Covering regular graphs

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Abstract

A covering projection from a graph G onto a graph H is a "local isomorphism": a mapping from the vertex set of G onto the vertex set of H such that, for every $v \in V(G)$, the neighborhood of v is mapped bijectively onto the neighborhood (in H) of the image of v. We investigate two concepts that concern graph covers of regular graphs. The first one is called "multicovers": we show that for any regular graph H there exists a graph G that allows many different covering projections onto H. Secondly, we consider *partial covers*, which require only that G be a subgraph of a cover of H. As an application of our results we show that there are infinitely many rigid regular graphs H for which the H-cover problem – deciding if a given graph G covers H – is NP-complete. This resolves an open problem related to the characterization of graphs H for which H-COVER is tractable.

1 Motivation and overview

For a fixed graph H, the H-cover problem admits a graph G as input and asks about the existence of a "local isomorphism": a labeling of vertices of G by vertices of H so that the label set of the neighborhood of every $v \in V(G)$ is equal to the neighborhood (in H) of the label of v and each neighbor of v is labeled by a different neighbor of the label of v. (Such a labeling is referred to as a *covering projection* from G onto H.)

We trace this concept to Conway and Biggs's construction of infinite classes of highly symmetric graphs, see Chapter 19 of [4]. Graph coverings are special cases of covering spaces from algebraic topology [17], and are used in many applications in topological graph theory

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[9]. We mention that Gross and Tucker [10] show that every covering projection of a graph G can be constructed by permutation voltage assignments, a construction based on labelling edges of G by permutations, while Hofmeister [12] uses similar constructions to classify isomorphism classes of covering projections and Biggs [5] uses covering graphs constructed from the homology group (cycle space) of G to classify cubic graphs with given symmetry types.

In a more applied setting, graph coverings have been used by Angluin [2] to study "local knowledge" in distributed computing environments, and by Courcelle and Métivier [6] to show that nontrivial minor closed classes of graphs cannot be recognized by local computations. In [1], Abello *et al.* raised the question of computational complexity of H-cover problems, noting that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. We have studied the question of complexity of graph covering problems in [15], where one of our main results was a complete catalogue of the complexity of this problem for graphs on at most 6 vertices. Though we have developed several general polynomial and NP-completeness theorems, all of our NP-completeness reductions depended heavily on symmetries of the parameter graph, thus leaving the case of regular rigid graphs (i.e., graphs with only trivial automorphisms) as one of the main open problems. Already these results indicate that the classification of computational complexity of the H-cover problem is much richer than for the related H-coloring problem, resolved completely by Hell and Nešetřil [13].

In this paper, we show that the H-cover problem is NP-complete for any k-regular graph H, provided $k > 2$ and H is k-edge colorable or $(1 + \lfloor \frac{k}{2} \rfloor)$-edge connected. There are infinitely many rigid regular graphs among such graphs. In proving NP-completeness of the problems, we present a reduction that is based on a nontrivial construction, whose properties are guaranteed by a general theorem. The cases that we are not able to resolve so far raise graph theoretical questions that are interesting on their own.

The paper is organized as follows. In Section 2 we give a construction of a graph that covers a given graph H in many different ways (we call such graphs multicovers). In Section 3 we introduce the notion of partial covers and prove our main results on the existence of partial covers of certain type. In section 4.1 we introduce an auxiliary NP-complete problem of chromatic index of k-regular s-uniform hypergraphs. Section 4.2 contains the main complexity result via a gadget construction, the existence of which is ensured by the results of the preceding sections. Final remarks and open problems are gathered in Section 5.

Our notation is standard and involves simple loopless graphs. If G is a graph, we denote by $V(G)$ ($E(G)$) its vertex set (edge set, respectively). If u is a vertex of a graph G, we denote by $N_G(u)$ the set of the neighbors of u, i.e., $N_G(u) = \{w | uw \in E(G)\}$, the degree of u, $deg_G(u)$, is the number of its neighbors. The distance between vertices u and v (i.e., the length of the shortest path connecting u and v) is denoted by $d_G(u, v)$. The only nonstandard notation that we introduce and use troughout the paper is G_u denoting the graph obtained from a graph G by splitting a vertex u into $deg_G(u)$ pendant vertices (of degree one), each adjacent to a distinct neighbor of u. For every $w \in N_G(u)$, the pendant vertex adjacent to w in G_u will be referred to as u_w.

2 Multicovers

Angluin and Gardner showed in [3] that pairs of regular graphs (of the same degree) have finite common covers. It follows from their proof, that for any regular graph H and any two vertices, say x and y, of H, there exists a graph G with a specified vertex u such that G allows two different covering projections onto H: one sending u onto x and the other one sending u onto y. We are interested in covers that not only allow a specified vertex to be mapped onto *any* vertex of H, but also allow the neighborhood of the specified vertex to be mapped onto the neighborhood of any vertex of H *in any possible permutation.* We call such graphs *multicovers.*

Theorem 1 *For every connected k-regular graph H, there exists a connected graph G with a specified vertex u, such that for every $x \in V(H)$ and every bijective mapping g from the neighbors of u (in G) onto the neighbors of x (in H), there exists a covering projection $f : V(G) \to V(H)$ such that $f(u) = x$ and $f(w) = g(w)$ for every $w \in N_G(u)$.*

Proof. This proof is a generalization of Angluin and Gardner's construction of common covers of regular graphs [3]. Note first that without loss of generality we may assume that H is k-edge colorable. (If it is not, we consider $\bar{H} = H \times K_2 = (V(H) \times \{0,1\}, \{(x,0)(y,1)|xy \in E(H)\})$ and $\bar{X} = \{(x,0)|x \in X\}$. This $\bar{H}$ is bipartite k-regular, and therefore 1-factorable, i.e., k-edge colorable. Further $\bar{f} : (x,i) \to x$ $(i = 0,1)$ is a covering projection from $\bar{H}$ onto H, and thus any covering projection $g : V(G) \to V(\bar{H})$ translates to a covering projection $g \circ f : V(G) \to V(H)$.)

Denote by X the vertex set of H. Fix a legal k-edge coloring of H, say $\varphi : E(H) \to \{1, 2, \ldots, k\}$. Denote by $S_{k,x}$ the set of all bijections from $\{1, 2, \ldots, k\}$ onto $N_H(x)$, for $x \in X$. For every $x \in X$ and $\pi \in S_{k,x}$, let $\varphi_{x,\pi}$ be the legal k-edge coloring of H obtained from φ by a suitable permutation of colors such that $\varphi_{x,\pi}(x\pi(i)) = i$ for every $i = 1, 2, \ldots, k$ (note that such coloring is unique).

We define a graph G' as a color product of $|X|k!$ copies of H. The vertices of G' are $(|X|k!)$-tuples of vertices of H. Since we will exploit the structure of the index set $S = \{(x,\pi)|\pi \in S_{k,x}, x \in X\}$, we will view the vertices as functions $A : S \to V(H)$. To define the edges of G', for every i, G' has a perfect matching (of color i) formed by the edges $E'_i = \{AB|A(x,\pi)B(x,\pi) \in E(H), \varphi_{x,\pi}(A(x,\pi)B(x,\pi)) = i$ for all $x \in X, \pi \in S_{k,x}\}$. Thus $E(G') = \bigcup_{i=1}^{k} E'_i$.

It is easily verified that G' is a k-regular k-edge colorable graph and that each projection $\psi_{x,\pi}(A) = A(x,\pi), x \in X, \pi \in S_{k,x}$ is a covering projection onto H. Note however, that G'

need not be connected.

Finally, consider vertices $u, u_1, \dots, u_k$ of G' defined by

$$u(x, \pi) = x, x \in X, \pi \in S_{k,x},$$

$$u_j(x, \pi) = \pi(j), x \in X, \pi \in S_{k,x}, j = 1, 2, \dots, k.$$

Since $uu_j \in E'_j$ for every $j = 1, 2, \dots, k$, these vertices belong to the same connected component of G'. This connected component will be our graph G. Indeed, every covering projection $\psi_{x,\pi}$ sends u onto x, and for every bijection g from $N_G(u)$ onto $N_H(x)$, $\psi_{x,\pi}(u_j) = g(u_j)$, provided $\pi \in S_{k,x}$ is the bijection that satisfies $g(u_j) = \pi(j)$ for all $j = 1, 2, \dots, k$. □

3 Partial covers

In this section, we consider partial covers, which are a natural generalization of covers of regular graphs for the case when the big graph is not necessarily regular. Particularly interesting questions arise when the big graph is 'almost' regular (i.e., it is obtained from a regular graph by splitting one vertex into pendant vertices).

Definition 3.1 *A mapping f from the set of vertices of a graph G into the set of vertices of a graph H is called a* partial cover *of H, if*
(1) for all $x, y \in V(G)$, if $xy \in E(G)$ then $f(x)f(y) \in E(H)$, and
(2) for all $x, y \in V(G)$, if $d_G(x, y) = 2$ then $f(x) \neq f(y)$.

Note that the first condition states that f is a homomorphism from G to H, and the second condition requires that vertices which have a common neighbor are mapped onto different vertices of H. One can easily show that a graph G allows a partial cover to H if and only if G is a subgraph of a cover of H. Partial covers were investigated already by Nešetřil in [18] in connection with monoids of graph endomorphisms. The following proposition is clear.

Proposition 3.2 *If G and H are regular connected graphs of the same degree, then every partial cover $f : V(G) \to V(H)$ is a covering projection from G onto H.* □

Now consider regular graphs G and H and the graph G_u obtained from G by splitting a vertex u into pendant vertices. If G covers H and $f : V(G) \to V(H)$ is a covering projection, then $f_u : V(G_u) \to V(H)$ defined by $f_u(v) = f(v), v \in V(G) - \{u\}$ and $f_u(u_w) = f(u), w \in N_G(u)$, is a partial cover of H. In this sense, let us call a partial cover $g : V(G_u) \to V(H)$ *good* if $g = f_u$ for some covering projection $f : V(G) \to V(H)$ and *bad* otherwise. Note that a partial cover g is good if and only if it maps all the pendant vertices $u_w, w \in N_G(u)$ onto the same vertex of H and their neighbors $w, w \in N_G(u)$ onto distinct vertices.

Given a fixed regular graph H, our aim will be to construct graphs that cover H possibly in many different ways, but allow only good partial covers. Since the latter is not easy to check, we will rather show that for many graphs H, no graph that covers H allows a bad partial cover. Two different sufficient conditions are given in the following subsection.

3.1 Good covers

Theorem 2 *Let G and H be connected k-regular graphs such that the number of vertices of G is a multiple of the number of vertices of H. If H is $(\lfloor \frac{k}{2} \rfloor + 1)$-edge connected, then for any vertex u of G, any partial cover from G_u onto H is good.*

Theorem 3 *Let G and H be connected k-regular graphs such that the number of vertices of G is a multiple of the number of vertices of H. If H is k-edge colorable, then for any vertex u of G, any partial cover from G_u onto H is good.*

The rest of this section is devoted to the proofs of the theorems. Let n be the number of vertices of H and let G have $h \cdot n$ vertices. Further let $f : V(G_u) \to V(H)$ be a partial cover. For $x \in V(H)$, denote by h_x the number of vertices of degree k in G_u that f maps onto x, i.e., $h_x = |f^{-1}(x) \cap (V(G) - \{u\})|$. For $x, y \in V(H)$, let $c_{x,y}$ be the number of pendant edges of G_u that f maps onto xy, more precisely $c_{x,y} = |\{w : f(w) = x, f(u_w) = y\}|$. (Note that $\sum_{x,y \in V(H)} c_{x,y} = k$ and that for $xy \notin E(H)$, $c_{x,y} = 0$.) Finally, let $C = \{xy : c_{x,y} \neq 0 \text{ or } c_{y,x} \neq 0\} \subset E(H)$ be the set of the images of the pendant edges of G_u, and let $H' = (V(H), E(H) - C)$ be the graph obtained from H by deleting the edges of C. We denote $A_1, A_2, \ldots, A_m$ the connected components of H'.

Lemma 3.3 *If xy is an edge of H, then $h_x - h_y = c_{x,y} - c_{y,x}$.*

Proof. Let F_x be the set of vertices of G_u which are mapped onto x and which have a neighbor mapped onto y (in the partial cover f). Similarly, let F_y be the set of vertices mapped onto y which have a neighbor mapped onto x. Since f is a partial cover, every vertex of degree k which is mapped onto x is in F_x, and a pendant vertex u_w is in F_x if and only if $f(u_w) = x$ and $f(w) = y$. Hence $|F_x| = h_x + c_{y,x}$. Similarly, $|F_y| = h_y + c_{x,y}$. Since the subgraph of G_u induced by $F_x \cup F_y$ is a perfect matching, we have $|F_x| = |F_y|$, from which our lemma follows □

Corollary 3.4 *If x and y belong to the same connected component of H', then $h_x = h_y$.*

Proof. If xy is an edge of H', we have $c_{x,y} = c_{y,x} = 0$ and thus $h_x = h_y$ follows from the preceding lemma. If xy is not an edge, then x and y are connected by a path in H', and all vertices z along this path have the same h_z. □

In view of the corollary, we will introduce $h_i = h_x$ for any $x \in A_i, i = 1, 2, \ldots, m$.

Lemma 3.5 *If $h_i \leq h$ for every $i = 1, 2, \ldots, m$ then f is good.*

Proof. Since $hn - 1 = \sum_{x \in V(H)} h_x$, we see that $h_x = h$ for all but one vertex of H, say z, and $h_z = h - 1$. This means that z itself forms a connected component of H', and $c_{x,z} = c_{z,x} + 1 \geq 1$ for every $x \in N_H(z)$. As $k = \sum_{x,y \in V(H)} c_{x,y} \geq \sum_{x \in N_H(z)} c_{x,z} \geq k$, we see that $c_{x,z} = 1$ and $c_{z,x} = 0$ for every $x \in N_H(z)$. Hence $f(u_w) = z$ for every $w \in N_G(u)$ and $f(w) \neq f(v)$ for distinct $w, v \in N_G(u)$. $\square$

Corollary 3.6 *The graph H' has at least two connected components.*

Proof. If H' had just one connected component, one would have $hn - 1 = |V(G) - \{u\}| = \sum_{x \in V(H)} h_x = nh_1$, a contradiction. $\square$

Proof of Theorem 2. We will follow the route prepared in Lemmas 3.3 and 3.5 and Corollaries 3.4 and 3.6. We know that H' has at least two connected components. Every connected component is incident with at least $\lfloor \frac{k}{2} \rfloor + 1$ edges of C (due to the assumption about the edge connectivity of H). If H' had $m \geq 4$ components, we would have $k \geq |C| \geq (\lfloor \frac{k}{2} \rfloor + 1)\frac{m}{2} \geq (\lfloor \frac{k}{2} \rfloor + 1)2 \geq k + 1$, a contradiction. Thus H' has 2 or 3 connected components.

Suppose H' has exactly two components and f is not good. Then we may assume without loss of generality that $h_1 \geq h + 1$ and consequently, since $hn - 1 = \sum_{x \in V(H)} h_x$, $h_2 \leq h - 1$. It follows from Lemma 3.3 that $c_{x,y} \geq 2$ for every edge $xy \in E(H)$ such that $x \in A_1$ and $y \in A_2$. Due to the assumed edge connectivity of H, there are at least $\lfloor \frac{k}{2} \rfloor + 1$ such edges and we have $k = \sum_{x,y \in V(H)} c_{x,y} \geq \sum_{x \in A_1, y \in A_2} c_{x,y} \geq 2(\lfloor \frac{k}{2} \rfloor + 1) \geq k + 1$, a contradiction.

If H' has three components, we may again assume $h_1 \geq h + 1$ and $h_2 \leq h - 1$. Similarly to the previous case, we have $c_{x,y} \geq 2$ for every edge $xy \in E(H)$ such that $x \in A_1$ and $y \in A_2$. Denoting $C_{ij} = \{xy | xy \in C, x \in A_i, y \in A_j\}$, the connectivity assumption implies $|C_{12}| + |C_{13}| \geq \lfloor \frac{k}{2} \rfloor + 1$ and $|C_{12}| + |C_{23}| \geq \lfloor \frac{k}{2} \rfloor + 1$. Thus $k = \sum_{x,y \in V(H)} c_{x,y} \geq 2|C_{12}| + |C_{13}| + |C_{23}| \geq 2(\lfloor \frac{k}{2} \rfloor + 1) \geq k + 1$, a contradiction. $\square$

We will next attend to the case of k-edge colorable graphs. The following Parity Lemma is well known: In any legal k-edge coloring of a k-regular graph, a minimal edge-cut has the same parity of the number of edges colored by a particular color (for all colors). (A proof may go as follows: Consider colors b and w and the subgraph induced by the edges of these two colors. This subgraph is 2-regular, i.e. a disjoint union of cycles, and each cycle uses even number of edges of the cut.) We have a similar lemma for our G_u.

Lemma 3.7 *Let G be a k-regular graph with even number of vertices, and let u be any of its vertices. Then any legal k-edge coloring of G_u uses different colors on the pendant edges $wu_w, w \in N_G(u)$.*

Proof. Since G_u has the same number of pendant edges as the number of colors, either each of the colors is used exactly once on the pendant edges, or there is a color, say b, which is missing. Suppose that the latter case applies. Consider a color, say c, which is present on at least one of the pendant edges, and consider the subgraph induced by edges of colors b and c. Each connected component of this graph is either a cycle within $V(G) - \{u\}$, or a path of odd length of type $u_w, w, \ldots, v, u_v$ for some $w, v \in N_G(u)$. (The length of the path is odd, because the colors of its edges alternate along the path and both edges $u_w w$ and $u_v v$ have color c.) It follows that each of these components uses even number of vertices of $V(G) - \{u\}$ and they span this set of vertices. This contradicts the assumption about the number of vertices of G. □

The following lemma is a direct consequence of the parity lemma.

Lemma 3.8 *Let H be a connected k-regular graph and let $C \subset V(G)$ be an edge-cut of size k such that in some legal k-edge coloring, all edges of C have different colors. Then C is a minimal cut.*

Proof. If C were not a minimal cut, H would have a nonempty edge cut $C' \subset C$ of size less than k. In the assumed coloring, the colors appearing on the edges of C' would appear exactly once each (i.e., each odd number of times) and the colors appearing on edges of $C - C'$ would be missing on C' (i.e., appearing on C' even number of times). That would contradict the parity lemma. □

Proof of Theorem 3. Note that any k-regular k-edge colorable graph has even number of vertices. Fix a legal k-edge coloring of H, say $\varphi : E(H) \to \{1, 2, \ldots, k\}$. A coloring $\psi : E(G_u) \to \{1, 2, \ldots, k\}$, defined by $\psi(xy) = \varphi(f(x)f(y))$ is a legal edge coloring of G_u.

Since all the pendant edges of G_u have different colors in ψ (Lemma 3.7), C is a set of k different edges of H, and these edges have different colors in φ. By Corollary 3.6 and Lemma 3.8, C is a minimal cut and H' has exactly two connected components. It follows that for every edge $xy \in C$, either $c_{x,y} = 0$ and $c_{y,x} = 1$ or vice versa. Hence, by Lemma 3.3, $|h_1 - h_2| = 1$. Since $\sum_{x \in V(H)} h_x = nh_1$, $\max\{h_1, h_2\} \leq h$ and the statement follows from Lemma 3.5. □

3.2 Bad covers

We will show an example of a cubic (not 2-connected) graph which allows a bad partial cover from some G_u.

Proposition 3.9 *Let H be a cubic graph with one articulation point v such that $G - v$ consists of three isomorphic components. Then there exists a graph G with a vertex u such that G covers H, G_u is connected and G_u allows a bad partial cover of H.*

Proof. Let A, B, C be the three components of $H - v$, $(A \cong B \cong C)$ and let a, b, c be the vertices of degree 2 in A, B, C, respectively. Let D be a 3-fold cover of A (i.e., D covers A and $|V(D)| = 3|V(A)|$). Then D has 3 vertices of degree 2 and all of them are mapped onto a by any covering projection of D onto A. Since A, B and C are isomorphic, D also covers B and C. Furthermore, take a 2-fold cover of A, say E.

Our graph G will consist of 3 disjoint copies of D (called D_1, D_2, D_3), 6 disjoint copies of E (called $E_1, \ldots, E_6$), 6 extra vertices $w_1, \ldots, w_6$ and a special vertex u. Figure 1 shows the graphs H and G_u together with a good and bad partial cover of G_u onto H.

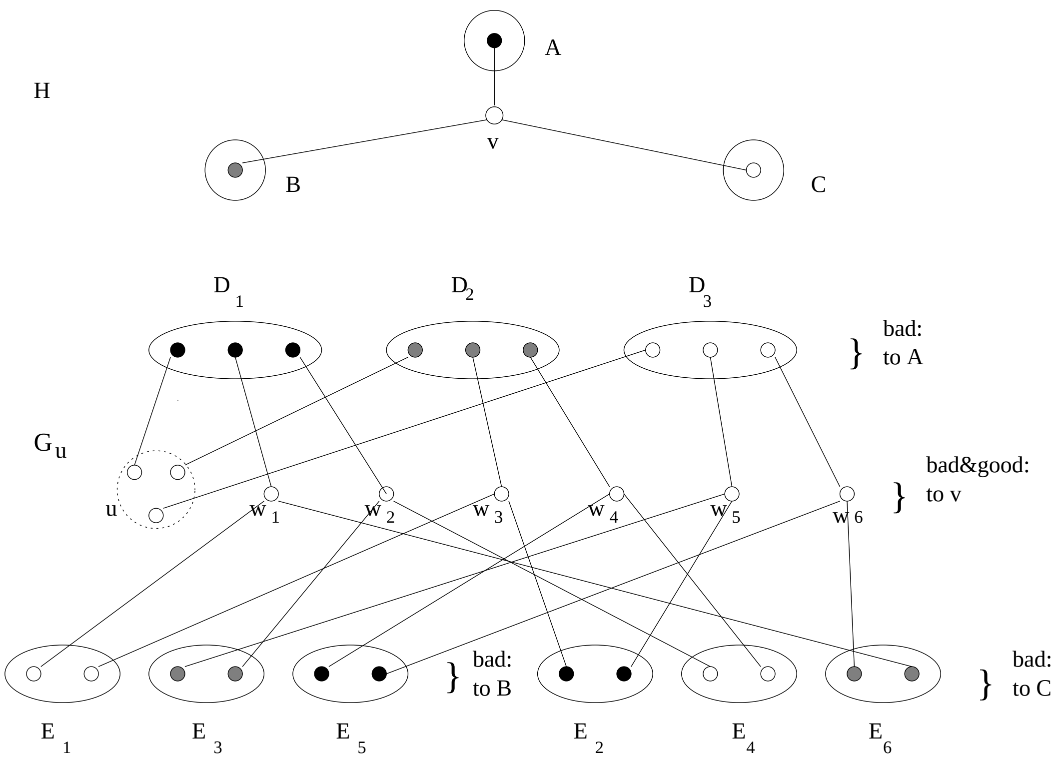

Figure 1: Both a good partial cover and a bad partial cover from G_u (bottom) onto H is shown. In both coverings the 3 pendant vertices from u and $w_1, \ldots, w_6$ are mapped to v. The good partial cover is otherwise specified by vertex shading (note neighbors of pendant vertices are mapped to different neighbors of v), and the bad partial cover by physical placement (note neighbors of pendant vertices are mapped to the same neighbor of v in A).

The vertices of degree 2 in D_i are x_{i1}, x_{i2}, x_{i3}, $i = 1,2,3$, and the vertices of degree 2 in E_j are y_{j1}, y_{j2}, $j = 1,2,\ldots,6$. The additional edges, making G a cubic graph, are $ux_{11}, ux_{21}, ux_{31}$, $w_1x_{12}, w_2x_{13}, w_3x_{22}, w_4x_{23}$, $w_5x_{32}, w_6x_{33}, w_1y_{11}$, $w_3y_{12}, w_3y_{21}, w_5y_{22}, w_5y_{31}$, $w_2y_{32}, w_2y_{41}, w_4y_{42}, w_4y_{51}, w_6y_{52}, w_6y_{61}, w_1y_{62}$. Obviously, G_u is connected.

A mapping obtained from covering projections of D_1, E_2 and E_5 onto A, D_2, E_3 and E_6 onto B, D_3, E_1 and E_4 onto C and which maps the additional vertices $u, w_1, \ldots, w_6$ onto v is a covering projection from G onto H.

Similarly, a mapping obtained from covering projections of D_1, D_2 and D_3 onto A, E_1, E_3 and E_5 onto B, E_2, E_4 and E_6 onto C and which maps the additional vertices $w_1, \ldots, w_6$ as well as the split vertices $u_{x_{11}}, u_{x_{21}}, u_{x_{31}}$ onto v is a bad partial covering projection from G_u onto H. $\square$

4 Complexity of regular covers

Abello et al. [1] first asked the question: "For which graphs H is the H-cover problem NP-complete and for which is it polynomially solvable?" (The H-cover problem takes a graph G as its input and asks if G covers H.) Fellows [personal communication] suggested that this problem might be polynomially solvable for rigid graphs H (i.e., for graphs with trivial automorphism group). We will show that this is (subject to P$\neq$NP) not the case, in fact, there are infinitely many rigid cubic graphs H for which H-COVER is NP-complete. Let us remark that until recently, all NP-completeness reductions were based on the symmetries of the parameter graph H and could not have been used for rigid graphs.

4.1 Auxiliary complexity lemma

A *hypergraph* is a pair (V, F), where V is a set of vertices and F is a family of subsets of V. The elements of F are called hyperedges. A hypergraph is *s-uniform* if every hyperedge has size s, and it is *k-regular* if every vertex belongs to exactly k-hyperedges. (Thus a 2-uniform hypergraph is a multigraph in the usual sense.) A coloring of the edges of a hypergraph is called *legal* if no vertex belongs to two or more edges of the same color. A hypergraph is called *k-edge colorable* if its edges can be legally colored by k colors.

Lemma 4.1 *For any fixed $k > 2$ and $s \geq 2$, it is NP-complete to decide if a given k-regular s-uniform hypergraph is k-edge colorable.*

Proof. For $k = 3$ and $s = 2$, the statement becomes the well known theorem of Holyer [14] about 3-edge colorablity of cubic graphs. Also for $s = 2$ and any fixed $k > 3$, the statement is known to be true, cf. [16]. We show a reduction from this latter problem to one involving a general $s > 2$. Given a k-regular graph $G = (V, E)$, we construct an s-uniform k-regular hypergraph H so that H is k-edge colorable if and only if G is. That will prove our lemma.

For the construction of H, first take s disjoint copies of G, and call them $G_i = (V_i, E_i)$, $1 \leq i \leq s$ (an edge $e \in E$ is considered a two-element set; the corresponding edge in $E(G_i)$ is denoted e_i). For every edge $e \in E$, introduce new vertices e_{ij}, $1 \leq i \leq s$, $\leq j \leq s-2$. The vertex set of H is thus

$$V(H) = \bigcup_{i=1}^{s} (V_i \cup \{e_{ij} | e \in E, 1 \leq j \leq s-2\}).$$

The s-uniform hyperedges of H are

$$S(e,i) = e_i \cup \{e_{ij} | 1 \leq j \leq s-2\}, \;\; 1 \leq i \leq s, \; e \in E$$

and

$$T_h(e,j) = \{e_{ij} | 1 \leq i \leq s\}, \;\; 1 \leq j \leq s-2, \; 1 \leq h \leq k-1, \; e \in E.$$

Obviously, if H is k-edge colorable, a restriction of any legal coloring to G_1 is a legal k-edge coloring of G.

If $\varphi : E \to \{1, 2, \ldots, k\}$ is a legal k-edge coloring of G, we define
$\psi(S(e,i)) = \varphi(e), i = 1, 2, \ldots, s, \; e \in E,$
$\psi(T_h(e,j)) = h, j = 1, 2, \ldots, s-2, \; h = 1, 2, \ldots, \varphi(e) - 1, \; e \in E,$
$\psi(T_h(e,j)) = h + 1, j = 1, 2, \ldots, s-2, \; h = \varphi(e), \ldots, k-1, \; e \in E.$
This ψ is a legal k-edge coloring of H. □

4.2 NP-completeness of regular covers

Theorem 4 *Let H be a connected k-regular graph. Then the H-cover problem is NP-complete, if there exists a vertex x of H and a k-regular graph G with a specified vertex u such that*

(1) for every bijective mapping g of the neighbors of u onto the neighbors of x, there exists a covering projection from G onto H that extends g and maps u onto x;

(2) for every neighbor y of x and for every bijective mapping g of the neighbors of u onto the neighbors of y, there exists a covering projection from G onto H that extends g and maps u onto y; and

(3) every partial cover $f : V(G_u) \to V(H)$ satisfies $f(u_w) = f(u_v)$ and $f(w) \neq f(v)$ for any two neighbors v, w of u.

Proof. We show a reduction from k-edge colorability of k-regular $(k-1)$-uniform hypergraphs to H-cover.

Let $K = (V, F)$ be a given k-regular $(k-1)$-uniform hypergraph. We will construct a k-regular graph G_K so that G_K covers H if and only if K is k-edge colorable. This construction will utilize the graph G from the assumptions of the theorem.

For every vertex $v \in V$, we introduce a vertex gadget G^v that will consist of one copy of G_u and the isolated vertex v. The pendant vertices of this copy of G_u will be denoted u_{ve} and their corresponding neighbors will be denoted w_{ve}, for each hyperedge e that contains v (note that there are exactly k such hyperedges).

For every hyperedge e and every vertex v of e, we introduce two copies of G_u, one called G_u^{ve} and the other one G_u^{ev}. The pendant vertices of G_u^{ve} will be denoted u_i^{ve} and their corresponding neighbors w_i^{ve} $(i = 1, 2, \ldots, k)$. Similarly, the pendant vertices of G_u^{ev} will be denoted u_i^{ev} and their neighbors w_i^{ev} $(i = 1, 2, \ldots, k)$. These $2k-2$ copies of G_u will be glued together by a series of unifications. For every $i = 2, 3, \ldots, k$, the $k-1$ vertices $u_i^{ve}, v \in e$ will be unified into one vertex called u_i^e, and the $k-1$ vertices $u_i^{ev}, v \in e$ will be unified into one vertex called $\bar{u}_i^e$. Finally, the edges $u_i^e \bar{u}_i^e, i = 2, 3, \ldots, k$ are added to form the edge gadget G^e.

The edge gadgets G^e and the vertex gadgets G^v are linked together by further unifications. For every hyperedge e and every vertex $v \in e$, we unify v with u_1^{ve}, w_{ve} with u_1^{ev} and u_{ve} with w_1^{ev}. This is our graph G_K. (Note that besides the explicitly shown unifications, the copies of G_u used in the construction are disjoint. An illustrative example of the construction appears in Figure 2.)

We first show that if G_K covers H then K is k-edge colorable. Suppose $f : V(G_K) \to V(H)$ is a covering projection. Note that for every copy of G_u that was used in the construction of G_K, the restriction of f to that copy of G_u is a partial cover of H. It thus follows, due to assumption (3), that all pendant vertices of that copy G_u are mapped onto the same vertex of H. Hence for any hyperedge e and two of its vertices v, t,

$$f(v) = f(u_1^{ve}) = f(u_2^{ve}) = f(u_2^e) = f(u_2^{te}) = f(u_1^{te}) = f(t),$$

and (assuming connectedness of K), $f(v) = f(v')$ for any two vertices v, v' of K. Let z be the vertex of H onto which all vertices of K are mapped (we have already seen that then $f(u_i^e) = z$ for every $e \in F$ and $i = 2, 3, \ldots, k$ as well).

Consider a hyperedge e and one of its vertices $v \in e$, and let $f(w_{ve}) = y$. It follows from G_u^{ev} that $f(\bar{u}_i^e) = y$ for $i = 2, 3, \ldots, k$ and hence $f(w_{te}) = y$ for every other $t \in e$. Since $u_2^e \bar{u}_2^e \in E(G_K)$, $y \in N_H(z)$. This allows us to color the hyperedges of K by neighbors of z in H (following the rule $\varphi(e) = f(\bar{u}_2^e) = f(w_{ve})$ for any $v \in e \in F$). This coloring is legal because of the ssumption (3) which (when applied to the copy of G_u in the vertex gadget G^v) implies that for every $v \in V$, $f(w_{ve}) \neq f(w_{vf})$ for any two distinct hyperedges e, f containing v.

For the other implication, suppose that K is k-edge colorable. Fix a legal coloring $\varphi : F \to N_H(x)$ (i.e., we identify the colors with the neighbors of vertex x in H). We predefine a partial covering projection on the connecting vertices of the copies of G_u using a mapping g.

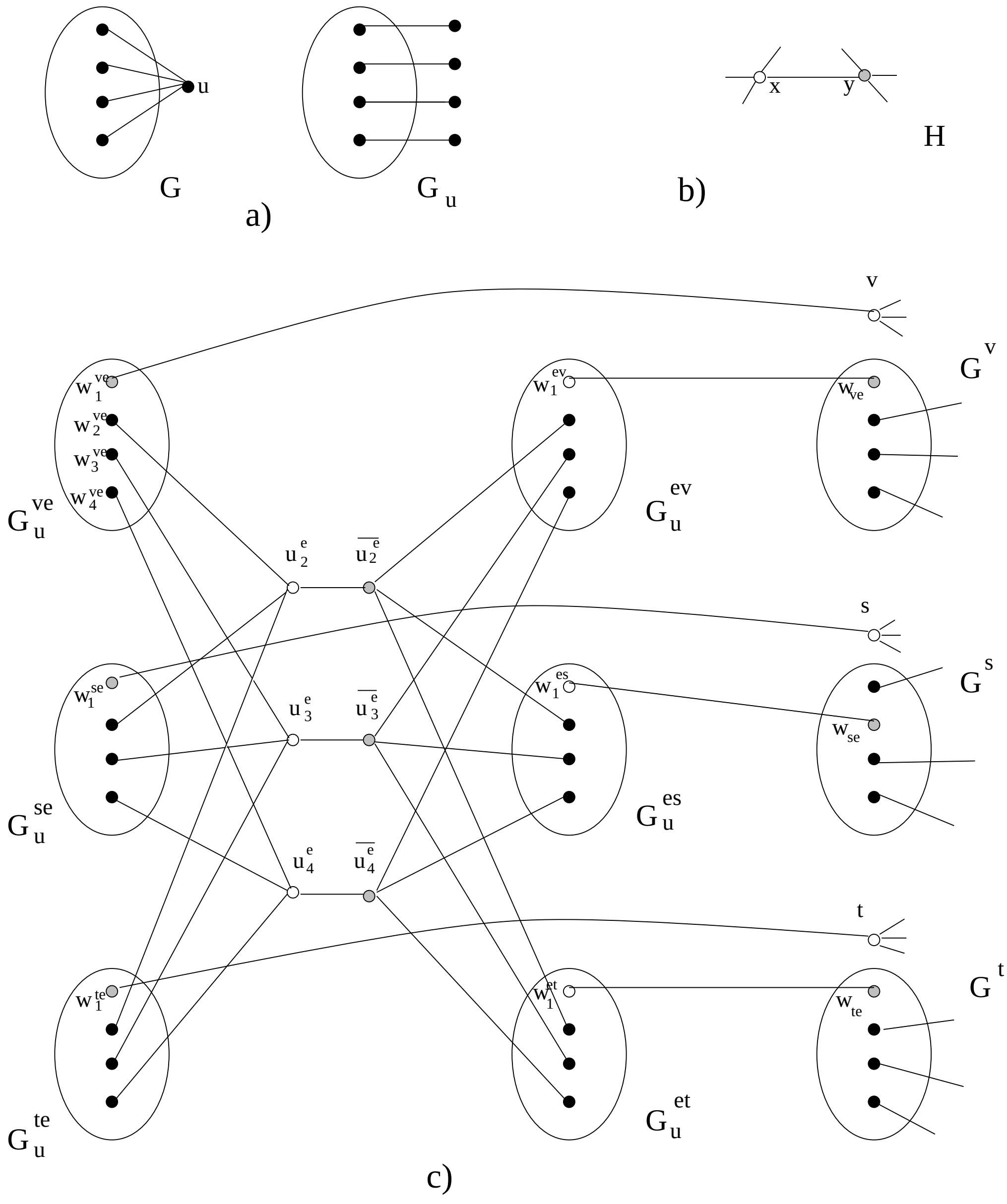

Figure 2: The construction for the case $k = 4$. a) Graphs G and G_u. b) Part of the 4-regular graph H. c) Vertex gadgets G^v, G^s, G^t of G_K connected by an edge gadget, resulting from the hyperedge $e = \{v, s, t\}$ of K. A covering from G_K to H is shown, with white vertices mapped to $x \in V(H)$, shaded vertices to $y \in V(H)$ and black vertices to remaining neighbors of y and x. The edge e is colored $y \in N_H(x)$.

For every particular copy of G_u, its pendant vertices will be mapped onto the same vertex of H (either x or a neighbor of x), and their neighbors will be mapped onto distinct neighbors of this vertex. Thus assumptions (1) and (2) will allow us to extend g to f locally within each copy of G_u. The mapping f will then be the desired covering projection of G_K onto H (the reader may verify that g fully satisfies all pendant vertices of particular copies of G_u). It only remains to show the mapping g.

For every $v \in V$, we set $g(v) = x$.

For every hyperedge $e = \{v_1, v_2, \ldots, v_{k-1}\} \in F$, we do the following. Let $\varphi(e) = y$, $(y \in N_H(x))$. Let $y_0, y_1, \ldots, y_{k-2}$ be the other neighbors of x and let $x_0, x_1, \ldots, x_{k-2}$ be the other neighbors of y (in H). For every $i = 1, 2, \ldots, k-1, j = 2, 3, \ldots, k$, we set

$g(w_{v_i e}) = y$,
$g(\bar{u}_j^e) = y$,
$g(w_1^{v_i e}) = y$,
$g(u_j^e) = x$,
$g(w_1^{e v_i}) = x$,
$g(w_j^{v_i e}) = y_{i+j}$,
$g(w_j^{e v_i}) = x_{i+j}$

(the addition in the subscripts of the last two assignments is modulo $k-1$). □

Corollary 4.2 *For every fixed $k > 2$ and every k-regular graph H, the H-cover problem is NP-complete if*

H is $(1 + \lfloor \frac{k}{2} \rfloor)$-edge connected, or

H is k-edge colorable.

Proof. Theorem 1 ensures the existence of gadgets satisfying conditions (1) and (2) of Theorem 4 and Theorems 2 and 3 guarantee that any such gadget satisfies the condition (3) as well (provided H is as in above). □

In [15], we have stated as one of the main open problems whether there exist rigid regular graphs H, for which the H-cover problem is NP-complete. Using methods similar to those developed in [11], one can construct infinitely many rigid k-edge colorable k-regular graphs for any $k \geq 3$ [J. Nešetřil: personal communication]. Therefore our question is answered affirmatively:

Corollary 4.3 *For every $k \geq 3$, there are infinitely many rigid k-regular graphs H for which the H-cover problem is NP-complete.* □

5 Final remarks and open problems

Our main conjecture, that we have solved only partially in this paper, is expressed as follows:

Problem 1. Show that the H-cover problem is NP-complete for every regular graph H of degree at least 3.

We are still optimistic in suggesting that the full conjecture can be settled with the same reduction that we gave above. It would be sufficient to prove:

Problem 2. For every regular graph H there exists a vertex $x \in V(H)$ and a regular graph G with a specified vertex u, such that the conditions (1), (2) and (3) of Theorem 4 are fulfilled.

It is still possible that the construction used in Theorem 1 satisfies our requirements.

The graph H of Section 3.2 shows that the condition of $(\lfloor \frac{k}{2} \rfloor + 1)$-edge connectedness in Theorem 2 is necessary for $k = 3$. Fiala [7] gave for each k divisible by 4 a construction of $\frac{k}{2}$-edge connected k-regular graphs which allow bad covers, thus showing that Theorem 2 is essentially tight. The following problem, which we believe is interesting in its own right, is still open:

Problem 3. Characterize the class of all connected regular graphs H such that for every graph G that covers H and every vertex u of G, all partial covers from G_u onto H are good.

References

[1] J. Abello, M.R. Fellows and J.C. Stillwell, On the complexity and combinatorics of covering finite complexes, *Australasian Journal of Combinatorics 4* (1991), 103-112

[2] D. Angluin, Local and global properties in networks of processors, in *Proceedings of the 12th ACM Symposium on Theory of Computing* (1980), 82-93

[3] D. Angluin and A. Gardner, Finite common coverings of pairs of regular graphs, *Journal of Combinatorial Theory B 30* (1981), 184-187

[4] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1974

[5] N. Biggs, Homological coverings of graphs, *J. London Math. Soc.* (2), 30 (1984), 1-14

[6] B. Courcelle and Y. Métivier, Coverings and minors: Applications to local computations in graphs, *European Journal of Combinatorics* 15 (1994), 127-138

[7] J. Fiala, Covers of graphs (in Czech), Master Thesis, Charles University, 1997 (in preparation)

[8] M.R. Garey and D.S. Johnson, *Computers and Intractability*, W.H.Freeman and Co., 1978

[9] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, J.Wiley and Sons, 1987

[10] J.L. Gross and T.W. Tucker, Generating all graph coverings by permutation voltage assignments, *Discrete Math.* 18 (1977) 273-283

[11] P. Hell and J. Nešetřil, Groups and monoids of regular graphs (and of graphs with bounded degrees), *Can. J. Math.* 25 (1973) 239-251

[12] M Hofmeister, Isomorphisms and automorphisms of graph coverings, *Discrete Math.* 98 (1991) 175-183

[13] P. Hell and J. Nešetřil, On the complexity of H-colouring, *Journal of Combinatorial Theory B 48* (1990), 92-110

[14] I. Holyer, The $\mathcal{NP}$-completeness of edge-coloring, *SIAM J. Computing 4* (1981) 718-720

[15] J. Kratochvíl, A. Proskurowski, J.A. Telle, On the complexity of graph covering problems, In: Graph-Theoretic Concepts in Computer Science, Proceedings of 20th International Workshop WG'94, Munchen, Germany, 1994, Lecture Notes in Computer Science 903, Springer Verlag, Berlin Heidelberg, 1995, pp. 93-105

[16] D. Leven, Z. Galil, NP-completeness of finding the chromatic index of regular graphs, J. Algoritms 4 (1983) 35-44

[17] W. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace and World, 1967

[18] J. Nešetřil, Homomorphism of derivative graphs, *Discrete Math.* 1 (1971) 257-268