contradicting the choice of a. Thus, $x \notin B_a$, and no vertex from $B_a \setminus B_b$ is in $N_H[x]$. Note that this follows from the choice of R as being a node of T whose corresponding maximal clique contains x. Due to the choice of a, B_a contains a vertex of C, and thus, all vertices in $B_a \setminus B_b$ are vertices of C.

Let u be a vertex from $B_a \setminus B_b$. Suppose for a contradiction that there is a node c of T with $c \neq a$ and $c \neq b$ and $u \in B_c$. The definition of clique trees shows that we can choose c as being adjacent to a in T. It follows that c must have larger rank than a, and B_c contains a vertex of C, namely u. This contradicts the choice of a. Thus, B_a is the unique maximal clique of H that contains u, i.e., $N_H[u] = B_a$. This means that u is a simplicial vertex of H. ∎

We are ready to prove the main result of this section.

Lemma 5.5. *Let F be a semi-complete digraph. Assume that the underlying graph of* $\text{bi}(F)$ *is chordal. Assume that F is not a perfect elimination digraph. Then,* $\text{uni}(F)$ *contains a copy of one of the four digraphs depicted in Figure 1 as an induced subgraph.*

Proof. For the proof, we distinguish between two main cases. As the first main case, we will assume that F contains no di-simplicial vertex, and as the second main case, we will assume that F contains a di-simplicial vertex.

1) F contains no di-simplicial vertex

Let $G =_{\text{def}} \text{uni}(F)$ be the uni-restriction of F, and let H be the underlying graph of the bi-restriction $\text{bi}(F)$ of F. Remember that F, G, H are graphs on the same vertex set. Due to the assumptions of the lemma, H is a chordal undirected graph. Lemma 5.3 shows that every vertex of H that is not simplicial in H is not di-simplicial in F, and the candidates for di-simplicial vertices of F are the simplicial vertices of H. Since H is chordal, H has simplicial vertices, and we analyse how none of these vertices is di-simplicial in F. Let S be the set of the vertices of F that are simplicial in H.

Claim A: Let u be a vertex from S. Then, u has only witness triples of the second or third type in F.

Proof. Since u is not di-simplicial in F due to our assumptions, u has a witness triple (u, v, w) in F due to Lemma 5.2. Suppose for a contradiction that (u, v, w) is a witness triple of the first type for u in F. Since v and w are not adjacent to u in G, it follows that v and w are adjacent to u in $\text{bi}(F)$ and thus in H. Since u is a simplicial vertex of H, v and w are adjacent in H and therefore are non-adjacent in G, a contradiction. □

If there is a vertex from S that has a witness triple of the third type in F then G contains a copy of the first digraph (digraph (a)) of Figure 1 as an induced subgraph. We henceforth assume that the witness triples for the vertices from S in F are all of the second type.

We construct an auxiliary digraph D as follows:

V) D has vertex set S, the set of the simplicial vertices of H

A) for every ordered vertex pair u, v of D, (u, v) is an arc of D if and only if there is a vertex w of F such that (u, v, w) is a witness triple for u in F.

Since (u, v, w) must be a witness triple of the second type, it particularly holds that u and v are adjacent in G, so that u and v are adjacent in D and in G. Note, however, that the arcs between u and v in D and G may be different. In particular, (u, v) and (v, u) may be arcs of D, due to different witness triples, while u and v are connected in G by exactly one of the two possible arcs.

Claim B: Let u be a vertex of D. Then, u has an out-neighbour in D, or u is contained in a copy of digraph (d) of Figure 1 in G.

Proof. Let (u, y, z) be a witness triple for u in F, that exists. Due to our assumptions, (u, y, z) is a witness triple of the second type. In particular, u and z are non-adjacent in G and y is adjacent to u and z in G. Thus, u and z are adjacent in H and y is non-adjacent to u and z in H. It follows that $N_H(u) \cup N_H(z) = N_H[u] \cup N_H[z]$ and $y \notin N_H[u] \cup N_H[z]$. Thus, y is a vertex of $H \setminus (N_H[u] \cup N_H[z])$. Since u is a simplicial vertex of H and u and z are adjacent in H, it holds that $N_H[u] \subseteq N_H[z]$. Thus, $H \setminus (N_H[u] \cup N_H[z]) = H \setminus N_H[z]$.

Let K be the connected component of $H \setminus N_H[z]$ that contains y. Since K contains only vertices of H that are non-adjacent to u and z in H, every vertex of K is adjacent to u and z in G. Due to Lemma 5.4, K contains a vertex v from S.

We want to show that (u, x, z) is a witness triple for u in F for every vertex x of K, in particular for v. We show the result inductively. Remember that y is a vertex of K and (u, y, z) is a witness triple for u in F. We mark y. Assume that vertices of K have been marked, which means that they constitute a desired witness triple for u. Assume that there is a still unmarked vertex x. Since K contains marked vertices, we can choose x such that K has a marked vertex x' that is adjacent to x in K. Since x and x' are adjacent in K, and therefore, x and x' are adjacent in H, it follows that x and x' are non-adjacent in G. We consider (u, x, z). If (u, x, z) is a witness triple for u in F then we mark x and hereby extend the set of the marked vertices. As the other case, assume that (u, x, z) is not a witness triple for u in F. Since u and z are adjacent to x in G and (u, x, z) is not a witness triple for u in F, one of the two cases must apply: either $(u, x), (z, x) \in A(G)$ or $(x, u), (x, z) \in A(G)$. Since (u, x', z) is a witness triple of the second type for u in F, one of the four situations as depicted in Figure 3 must appear in G. Thus, $\{u, z, x, x'\}$ induces a copy of the fourth digraph (digraph (d)) of Figure 1 in G. □

If there is a vertex of D that has no out-neighbour in D then G contains a copy of a digraph of Figure 1 as an induced subgraph. We henceforth assume that every vertex of D has an out-neighbour in D, which means that D has a directed cycle.

Let $k \geq 2$ be the smallest integer such that D has a directed cycle of length k. Let $C = (u_1, \ldots, u_k)$ be a directed cycle of D of length k. We distinguish between cases about the value of k. As the first case, assume that $k = 2$. This means that $C = (u_1, u_2)$, and $(u_1, u_2), (u_2, u_1) \in A(D)$. Remember that u_1 and u_2 are adjacent in G. Due to the symmetry of u_1 and u_2, we can assume without loss of generality that $(u_1, u_2) \in A(G)$. Due to the definition of the arcs of D, there are vertices a and b of F such that (u_1, u_2, a) is a witness triple for u_1 in F and

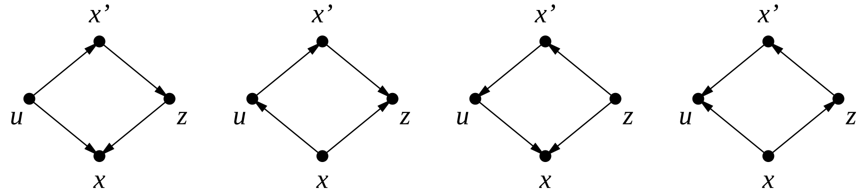

Figure 3: The construction of the digraph D in the proof of Lemma 5.5 yields a digraph with a cycle, or uni(F) contains a copy of one the four depicted digraphs as an induced subgraph.

(u_2, u_1, b) is a witness triple for u_2 in F. Since (u_1, u_2, a) and (u_2, u_1, b) are witness triples of the second type, u_1 and a are non-adjacent in G and u_2 and b are non-adjacent in G. Since u_2 and a are adjacent in G and u_1 and b are adjacent in G, a and b are different vertices. If a and b are non-adjacent in G then $\{u_1, u_2, a, b\}$ induces a copy of the second digraph (digraph (b)) of Figure 1 in G, and if a and b are adjacent in G then $\{u_1, u_2, a, b\}$ induces a copy of the third or fourth digraph (digraph (c) or (d)) of Figure 1 in G. This completes the proof for $k = 2$. We henceforth assume that $k \geq 3$.

As an intermediate case, assume that $(u_1, \dots, u_k)$ is a directed cycle of G or that $(u_k, \dots, u_1)$ is a directed cycle of G. Since H is a chordal undirected graph, the underlying graph of G has no chordless cycle of length more than 4. Thus, G contains a directed cycle of length 3 or 4, which means that G contains a copy of the first or the third digraph (digraph (a) or (c)) of Figure 1 as an induced subgraph. As the other case, assume that $(u_1, \dots, u_k)$ and $(u_k, \dots, u_1)$ are not directed cycles of G.

Claim C: No witness triple has all three vertices on C.

Proof. Suppose for a contradiction that there is a witness triple that has its three vertices on C. This means there are indices i, i', i'' with $1 \leq i, i', i'' \leq k$ such that $(u_i, u_{i'}, u_{i''})$ is a witness triple for u_i in F. Since $(u_i, u_{i'}, u_{i''})$ is a witness triple of the second type, it follows that $(u_{i''}, u_{i'}, u_i)$ is a witness triple for $u_{i''}$ in F. Due to the symmetry of the two witness triples, we can assume without loss of generality that $i < i''$. Since $u_i, u_{i'}, u_{i''}$ are vertices from S, it follows that $(u_i, u_{i'})$ and $(u_{i''}, u_{i'})$ are arcs of D. Remember that C is a directed cycle of D of shortest length, and the length is larger than 2. Thus, $(u_{i'}, u_i)$ and $(u_{i'}, u_{i''})$ are not arcs of D, and neither u_i nor $u_{i''}$ is an out-neighbour of $u_{i'}$ in D. Thus, $u_i, u_{i'}, u_{i'+1}, u_{i''}$ are pairwise different vertices of D, and one of the three cases applies: (1) $i' < i' + 1 < i < i''$, or (2) $i < i' < i' + 1 < i''$, or (3) $i < i'' < i' < i' + 1$. Then, D has a directed cycle of length at most $k - 1$ due to: (1) $(u_i, u_{i'})$, and (2) $(u_{i''}, u_{i'})$, and (3) $(u_i, u_{i'})$, a contradiction to the choice of k in each case. □

Without loss of generality, we can assume that (u_1, u_2) and (u_1, u_k) are arcs of G.

Claim D: $k = 3$

Proof. Suppose for a contradiction that $k \geq 4$. We consider the vertex triple u_1, u_2, u_k of D. Let a, b be vertices of F such that (u_1, u_2, b) is a witness triple for u_1 in F and (u_k, u_1, a) is a witness triple for u_k in F. With our assumptions about (u_1, u_2) and (u_1, u_k), it follows that (u_1, u_2) and (u_2, b) are arcs of G and (a, u_1) and (u_1, u_k) are arcs of G. Due to the result of Claim C, a and b are not vertices from C. Since a is adjacent to u_1 in G and b is non-adjacent to u_1 in G, a and b are different vertices of F. If a and u_2 are non-adjacent in G then (u_2, u_1, a) is a witness triple for u_2 in F, which means that (u_2, u_1) is an arc of D, and D has a directed cycle of length 2, a contradiction. Thus, a and u_2 must be adjacent in G.

If (u_2, a) is an arc of G then (u_1, a, u_2) is a witness triple of the third type for u_1 in F, which does not exist, so that (a, u_2) is an arc of G. We consider the vertices u_2 and u_k. Remember

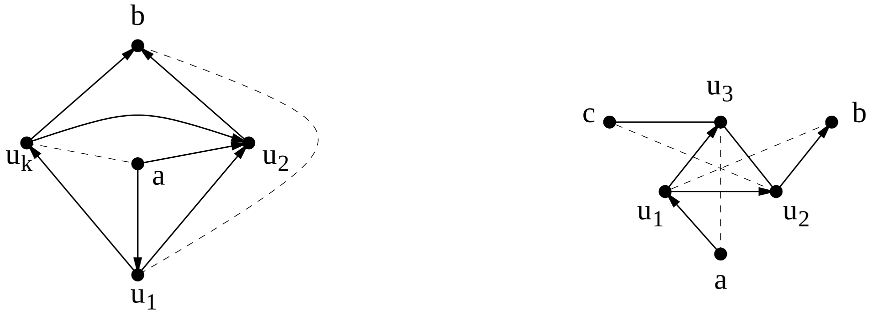

Figure 4: The two digraphs illustrate situations in the proof of Lemma 5.5. The left hand side digraph shows the situation in G at the end of the proof of Claim D. Dashed line segments connect non-adjacent vertices, vertex pairs that are not connected by a line segment may or may not be adjacent in G. The right hand side digraph summarises a situation at the end of the proof. We know that u_3 is adjacent to c and u_2 in G but we do not know by which arcs.

that a and u_k are non-adjacent in G. If u_2 and u_k are adjacent in H then, since u_2 and u_k are simplicial vertices of H, we obtain a contradiction from the first statement of Lemma 5.4, so that u_2 and u_k are non-adjacent in H, thus, u_2 and u_k are adjacent in G.

If (u_2, u_k) is an arc of G then (u_k, u_2, a) is a witness triple for u_k in F, and (u_k, u_2) is an arc of D, and $(u_2, \ldots, u_k)$ is a directed cycle of length at most $k-1$ in D, a contradiction to the choice of k. Thus, (u_k, u_2) is an arc of G. If u_k and b are non-adjacent in G then (u_k, u_2, b) is a witness triple for u_k in G, and (u_k, u_2) is an arc of D, a contradiction. Thus, u_k and b are adjacent in G.

If (b, u_k) is an arc of G then (u_k, b, u_2) is a witness triple of the third type for u_k in F, which does not exist. So, (u_k, b) is an arc of G. The situation in G is depicted in the left hand side digraph of Figure 4. Then, (u_1, u_k, b) is a witness triple for u_1 in F. Thus, (u_1, u_k) is an arc of D, and (u_1, u_k) is a directed cycle of length 2 in D, a contradiction to the choice of k. □

We summarise: the vertices u_1, u_2 and $u_k = u_3$ are pairwise adjacent in G, and (u_1, u_2) and (u_1, u_3) are arcs of G, and either (u_2, u_3) is an arc of G or (u_3, u_2) is an arc of G. We distinguish between the two cases. Let a, b, c be vertices of F such that (u_1, u_2, b), (u_2, u_3, c) and (u_3, u_1, a) are witness triples for respectively u_1, u_2 and u_3 in F. Due to Claim C, a, b, c are vertices that do not appear in C, so they are different from u_1, u_2, u_3. Observe that a, b, c are pairwise different vertices: u_1 is adjacent to a and non-adjacent to b in G, u_2 is adjacent to b and non-adjacent to c in G, and u_3 is adjacent to c and non-adjacent to a in G. The situation in G is depicted in the right hand side digraph of Figure 4.

As the first case, assume that (u_2, u_3) is an arc of G. This means that (u_3, c) is an arc of G. We consider a and u_2. Remember that u_1 and b are non-adjacent in G. If a and u_2 are non-adjacent in G then $\{a, u_1, u_2, b\}$ induces a copy of the second, third or fourth digraph (digraph (b), (c) or (d)) of Figure 1 in G. If a and u_2 are adjacent in G and (u_2, a) is an arc of G then (u_1, a, u_2) is a witness triple of the third type for u_1, a contradiction. If a and u_2 are adjacent in G and (a, u_2) is an arc of G then $\{a, u_2, u_3, c\}$ induces a copy of the second, third or fourth digraph (digraph (b), (c) or (d)) of Figure 1 in G.

As the second case, assume that (u_3, u_2) is an arc of G. This means that (c, u_3) is an arc of G. We consider b and u_3. If b and u_3 are non-adjacent in G then $\{c, u_3, u_2, b\}$ induces a copy of the second, third or fourth digraph of Figure 1 in G. If b and u_3 are adjacent in G and (b, u_3) is an arc of G then (u_3, b, u_2) is a witness triple of the third type for u_3 in F, a contradiction.

If b and u_3 are adjacent in G and (u_3, b) is an arc of G then $\{a, u_1, u_3, b\}$ induces a copy of the second, third or fourth digraph of Figure 1 in G.

We have shown that if F has no di-simplicial vertex then the uni-restriction of F must contain a copy of one of the four digraphs depicted in Figure 1 as an induced subgraph. This completes the proof of the first main case.

2) F contains a di-simplicial vertex
We apply Proposition 3.6. Since F is not a perfect elimination digraph due to the assumptions of the lemma, there is a set X of at least three vertices of F such that $F[X]$ does not contain a di-simplicial vertex. Observe that the underlying graph of $\text{bi}(F[X])$ is chordal. Then, $F[X]$ satisfies the assumptions of the lemma and of the first main case, and we conclude that the uni-restriction of $F[X]$, and thus of F, contains a copy of a digraph depicted in Figure 1 as an induced subgraph. ▪

Theorem 5.6. *Let F be a semi-complete digraph. Then, F is a perfect elimination digraph if and only if the underlying graph of* $\text{bi}(F)$ *is chordal and* $\text{uni}(F)$ *does not contain a copy of any of the digraphs depicted in Figure 1 as an induced subgraph.*

Proof. If the underlying graph of $\text{bi}(F)$ is not chordal then $\text{bi}(F)$ contains a chordless directed cycle of length at least 4, and F is not a perfect elimination digraph due to Lemma 4.2. If the underlying graph of $\text{bi}(F)$ is chordal and F is not a perfect elimination digraph then $\text{uni}(F)$ contains a copy of one of the four digraphs depicted in Figure 1 as an induced subgraph due to Lemma 5.5. If the underlying graph of $\text{bi}(F)$ is chordal and F is a perfect elimination digraph then $\text{uni}(F)$ does not contain a copy of any of the digraphs depicted in Figure 1 as an induced subgraph, since each of them is the uni-restriction of a semi-complete digraph that is not a perfect elimination digraph. ▪

6 Conclusion and open problems

The main result of this paper is a characterisation of the semi-complete perfect elimination digraphs by forbidden induced subgraphs. Combining the results of Lemma 4.2 and Theorem 5.6, the minimal such set of forbidden induced subgraphs for semi-complete perfect elimination digraphs is shown in Figure 5. This set of forbidden induced subgraphs contains only four digraphs that do not correspond to the corresponding forbidden induced subgraphs for chordal graphs, namely the chordless cycles of length 4 or larger.

Note that even though the actual set of minimal forbidden induced subgraphs for semi-complete perfect elimination digraphs is much bigger than the set of minimal forbidden induced subgraphs for chordal graphs (due to the many different orientations), the structure of semi-complete perfect elimination digraphs is already much richer than the structure of the whole class of chordal graphs. This can give a first impression of the significant difference between directed and undirected graphs. Secondly, it is an interesting observation that each digraph of Figure 1 is isomorphic to its own reverse graph. Motivated by an open problem by Haskins and Rose [9], Kleitman showed that perfect elimination digraphs cannot be characterised by their behaviour on a finite set of paths [11]. This particularly shows that a forbidden induced subgraph characterisation of the perfect elimination digraphs cannot be easy.

Chordal graphs are important both in graph theory and for graph algorithms. Several central graph notions show a specific behaviour on chordal graphs, such as cliques and minimal

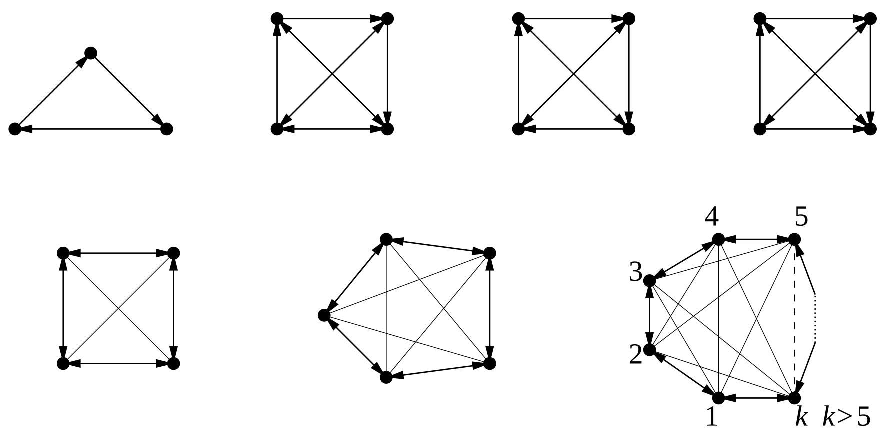

Figure 5: The figure shows a minimal set of forbidden induced subgraphs for the semi-complete perfect elimination digraphs. The upper line shows the four digraphs that are forbidden for semi-complete perfect elimination digraphs where the underlying graph of its bi-restriction is chordal. The lower line shows semi-complete digraphs where the underlying graph of its bi-restriction is not chordal. The connections without an arrow stand for arcs that may be oriented either way, however not both ways.

vertex separators [4]. Is there a directed notion of minimal separator that is related to d-cliques and perfect elimination digraphs? A famous characterisation of chordal graphs is as the intersection graphs of subtrees of trees [3, 6, 17]. Can this be generalised to perfect elimination digraphs? Let us remark here that there is a generalisation of intersection graphs to 'intersection digraphs', that results in an interesting directed analogue of interval graphs [16, 18]. However, if using this definition of 'intersection digraphs' then all digraphs become representable as 'subtree intersection digraphs' [8], see also Bang-Jensen and Gutin [1], Proposition 4.13.2. Thus, a different approach is needed to define the proper directed analogue of 'intersection graph of subtrees of a tree'. Feder et al. defined *adjusted interval digraphs* as a directed generalisation of interval graphs [5]. They give a linear ordering characterisation of the adjusted interval digraphs, that generalises interval orderings for interval graphs and that shows that adjusted interval digraphs are in fact a class of perfect elimination digraphs. They also provide a forbidden structure characterisation of the adjusted interval digraphs, however not a forbidden induced subgraph characterisation.

Chordal graphs are closely related to the treewidth parameter. Hunter and Kreutzer introduced the Kelly-width parameter for digraphs as a generalisation of treewidth [10], and Kelly-width is in a similar way related to perfect elimination digraphs. For a given integer k with $k \geq 0$, a d-clique (A, B) of a digraph G has *width at most* k if $|B| \leq k$. A perfect elimination digraph that is obtained as in Definition 3.3 by choosing only d-cliques of width at most k is called a k-DAG, and a digraph G has Kelly-width at most $k+1$ if it is a subgraph of a k-DAG [10], see also [12, 13]. For the purpose of investigating problems that are tractable on digraphs of bounded Kelly-width, a first step may be to study their complexity on perfect elimination digraphs or k-DAGs.

On the algorithmic side, there are many interesting problems for perfect elimination digraphs. How fast can a di-simplicial vertex be found in an arbitrary digraph or in a perfect elimination digraph? The straightforward algorithm, based on the definition, gives an $\mathcal{O}(nm)$-time upper bound for the problem. This is the best running time for perfect elimination digraph recognition [15]. Since the first vertex of a directed transitive vertex layout is a di-simplicial vertex, the currently best times for finding a di-simplicial vertex and for recognising perfect elimination digraphs are equal. It seems likely that finding a di-simplicial vertex can be done faster. Another interesting problem is to improve the running time for verifying that a vertex layout is directed transitive. Can this be done in $\mathcal{O}(n^2)$ time?

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