# Generalized *H*-coloring of Graphs

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**Abstract** For fixed simple graph $H$ and subsets of natural numbers $\sigma$ and $\rho$, we introduce $(H, \sigma, \rho)$-colorings as generalizations of $H$-colorings of graphs. An $(H, \sigma, \rho)$-coloring of a graph $G$ can be seen as a mapping $f : V(G) \to V(H)$, such that the neighbors of any $v \in V(G)$ are mapped to the closed neighborhood of $f(v)$, with $\sigma$ constraining the number of neighbors mapped to $f(v)$, and $\rho$ constraining the number of neighbors mapped to each neighbor of $f(v)$. A traditional $H$-coloring is in this sense an $(H, \{0\}, \{0, 1, \ldots\})$-coloring. We initiate the study of how these colorings are related and then focus on the problem of deciding if an input graph $G$ has an $(H, \{0\}, \{1, 2, \ldots\})$-coloring. This $H$-COLORDOMINATION problem is shown to be no easier than the $H$-COVER problem and $\mathcal{NP}$-complete for various infinite classes of graphs.

## 1 Introduction

Let $H$ be a fixed simple graph with $k$ vertices $V(H) = \{h_1, h_2, \ldots, h_k\}$, and let $\sigma$ and $\rho$ be fixed subsets of natural numbers. We define an $(H, \sigma, \rho)$-*coloring* of a graph $G$ to be a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ such that for all $1 \leq i, j \leq k$

$$\forall v \in V_i : |N_G(v) \cap V_j| \in \begin{cases} \sigma & \text{if } i = j \\ \rho & \text{if } h_i h_j \in E(H) \\ \{0\} & \text{otherwise} \end{cases} ,$$

where $N_G(v)$ denotes the (open) neighborhood of $v$ in $G$ and $E(H)$ the edges of $H$. We will also view the partition as given by a function $f : V(G) \to V(H)$, with $f(v) = h_i$ for $v \in V_i$. We refer to the vertices of $H$ as 'colors' and denote $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{N}^+ = \{1, 2, \ldots\}$. The well-known $H$-COLORING, also known as $H$-HOMOMORPHISM, problem asks for an assignment of 'colors' to the vertices of an input graph $G$ such that adjacent vertices of $G$ obtain adjacent 'colors'. This corresponds to asking if an input graph $G$ has an $(H, \{0\}, \mathbb{N})$-coloring. Similarly, an $H$-cover of a graph $G$ is a 'local isomorphism' between $G$ and $H$, a degree-preserving mapping of vertices where the set of 'colors' assigned to the neighbors of a vertex 'colored' $h$ is exactly equal to the set of 'colors' adjacent to $h$, and corresponds to an $(H, \{0\}, \{1\})$-coloring. A third example is given by a so-called $H$-partial cover of a graph $G$ which exists if and only if $G$ is the subgraph of a graph having an $H$-cover, and corresponds precisely to the existence of an $(H, \{0\}, \{0, 1\})$-coloring.

For an arbitrary input graph $G$, the $H$-COLORING problem is known to be solvable in polynomial time whenever the fixed graph $H$ is bipartite, and $\mathcal{NP}$-complete for all other $H$ [5]. For the $H$-COVER problem, i.e. deciding if an input graph $G$ has an $H$-cover, even if a variety of results are known about its complexity, see e.g. [1,9,10,8], it is still unclear what characterizes the class of graphs $H$ that lead to polynomial-time $H$-COVER problems. Recently there has been some interest also in the $H$-PARTIAL COVER problem [9,2,7], deciding if an input graph $G$ has an $H$-partial cover, and again the complexity of the problem seems quite rich and hard to settle up to $\mathcal{P}$ versus $\mathcal{NP}$-complete.

In this paper we view the $H$-COLORING, $H$-COVER, and $H$-PARTIAL COVER problems as instances of a more general problem parameterized not only by $H$, but also by $\sigma, \rho \subseteq \mathbb{N}$. A $(\sigma, \rho)$-set in a graph $G$ [13] is a subset of vertices $S \subseteq V(G)$ such that for any vertex $v \in V(G)$ we have

$$|N(v) \cap S| \in \begin{cases} \sigma \text{ if } v \in S \\ \rho \text{ if } v \notin S \end{cases} .$$

In this sense $H$-colorings arise from independent sets ($\sigma = \{0\}, \rho = \mathbb{N}$), $H$-covers from perfect codes ($\sigma = \{0\}, \rho = \{1\}$), and $H$-partial covers from 2-packings, also called strong stable sets, ($\sigma = \{0\}, \rho = \{0, 1\}$) [3]. See [13] for a list of other vertex subset properties from the literature expressed as $(\sigma, \rho)$-sets.

Asking about the existence of a partitioning of the vertices of a graph $G$ into $k$ $(\sigma, \rho)$-sets corresponds in this setting to asking for a $(K_k, \sigma, \rho)$-coloring, and the complexity of this question has been resolved for most values of $k \in \mathbb{N}$ and $\sigma, \rho \in \{\{0\}, \{0, 1\}, \{1\}, \mathbb{N}, \mathbb{N}^+\}$ [4]. We mention that the minimum value of $k$ such that a graph $G$ has a $(K_k, \{0\}, \mathbb{N})$-coloring is known as the chromatic number of $G$, while the maximum value of $k$ such that $G$ has a $(K_k, \mathbb{N}, \mathbb{N}^+)$-coloring is known as its domatic number, and the maximum value of $k$ such that it has a $(K_k, \mathbb{N}^+, \mathbb{N}^+)$-coloring its total domatic number. See [3] for an in-depth treatment of domination and related subset problems in graphs.

In the next section we initiate the study of $(H, \sigma, \rho)$-colorings of graphs by giving several observations on their interconnectedness. We then focus on the problem of deciding if an input graph $G$ has an $(H, \{0\}, \mathbb{N}^+)$-coloring. Since ($\sigma = \{0\}, \rho = \mathbb{N}^+$)-sets are exactly the independent dominating sets of a graph, we call this the $H$-COLORDOMINATION problem. The complexity of $H$-COLORDOMINATION is related to the $H$-COVER problem, and moreover we show it to be $\mathcal{NP}$-complete for infinite classes of graphs such as $H$ a cycle on at least three vertices, $H$ a star with at least two leaves, or $H$ a path of at least three vertices.

## 2 Observations

We first state, without proof, some general facts about the existence of $(H, \sigma, \rho)$-colorings.

**Fact 1** *The trivial partition of $V(H)$ into singleton sets is an $(H, \{0\}, \{1\})$-coloring of $H$ (in fact it is an $(H, \sigma, \rho)$-coloring as long as $0 \in \sigma, 1 \in \rho$.) At the other extreme, the trivial partition of $V(G)$ into one block is a $(K_1, \sigma, \rho)$-coloring of $G$ as long as $\forall v \in V(G) : |N_G(v)| \in \sigma$ (adding empty blocks as needed, it is in fact an $(H, \sigma, \rho)$-coloring for any $H$, if in addition $0 \in \rho$.)*

**Fact 2** *If $r$ is a positive integer and $H$ is a connected graph, then all blocks of an $(H, \sigma, \{r\})$-coloring $V_1, \ldots, V_{|V(H)|}$ of a graph $G$ must be of the same cardinality, $|V_i| = |V(G)|/|V(H)|$ for $i = 1, \ldots, |V(H)|$.*

To investigate how $(H, \sigma, \rho)$-colorings in $G$ interact with $(H', \sigma', \rho')$-colorings in $H$, we first define, for two non-empty sets of natural numbers, $A$ and $B$

$$
\begin{aligned}
A \oplus B &\stackrel{\text{def}}{=} \{a + b : a \in A, b \in B\} \\
A \otimes B &\stackrel{\text{def}}{=} \{b_1 + \cdots + b_a : a \in A, b_i \in B\} \\
\forall B : \{0\} \otimes B &\stackrel{\text{def}}{=} \{0\} \ .
\end{aligned}
$$

**Fact 3** $\forall A \in \{\{0\}, \{1\}, \{0, 1\}, \mathbb{N}, \mathbb{N}^+\} : A \otimes A = A.$

Viewed as a relation on graphs $(H, \sigma, \rho)$-coloring exhibits transitive properties for certain values of $\sigma$ and $\rho$. For these values $(H, \sigma, \rho)$-colorability of $G$ and $(H', \sigma, \rho)$-colorability of $H$ will imply $(H', \sigma, \rho)$-colorabiliy of $G$. We first state a more general result.

**Theorem 1.** *If a graph $G$ is $(H, \sigma, \rho)$-colorable, and $H$ is $(H', \sigma', \rho')$-colorable, then $G$ is $(H', \sigma \oplus (\sigma' \otimes \rho), \rho' \otimes \rho)$-colorable.*

*Proof.* Let the functions $f : V(G) \to V(H)$ and $g : V(H) \to V(H')$ be the $(H, \sigma, \rho)$-coloring of $G$ and the $(H', \sigma', \rho')$-coloring of $H$, respectively. We show that $f$ composed with $g$ is an $(H', \sigma \oplus (\sigma' \otimes \rho), \rho' \otimes \rho)$-coloring of $G$. For vertices $v', u' \in V(H')$ we let $V'_H = \{w \in V(H) : g(w) = v'\}$ and $U'_H = \{w \in V(H) : g(w) = u'\}$, and let $V'_G = \{v \in V(G) : f(v) \in V'_H\}$ and $U'_G = \{v \in V(G) : f(v) \in U'_H\}$. For any vertex $v \in V(G)$ we want to count the number of neighbors of $v$ mapped to a vertex $u' \in V(H')$, which is $|N_G(v) \cap U'_G|$. We assume without loss of generality that $v \in V'_G$. There are three cases to consider: $v' = u'$, $v'u' \in E(H')$, and $v'u' \notin E(H')$.

If $v' = u'$ then $V'_G = U'_G$, and the number of neighbors of $v$ mapped to $u'$ consists of (i) the number of neighbors of $v$ mapped to $f(v)$ plus (ii) the number of neighbors of $v$ mapped to each vertex in $N_H(f(v)) \cap U'_H$. Since $|N_G(v) \cap \{u \in V(G) : f(u) = f(v)\}| \in \sigma$, we have that (i) is an element-of-$\sigma$. And since $|N_H(f(v)) \cap U'_H| \in \sigma'$, and for any $w \in N_H(f(v))$ we have $|N_G(v) \cap \{u \in V(G) : f(u) = w\}| \in \rho$ so that (ii) is the sum of some element-of-$\sigma'$ terms from $\rho$. This gives $|N_G(v) \cap U'_G| \in \sigma \oplus (\sigma' \otimes \rho)$ for $v' = u'$.

If $v'u' \in E(H')$ the number of neighbors of $v$ mapped to $u'$ consists simply of part (ii) above i.e. the number of neighbors of $v$ mapped to each vertex in $U'_H$. We now have $|N_H(f(v)) \cap U'_H| \in \rho'$, and for any $w \in N_H(f(v))$ we have $|N_G(v) \cap \{u \in V(G) : f(u) = w\}| \in \rho$. Thus $|N_G(v) \cap U'_G| \in \rho' \otimes \rho$ for $v'u' \in E(H')$.

If $v'u' \notin E(H')$ the number of neighbors of $v$ mapped to $u'$ is 0. □

The following result which follows from Theorem 1 and Fact 3, shows that $(H, \sigma, \rho)$-coloring is a transitive relation on graphs for certain values of $\sigma$ and $\rho$.

**Corollary 1.** *If $G$ is $(H, \sigma, \rho)$-colorable and $H$ is $(H', \sigma, \rho)$-colorable, with $\sigma = \{0\}$ and $\rho \in \{\{0\}, \{1\}, \{0, 1\}, \mathbb{N}, \mathbb{N}^+\}$, or $\sigma = \rho = \mathbb{N}$, then $G$ is $(H', \sigma, \rho)$-colorable.*

From this it follows that $H$-coloring, $H$-covering, $H$-partial covering and $H$-colordomination are all transitive relations on graphs. In the following we consider how $(H, \sigma, \rho)$-colorings of $G$ interact with $(\sigma', \rho')$-sets in $H$.

**Theorem 2.** *If a graph $G$ has an $(H, \sigma, \rho)$-coloring and $S$ is a $(\sigma', \rho')$-set in $H$, then $S' = \{v \in V(G) : f(v) \in S\}$ is a $(\sigma \oplus (\sigma' \otimes \rho), \rho' \otimes \rho)$-set in $G$.*

*Proof.* For space reasons we only sketch the proof, as it is similar to that of Theorem 1. Here we want to count the number of neighbors of an arbitrary vertex $v \in V(G)$ mapped to a vertex in $S$, which is $|N_G(v) \cap \{u \in V(G) : f(u) \in S\}|$. There are two cases to consider: $v \in S'$ and $v \notin S'$. The argument for the first case is similar to the argument for the case $v' = u'$ in the proof of Theorem 1, and the argument for the second case similar to that of the case $v'u' \in V(H')$. □

Let us assume that a graph $G$ is $(H, \sigma, \rho)$-colorable. The following result, which follows from Theorem 2 and Fact 3, shows that a $(\sigma, \rho)$-set in $H$ will induce a $(\sigma, \rho)$-set in $G$, for certain values of $\sigma$ and $\rho$.

**Corollary 2.** *If $G$ is $(H, \sigma, \rho)$-colorable, with $\sigma = \{0\}$ and $\rho \in \{\{0\}, \{1\}, \{0, 1\}, \mathbb{N}, \mathbb{N}^+\}$, or $\sigma = \rho = \mathbb{N}$, then a $(\sigma, \rho)$-set in $H$ will induce a $(\sigma, \rho)$-set in $G$.*

We observe that Corollary 2 holds for some of the most common variants of $(\sigma, \rho)$-sets, such as perfect codes, 2-packings, independent sets, and independent dominating sets, as defined in the introduction.

## 3 The Complexity of $H$-COLORDOMINATION

The complexity of deciding if an arbitrary input graph $G$ has an $(H, \sigma, \rho)$-coloring will depend on the three fixed values $H$, $\sigma$, and $\rho$. As mentioned in the introduction, for values of $\sigma$ and $\rho$ that arise from independent sets ($\sigma = \{0\}, \rho = \mathbb{N}$), perfect codes ($\sigma = \{0\}, \rho = \{1\}$), and 2-packings ($\sigma = \{0\}, \rho = \{0, 1\}$) the complexity of the corresponding $(H, \sigma, \rho)$-problems, respectively named $H$-COLORING, $H$-COVER, and $H$-PARTIAL COVER, have been investigated for varying $H$.

Several of the $(\sigma, \rho)$-sets that have been studied in the literature have $\sigma \neq \{0\}$, but here we continue the setting from the already studied $(H, \sigma, \rho)$-colorings and focus on the case $\sigma = \{0\}$. The most natural $(H, \{0\}, \rho)$-problem that to our knowledge has not been studied in general, is maybe the case where $\rho = \mathbb{N}^+$. In this section we therefore initiate the investigation of the complexity of the $(H, \sigma, \rho)$-coloring problem that arises from independent dominating sets ($\sigma = \{0\}, \rho = \mathbb{N}^+$). We call this the $H$-COLORDOMINATION problem.

We show that $H$-COLORDOMINATION is no easier than $H$-COVER, and also present complexity results for $H$-COLORDOMINATION for classes of graphs for which $H$-COVER is in $\mathcal{P}$. We will in the following consider the graphs to be connected, and without loops.

We first mention the following result on cliques $K_k$ which, albeit with different terminology, can be found in [4].

**Theorem 3.** [4] *For every $k \geq 3$ the $K_k$-COLORDOMINATION problem is $\mathcal{NP}$-complete.*

### 3.1 No Easier than $H$-COVER

The *degree partition* of a graph $G$ is the partition of its vertices, $V(G)$, into the minimum number of blocks $\mathcal{B}_G = \{B_1(G), \ldots, B_k(G)\}$, for which there are constants $r_{ij}$ such that for each $i, j (1 \leq i, j \leq k)$ each vertex in $B_i$ is adjacent to exactly $r_{ij}$ vertices in $B_j$. For a given ordering of degree partition blocks, the $k \times k$ matrix $R, R[i, j] = r_{ij}$, is called the *degree refinement.*

The degree partition and degree refinement matrix of a graph can be computed in polynomial time by stepwise refinement with the following algorithm:

1. Partition the vertices into blocks by their degree values, and arrange the blocks in ascending order (by degree value).
2. For every vertex compute the number of neighbours it has in the current blocks of the partition. These numbers, maintaining the order of the blocks, make up the degree vector of the vertex.
3. If a block contains vertices with different degree vectors it is split into as many new blocks as there are different degree vectors. The new blocks are ordered in lexicographically ascending order by the degree vectors, and they maintain their order relative to the other blocks of the partition. If no block contains vertices with different degree vectors we are done.
4. Goto 2.

The above algorithm gives a unique ordering of the degree partition blocks, and gives us a well defined degree refinement matrix $R$ for every graph $G$.

While computing the degree partition of a graph $G$, let $\mathcal{B}_G^i = \{B_1^i(G), \ldots, B_{k^i}^i(G)\}$ be the preliminary degree partition at the start of the $i$th iteration of the degree partition procedure, with $\mathcal{B}_G^1$ the starting partition of $V(G)$ by its degree values. Let $b_G(v) = j$ be the index such that vertex $v$ belongs to block $B_j(G)$ in the degree partition, and let $b_G^i(v) = j$ be the index such that vertex $v$ belongs to block $B_j^i(G)$ at the start of the $i$th iteration. Also, let $n_G^i(v)$ be the the degree vector of vertex $v$ computed in the $i$th iteration. The following result is similar to one from [7], where it was shown that if $G$ has an $H$-partial cover and $G$ and $H$ have the same degree partition, then $G$ has an $H$-cover.

**Lemma 1.** *If two graphs $G$ and $H$ have the same degree partition, $\mathcal{B}_G = \mathcal{B}_H$, then for every $i \geq 0$ we have $n_G^i(u) = n_H^i(v)$ if $b_G(u) = b_H(v)$.*

*Proof.* If $b_G(u) = b_H(v)$, we have $n_G^i(u) = n_H^i(v)$ in the last iteration of the degree partition procedure. We must also have had $n_G^{i-1}(u) = n_H^{i-1}(v)$, otherwise $u$ and $v$ would have been separated. This implies that for every $i \geq 0$ we have $n_G^i(u) = n_H^i(v)$ if $b_G(u) = b_H(v)$.

**Lemma 2.** *If $G$ and $H$ are connected graphs with the same degree partition, $\mathcal{B}_G = \mathcal{B}_H$, and $G$ has an $H(\{0\}, \mathbb{N}^+)$-colouring, then $G$ has an $H(\{0\}, \{1\})$-colouring, or $H$-cover.*

*Proof.* Assume $G$ has an $H(\{0\}, \mathbb{N}^+)$-colouring $f : V(G) \to V(H)$, and that $G$ and $H$ have the same degree partition. We will show that we necessarily have $b_G(v) = b_H(f(v))$, to prove the lemma.

We first show that we must have $b_G(v) \geq b_H(f(v))$ for all $v \in V(G)$. Since $f$ is an $H(\{0\}, \mathbb{N}^+)$-colouring, $\deg_G(v) \geq \deg_H(f(v))$ holds for all $v \in V(G)$, because $\rho = \mathbb{N}^+$. This implies that $b_G^1(v) \geq b_H^1(f(v))$ for all $v \in V(G)$. We show, by induction on $i$, that we must have $b_G^i(v) \geq b_H^i(f(v))$ for all $v \in V(G)$ and all $i \geq 1$. For $i \geq 1$, $b_G(v)$ and $b_H(f(v))$ depend on the degree vectors $n_G^{i-1}(v)$ and $n_H^{i-1}(f(v))$. For every $u \in N_G(v)$, $f(u) \in N_H(f(v))$, and by the induction hypothesis, $b_G^{i-1}(u) \geq b_H^{i-1}(f(u))$. Therefore, in the lexicographic ordering, $n_G^{i-1}(v) \geq_{\text{lex}} n_H^{i-1}(f(v))$, this means that we will have $b_G^i(v) \geq b_H^i(f(v))$. Note that we are implicitly using Lemma 1, as we are comparing degree vectors in $G$ and $H$.

We next show that we must have $b_G(v) = b_H(f(v))$ for all $v \in V(G)$. For $v \in B_1(G)$, $b_G(v) = 1$, and since $b_G(v) \geq b_H(f(v))$, this implies that $b_H(f(v)) = 1$. Assume a vertex $v \in V(G)$ exists with $b_G(v) > b_H(f(v))$, and let $u$ be an arbitrary vertex from $B_1(G)$. Consider a path $P$ from $u$ to $v$ in $G$. $P$ must contain an edge $u'v' \in E(G)$ such that $b_G(u') = b_H(f(u'))$ and $b_G(v') > b_H(f(v'))$. Since $u'$ and $f(u')$ have the same number of neighbors in blocks with the same index, and neighbor $v'$ of $u'$ has been sent to a block numbered lower, there must exist a neighbor $w'$ of $u'$ that is sent to a block numbered higher, $b_G(w') < b_H(f(w'))$, a contradiction. This implies that for all $v \in V(G)$, $b_G(v) = b_H(f(v))$, and that for all $v \in V(G)$, $\deg_G(v) = \deg_H(f(v))$. This in turn implies that $f$ is a valid $H(\{0\}, \{1\})$-colouring, or $H$-cover, of $G$.

**Theorem 4.** *If $H$-COVER is $\mathcal{NP}$-complete, then $H$-COLOURDOMINATION is $\mathcal{NP}$-complete.*

*Proof.* If there is a polynomial-time algorithm for $H$-COLOURDOMINATION then Lemma 2 gives us the following polynomial-time algorithm for $H$-COVER: Given a graph $G$, answer YES if $G$ and $H$ have the same degree refinement and $G$ has an $H(\{0\}, \mathbb{N}^+)$-colouring; otherwise answer NO.

### 3.2 Cycles

In this section we show the following for the cycles $C_k$:

**Theorem 5.** *For every $k \geq 3$ the $C_k$-COLORDOMINATION problem is $\mathcal{NP}$-complete.*

The result will follow from three lemmata.

**Lemma 3.** *The $(C_k, \{0\}, \mathbb{N}^+)$-coloring problem is $\mathcal{NP}$-complete for all $k = 2i+1, i \geq 1$.*

*Proof.* We use a reduction from $C_k$-COLORING, $\mathcal{NP}$-complete for all odd $k$ [5]. Given a graph $G$ we construct a graph $G'$ which will have a $(C_k, \{0\}, \mathbb{N}^+)$-coloring if and only if $G$ has a $C_k$-coloring. $G'$ is constructed by replacing the vertices $v \in V(G)$ with a cycle of length $k$, ${C_k}^{(v)}$, each such cycle with a designated vertex ${c_1}^{(v)}$, and all edges $uv \in E(G)$ by an edge ${c_1}^{(u)}{c_1}^{(v)}$.

The cycles ${C_k}^{(v)}$ ensure that $G'$ has a $(C_k, \{0\}, \mathbb{N}^+)$-coloring whenever $G$ is $C_k$-cororable. If $G$ has no $C_k$-coloring the subgraph induced by the designated vertices prevents a $(C_k, \{0\}, \mathbb{N}^+)$-coloring of $G'$. □

In the next reduction we use the following problem which was shown to be $\mathcal{NP}$-complete in [12].

[NAESAT] **NOT-ALL-EQUAL SATISFIABILITY**
INSTANCE: A collection $C$ of clauses on a finite set $U$ of variables such that each clause $c \in C$ has $|c| = 3$.
QUESTION: Is there a truth assignment for $U$ such that each clause in $C$ has at least one TRUE literal and one FALSE literal?

**Lemma 4.** *The $(C_4, \{0\}, \mathbb{N}^+)$-coloring problem is $\mathcal{NP}$-complete.*

*Proof.* The reduction is from NAESAT. Let $U$ be the set of variables and $C$ be the set of clauses. We can assume that all literals, $u$ and $\bar{u}$, occur in some clause, otherwise for each literal that does not occur, we find a clause where the opposite literal occurs, and add a copy of this clause with all literals negated. We construct a bipartite graph $G$ which will have a $(C_k, \{0\}, \mathbb{N}^+)$-coloring if and only if the variables of $U$ can be assigned values TRUE or FALSE, such that all clauses in $C$ have at least one literal that is TRUE and one that is FALSE. For each variable $u$ there is a variable gadget, $P_3^{(u)}$, with literal vertices $v_u$ and $v_{\bar{u}}$ as the endpoints, and a center vertex $v_{u\bar{u}}$. For each clause $c$ there is a vertex $v_c$ with edges to the literal vertices corresponding to the literals occurring in this clause.

Let $T$ be a valid truth assignment for the NAESAT instance, and label the vertices of $C_4$ $A$, $B$, $C$, and $D$, following the cycle. We define a mapping $f : V(G) \to \{A, B, C, D\}$ giving a $(C_4, \{0\}, \mathbb{N}^+)$-coloring of $G$. Let $f(v_c) = A$, for all $v_c$. And let $f(v_u) = B$ if $T(u) =$ TRUE or $f(v_u) = D$ if $T(u) =$ FALSE, for all literal vertices $v_u$ and $v_{\bar{u}}$. Let $f(u_{u\bar{u}}) = C$, for all variable vertices. Since all clauses have at least one literal set to TRUE and one set to FALSE, $f$ is a $(C_4, \{0\}, \mathbb{N}^+)$-coloring of $G$.

For the other direction of the proof we assume $f$ is a valid $(C_4, \{0\}, \mathbb{N}^+)$-coloring of $G$. A clause vertex $v_c$ is mapped to a vertex of $C_4$, call this vertex $A$.

This forces the literal vertices to be mapped to $B$ or $C$, in such a way that if $v_u$ is mapped to $B$, then $v_{\bar{u}}$ is mapped to $C$. The other clause vertices are mapped to either $A$ or $D$, but in both cases the literal vertices are mapped to $B$ or $C$. Since $f$ is a valid $(C_4, \{0\}, \mathbb{N}^+)$-coloring each clause vertex must have at least one neighbor in each of $B$ and $C$. We define a valid truth assignment $T$ for the NAESAT instance by taking $T(u) =$ TRUE if $f(v_u) = B$, and $T(u) =$ FALSE if $f(v_u) = C$. □

**Lemma 5.** *The* $(C_{2k}, \{0\}, \mathbb{N}^+)$*-coloring problem is* $\mathcal{NP}$*-complete if the* $(C_k, \{0\}, \mathbb{N}^+)$*-coloring problem is* $\mathcal{NP}$*-complete.*

*Proof.* Given a graph $G$ we construct a graph $G'$ which will be $(C_{2k}, \{0\}, \mathbb{N}^+)$-colorable if and only if $G$ is $(C_k, \{0\}, \mathbb{N}^+)$-colorable, by subdividing all the edges of $G$ once. □

### 3.3 Paths

Let $P_k$ denote a path with $k$ vertices and $k-1$ edges. We first look at the case $k = 2$ and observe that the $(P_2, \{0\}, \mathbb{N}^+)$-coloring problem is easily solvable in polynomial time.

**Observation 1** *A graph $G$ has a* $(P_2, \{0\}, \mathbb{N}^+)$*-coloring if and only if it is bipartite.*

For $k \geq 3$ the situation is different. We show the following result for paths $P_k$:

**Theorem 6.** *For every $k \geq 3$ the $P_k$-COLORDOMINATION problem is* $\mathcal{NP}$*-complete.*

The result will follow from four lemmata.

**Lemma 6.** *The* $(P_3, \{0\}, \mathbb{N}^+)$*-coloring problem is* $\mathcal{NP}$*-complete.*

*Proof.* The reduction is from NAESAT. Let $U$ be the set of variables and $C$ the set of clauses. We construct a bipartite graph $G$ which will have a $(P_3, \{0\}, \mathbb{N}^+)$-coloring if and only if the variables of $U$ can be assigned values TRUE or FALSE, such that all clauses of $C$ have at least one literal that is TRUE and one that is FALSE. For each variable $u$ there is a variable gadget, ${P_3}^{(u)}$, with literal vertices $v_u$ and $v_{\bar{u}}$ as the endpoints, and a center vertex $v_{u\bar{u}}$. For each clause $c$ there is a vertex $v_c$ with edges to the literal vertices corresponding to the literals occurring in this clause. In addition to this we add a new clause vertex $v_x$, and a new variable gadget ${P_3}^{(y)}$. We connect $v_x$ to both $v_u$ and $v_{\bar{u}}$ of one already existing variable $u$, and to $v_y$ of the added variable gadget ${P_3}^{(y)}$. This augmentation will not affect the satisfiability of the original instance.

Let $T$ be a valid truth assignment for the NAESAT instance, and label the vertices of $P_3$ $A$, $B$, and $C$, with $B$ as the center vertex. We define a mapping $f : V(G) \to \{A, B, C\}$ giving a $(P_3, \{0\}, \mathbb{N}^+)$-coloring of $G$. Let $f(v_c) = f(v_{u\bar{u}}) =$

$B$, for all $u \in U$ and all $c \in C$. And let $f(v_u) = A$ and $f(v_{\bar{u}}) = C$ if $T(u) =$ TRUE, or $f(v_{\bar{u}}) = A$ and $f(v_u) = C$ if $T(u) =$ FALSE. Since all clauses have at least one literal set to TRUE and one set to FALSE, $f$ is a $(P_3, \{0\}, \mathbb{N}^+)$-coloring.

For the other direction of the proof we assume $f$ is a valid $(P_3, \{0\}, \mathbb{N}^+)$-coloring of $G$. Since $v_y$ has degree one, it must map to either $A$ or $C$. As $G$ is bipartite this forces all clause vertices to be mapped to $B$. Since $f$ is a valid $(P_3, \{0\}, \mathbb{N}^+)$-coloring each clause vertex must have at least one neighbor in each of $A$ and $C$. We define a valid truth assignment $T$ for the NAESAT instance by taking $T(u) =$ TRUE if $f(v_u) = A$, and $T(u) =$ FALSE if $f(v_u) = C$. □

The same technique can be applied to all paths of odd length.

**Lemma 7.** *The $P_k$-COLORDOMINATION problem is $\mathcal{NP}$-complete for all $k = 2i + 1, i \geq 1$.*

*Proof.* The proof is essentially the same as that of Lemma 6. We modify it to hold for all $k = 2i + 1$ by replacing the variable gadgets with paths of length $k$, and connecting the clause vertices to the literal vertices using paths with $\lfloor k/2 \rfloor$ edges. □

For paths of even length we apply a variation of the same technique.

**Lemma 8.** *The $(P_4, \{0\}, \mathbb{N}^+)$-coloring problem is $\mathcal{NP}$-complete.*

*Proof.* The reduction is again from NAESAT. Let $U$ be the set of variables and $C$ the set of clauses. We construct a graph $G$ which will have a $(P_4, \{0\}, \mathbb{N}^+)$-coloring if and only if the variables of $U$ can be assigned values TRUE or FALSE, such that all clauses of $C$ have at least one literal that is TRUE and one that is FALSE. For each variable $u$ there is a variable gadget, ${P_5}^{(u)}$, with literal vertices $v_u$ and $v_{\bar{u}}$ as the endpoints. For each clause $c$ there is a clause gadget consisting of two components: (i) ${P_2}^{(c)}$ with a designated vertex $v_c$ as one of its endpoints, and (ii) a vertex $v_{c'}$. For each literal that occurs in a clause $c$ there is an edge between the corresponding literal vertex and the vertex $v_c$, and an edge between the corresponding negated literal vertex (the other end of ${P_5}^{(u)}$) and $v_{c'}$.

Let $T$ be a valid truth assignment for the NAESAT instance, and label the vertices of $P_4$ in order, $A$, $B$, $C$, and $D$. We define a mapping $f : V(G) \rightarrow \{A, B, C, D\}$ giving a $(P_4, \{0\}, \mathbb{N}^+)$-coloring of $G$. Let the clause gadget ${P_2}^{(c)}$ map to $A, B$ with $f(v_c) = B$, for all $v_c$. Let $f(v_u) = A$ and $f(v_{\bar{u}}) = C$ if $T(u) =$ TRUE, or let $f(v_{\bar{u}}) = A$ and $f(v_u) = C$ if $T(u) =$ FALSE. Finally let $f(V_{c'}) = B$, for all $v_{c'}$. This enforces a mapping of the remaining vertices of the variable gadgets. The path ${P_5}^{(u)}$ of the variable gadget is either mapped $A, B, C, D, C$, if $T(u) =$ TRUE, or $C, D, C, B, A$, if $T(u) =$ FALSE. Since all clauses have at least one literal set to TRUE and one set to FALSE, $f$ is a $(P_4, \{0\}, \mathbb{N}^+)$-coloring.

For the other direction of the proof we assume $f$ is a valid $(P_4, \{0\}, \mathbb{N}^+)$-coloring of $G$. Since each clause gadget ${P_2}^{(c)}$ has one endpoint of degree one, $A, B$ or $C, D$ are the only possible mappings of the clause gadgets. Without loss of generality we may assume that $G$ is connected, in which case the lengths of the variable gadget paths ensure that all clause gadgets will map to the same pair of vertices, say $A, B$. So we must have $f(v_c) = B$. The clause gadgets are paths of length five, and they must be mapped either $A, B, C, D, C$ or $C, D, C, B, A$, so we must have $f(v_{c'}) = B$. We define a valid truth assignment $T$ for the NAESAT instance by taking $T(u) =$ TRUE if $f(v_u) = A$, and $T(u) =$ FALSE if $f(v_u) = C$. □

The technique used for $P_4$ can be applied for all paths of even length.

**Lemma 9.** *The $P_k$-COLORDOMINATION problem is $\mathcal{NP}$-complete for all $k = 2i, i \geq 2$.*

*Proof.* The proof is essentially the same as that of Lemma 8. We modify it to hold for all $k = 2i, i \geq 2$ by replacing the variable gadgets with paths of length $2k - 3$. □

### 3.4 Stars

Let $S_k$ denote the graph $K_{1,k}$, a star with $k$ leaves. In the reduction below we use the following problem.

[$k$-EC] **$k$-EDGE-COLORING**
INSTANCE: A graph $G = (V, E)$.
QUESTION: Can $E(G)$ be partitioned into $k'$ disjoint sets $E_1, E_2, \dots, E_{k'}$, with $k' \leq k$, such that, for $1 \leq i \leq k'$, no two edges in $E_i$ share a common endpoint in $G$?

If $G$ is a $k$-regular graph, the question becomes whether each vertex is incident to $k$ distinctly colored edges. This last problem was shown to be $\mathcal{NP}$-complete for $k = 3$ in [6], and for $k \geq 3$ in [11]. We get the following result for the complexity of $S_k$-COLORDOMINATION.

**Theorem 7.** *For all $k \geq 2$ the $S_k$-COLORDOMINATION problem is $\mathcal{NP}$-complete.*

*Proof.* Since $P_3 = S_2$ we use Lemma 6 for this case. For $k \geq 3$ the reduction is from $k$-EC on $k$-regular graphs, defined above. Let $G$ be an instance of $k$-EC, such that $G$ is $k$-regular. We construct a $k$-regular graph $G'$, such that $G'$ has an $(S_k, \{0\}, \mathbb{N}^+)$-coloring if and only if $G$ is $k$-edge-colorable. $G'$ is constructed by replacing vertices and edges by simple gadgets as shown in Figure 1.

We let $c$ be the center vertex of the star $S_k$, $l_1, l_2, \dots, l_k$ the leaves, and assume there exists a mapping $f : V(G') \to \{c, l_1, l_2, \dots, l_k\}$ that is a valid $(S_k, \{0\}, \mathbb{N}^+)$-coloring of $G'$. We must have $f(u_c) = c$, since $u_c$ has neighbors of degree two. This implies that $f(u_{li}) = \{l_1, l_2, \dots, l_k\}, i = 1, 2, \dots, k$, and that $f(e_u) = c$. This gives $f(e_j) = \{l_1, l_2, \dots, l_k\} \setminus f(u_{l1}), j = 1, 2, \dots, k-1$, and $f(e_v) = c$. This enforces $f(v_{l1}) = f(u_{l1})$, and thus $f(v_c) = c$. In $G$ we can color the edge $e = uv$ with the color $f(u_{l1})$. Coloring every edge in the same manner, we can conclude that $G$ is $k$-edge-colorable if $G'$ is $(S_k, \{0\}, \mathbb{N}^+)$-colorable.

For the other direction of the proof we assume $G'$ is $k$-edge-colorable and apply a reversal of the mapping described above. □

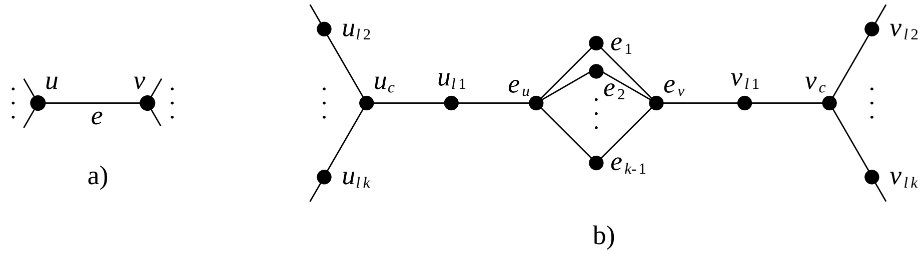


**Figure 1.** a) An edge in $G$, and b) the corresponding subgraph in $G'$

## 4 Conclusion

We have introduced a generalization of $H$-colorings of graphs, initiated the study of these colorings, and given some results on the complexity of one class of problems that it gives rise to. We leave as an open problem the question of whether $H$-COLORDOMINATION is $\mathcal{NP}$-complete for all connected $H$ on at least three vertices.

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