Complexity of Colored Graph Covers I. Colored Directed Multigraphs.

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Abstract. A covering projection from a graph G onto a graph H is a "local isomorphism": a mapping from the vertex set of G onto the vertex set of H such that, for every $v \in V(G)$, the neighborhood of v is mapped bijectively onto the neighborhood (in H) of the image of v. We continue the investigation of the computational complexity of the H-cover problem – deciding if a given graph G covers H. We introduce a more general notion of covers of directed colored multigraphs (cdm-graphs) and show that a complete characterization of the complexity of covering of simple undirected graphs would necessarily resolve the complexity of covering of cdm-graphs as well. On the other hand, we introduce reductions that will enable to consider only multigraphs with minimum degree ≥ 3. We illustrate the methodology by presenting a complete characterization of the complexity of covering problems for two-vertex cdm-graphs.

1 Motivation and Overview

For a fixed graph H, the H-cover problem admits a graph G as input and asks about the existence of a "local isomorphism": a labeling of the vertices of G by vertices of H so that the label set of the neighborhood of every $v \in V(G)$ is equal to the neighborhood (in H) of the label of v and each neighbor of v is labeled by a different neighbor of the label of v. Such a labeling is referred to as a *covering projection* from G onto H. We trace this concept to Biggs' construction of highly symmetric graphs in [4], and to Angluin's discussion of "local knowledge" in distributed computing environment in [2]. More recently, Abello *et al.* [1] raised the question of computational complexity of H-cover problems, noting that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. We have studied the question of complexity of graph covering problems in [8], where one of our main results was a complete catalogue of the complexity of this problem for simple graphs on at most 6 vertices. In [9], we proved that the

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H-cover problem is NP-complete for any k-regular graph H, provided $k > 2$ and H is k-edge colorable or $(1 + \lfloor \frac{k}{2} \rfloor)$-edge connected. This is a significant headway towards the more general conjecture stating that the H-cover problem is NP-complete for every k-regular graph, $k > 2$. In particular, it follows that there are infnitely many rigid graphs for which the covering problem is NP-complete.

In this paper we introduce covers of colored directed multigraphs (cdm-graphs). Though this notion may seem too general at first sight, it is readily seen that covers of colored directed multigraphs can be encoded in terms of covers of simple graphs. On the other hand the language of cdm-graphs enables more compact description of the results. We will show that it suffices to consider covers of cdm-graphs with minimum degree ≥ 3.

2 Colored Directed Multigraphs

In this paper, we consider *colored directed multigraphs*, shortly *cdm-graphs.* A directed multigraph with vertex set V has edge set $E = D \cup F \cup L$, where D is the set of directed edges (including directed loops), F is the set of undirected edges and L is the set of undirected loops. A function $\mu : E \to (V \times V) \cup \binom{V}{2} \cup V$ describes the incidencies, i.e., for an undirected edge $e \in F$, $\mu(e) \in \binom{V}{2}$ is the pair of vertices connected by e, for a loop $e \in L$, $\mu(e) \in V$ is the vertex hosting the loop e and for a directed edge $e \in D$, $\mu(e) \in V \times V$ is the ordered pair of vertices connected by e. Vertices and edges are colored by a coloring $C : V \cup E \to C(V \cup E)$ (this coloring need not be proper in the sense that adjacent vertices and/or edges may receive the same color). Since vertices, directed and undirected edges may be distinguished regardless the color, we may assume that $C(V)$, $C(D)$ and $C(F \cup L)$ are disjoint. A cdm-graph is uncolored if $C(V)$, $C(D)$ and $C(F \cup L)$ are one-element set each.

For an edge-color c, the *c-colored degree* of a vertex $x \in V(G)$ is defined as

$$deg_G^c(x) = |\{e : x \in \mu(e), e \in F(G)\}| + 2|\{e : x = \mu(e), e \in L(G), C(e) = c\}|$$

for $c \in C(F \cup L)$, and

$$deg_G^c(x) = (deg_G^{c-}(x), deg_G^{c+}(x))$$

where

$$deg_G^{c-}(x) = |\{e : \mu(e) = (x,u) \text{ for some } u, e \in D(G), C(e) = c\}|$$

and

$$deg_G^{c+}(x) = |\{e : \mu(e) = (u,x) \text{ for some } u, e \in D(G), C(e) = c\}|.$$

The *total degree* of a vertex u is

$$deg_G u = \sum_{c \in C(F \cup L)} deg^c(u) + \sum_{c \in C(D)} (deg^{c+}(u) + deg^{c-}(u)).$$

Definition 1. A *covering projection of a cdm-graph G onto a cdm-graph H* is a mapping $f : V(G) \cup E(G) \to V(H) \cup E(H)$ such that

(1) $f(u) \in V(H)$ and $C(f(u)) = C(u)$ for every $u \in V(G)$,

(2) $f(e) \in D(H)$, $C(f(e)) = C(e)$ and $\mu(f(e)) = (f(u), f(v))$ for every $e \in D(G)$ such that $\mu(e) = (u, v)$,

(3) $f(e) \in F(H) \cup L(H)$, $C(f(e)) = C(e)$ and $\mu(f(e)) = \{f(u), f(v)\}$ for every $e \in F(G) \cup L(G)$ such that $\mu(e) = \{u, v\}$,

(4) for every $u \in V(G)$ and every $e \in D(H)$ such that $\mu(e) = (f(u), w)$ ($\mu(e) = (w, f(u))$) for some w, there is exactly one arc $e' \in D(G)$ such that $\mu(e') = (u, w')$ ($\mu(e') = (w', u)$, respectively) for some w' and $f(e') = e$,

(5) for every $u \in V(G)$ and every $e \in F(H)$ such that $f(u) \in \mu(e)$, there is exactly one edge $e' \in F(G)$ such that $u \in \mu(e')$ and $f(e') = e$,

(6) for every $u \in V(G)$ and every $e \in L(H)$ such that $\mu(e) = f(u)$, there is either exactly one loop $e' \in L(G)$ such that $\mu(e') = u$ and $f(e') = e$, or there are exactly two edges $e', e'' \in F(G)$ such that $u \in \mu(e'), u \in \mu(e'')$ and $f(e') = f(e'') = e$.

The above definition follows the usual definition of topological covering spaces. It is somewhat lengthy because of the presence of edges of different types. Note that (3) implies that every undirected loop of G is mapped (in a covering projection) again onto a loop in H, however the preimage of a loop need not be a loop itself. In general, the preimage of a loop in H is a disjoint union of cycles in G. In the case of simple undirected graphs a covering projection is obviously uniquely determined by its restriction to the vertex set. Theorem 3 shows that as far as the existence of a covering projection is concerned, this is true also for cdm-graphs.

Definition 2. A *vertex-covering projection* is a mapping $g : V(G) \to V(H)$ such that

(1') $C(g(u)) = C(u)$ for every $u \in V(G)$,

(4') for every $u \in V(G)$, $w \in V(H)$ and every edge color $c \in C(D)$,

$$|\{e \in D(G) : \mu(e) = (u, x), g(x) = w, C(e) = c\}| =$$

$$|\{e' \in D(H) : \mu(e') = (g(u), w), C(e') = c\}|$$

and

$$|\{e \in D(G) : \mu(e) = (x, u), g(x) = w, C(e) = c\}| =$$

$$|\{e' \in D(H) : \mu(e') = (w, g(u)), C(e') = c\}|$$

(5') for every $u \in V(G)$, $w \in V(H)$ such that $w \neq g(u)$, and every $c \in C(F \cup L)$,

$$|\{e \in F(G) : \mu(e) = \{x, u\}, g(x) = w, C(e) = c\}| =$$

$$|\{e' \in F(H) : \mu(e') = \{w, g(u)\}, C(e') = c\}|$$

(6') for every $u \in V(G)$ and every $c \in C(F \cup L)$,

$$|\{e \in F(G) : \mu(e) = \{x, u\}, g(x) = g(u), C(e) = c\}| +$$
$$2|\{e \in L(G) : \mu(e) = u, C(e) = c\}| =$$
$$2|\{e' \in L(H) : \mu(e') = g(u), C(e') = c\}|.$$

Theorem 3. *A cdm-graph G covers a cdm-graph H if and only if there exists a vertex-covering projection of G onto H.*

Proof. If f is a covering projection of G onto H then the restriction of f to the vertex set of G is a color-preserving projection satisfying (4'-6').

Suppose on the other hand that $g : V(G) \to V(H)$ is a color preserving mapping satisfying (4'-6'). In particular, (1) is fulfilled for g. We will show how to extend g to a mapping f defined also on the edges of G so that (2-6) are fulfilled as well.

Fix an edge-color, say c, and consider only the edges of color c (in G and in H as well). Suppose first that the edges of color c are undirected. Consider a vertex $u \in V(H)$ and let there be k loops $l_1, l_2, \dots, l_k$ of color c around u in H. Let G^u be the subgraph of G induced by the vertices mapped onto u and edges among them of color c. (I.e., $V(G^u) = g^{-1}(u)$ and $F(G^u) \cup L(G^u) = \{e \in F(G) \cup L(G) : C(e) = c, \mu(e) \subset g^{-1}(u)\}$.) It follows from (6') that G^u is a $2k$-regular multigraph, and by Petersen theorem, G^u is 2-factorable. Let $E_1, \dots, E_k$ be a collection of 2-factors that partitions $F(G^u) \cup L(G^u)$. We define

$$f(e) = l_i \text{ iff } e \in E_i.$$

Straightforwardly, (6) is satisfied for f.

Next consider vertices $u \neq v \in V(H)$ and let there be k edges $e_1, e_2, \dots, e_k$ of color c in H such that $\mu(e_i) = \{u, v\}$. Let G^{uv} be the subgraph of G induced by the vertices mapped onto u or v and edges among them of color c. (I.e., $V(G^{uv}) = g^{-1}(\{u, v\})$ and $F(G^{uv}) = \{e \in F(G) : C(e) = c, \mu(e) = \{x, y\}$ for some $x \in g^{-1}(u), y \in g^{-1}(v)\}$.) It follows from (5') that G^{uv} is a k-regular multigraph, and since it is also bipartite, G^{uv} is 1-factorable (König-Hall theorem). Let $E_1, \dots, E_k$ be a collection of perfect matchings that partitions $F(G^{uv})$. We define

$$f(e) = e_i \text{ iff } e \in E_i.$$

Straightforwardly, (5) is satisfied for f.

Now suppose that the edges of color c are directed. Though we formally do not distinguish directed edges and directed loops, we need to make the distinction for this proof. Consider a vertex $u \in V(H)$ and let there be k directed loops $l_1, l_2, \dots, l_k$ of color c around u in H. Let G^u be the subgraph of G induced by the vertices mapped onto u and edges among them of color c. (I.e., $V(G^u) = g^{-1}(u)$ and $D(G^u) = \{e \in D(G) : C(e) = c, \mu(e) \in g^{-1}(u) \times g^{-1}(u)\}$.) It follows from (4') that G^u has all indegrees and outdegrees k. Similarly as in the undirected case, the edge set of G^u can be partitioned into sets $E_1, \ldots, E_k$, each of which is a disjoint union of directed cycles. We then set

$$f(e) = l_i \text{ iff } e \in E_i.$$

For non-loop directed edges, consider vertices $u \neq v \in V(H)$ and let there be k directed edges $e_1, e_2, \ldots, e_k$ of color c in H such that $\mu(e_i) = (u, v)$. Let G^{uv} be the subgraph of G induced by the vertices mapped onto u or v and edges $D(G^u) = \{e \in D(G) : C(e) = c, \mu(e) \in g^{-1}(u) \times g^{-1}(v)\}$.) It follows from (4') that each vertex of G^{uv} is either a sink of indegree k or a source of outdegree k. Thus G^{uv} is bipartite and, by König-Hall theorem, G^{uv} is 1-factorable. Let $E_1, \ldots, E_k$ be a collection of perfect matchings that partitions $F(G^{uv})$. We set

$$f(e) = e_i \text{ iff } e \in E_i.$$

Straightforwardly, (4) is satisfied for f.

It follows from the construction that f preserves also the edge-colors and that the mapping of the edges is compatible with the mapping of their endpoints, i.e., f satisfies (2-3) as well.

Let us note that the existence of a vertex-covering projection is an obvious necessary condition for a cdm-graph G to cover a cdm-graph H. Thus Theorem 3 describes an "oncas" situation ("obvious necessary conditions are sufficient").

It is clear that a nonconnected cdm-graph G covers H if and only if every connected component of G covers H, and a connected G covers a nonconnected H if and only if G covers at least one connected component of H. Therefore we assume in the rest of the paper that both G and H are connected. It is then easy to see that the preimages of the vertices of H have the same size, and every cdm-graph G that covers H is an h-fold cover for some h ($h = |\{x \in V(G) : g(x) = u\}|$ for any $u \in V(H)$).

Definition 4. The *degree partition* of a cdm-graph G is the coarsest partition of $V(G)$ into monochromatic equivalence classes $B_1, \ldots, B_k$ such that there exist numbers $r_{ij}^c, d_{ij}^{c+}, d_{ij}^{c-}$ ($i, j = 1, 2, \ldots, k$, $c \in C(D) \cup C(F) \cup C(L)$) such that

(i) for every i, j and every $u \in B_i$,

$$|\{e \in D(G) : \mu(e) \in \{u\} \times B_j, C(e) = c\}| = d_{ij}^{c+},$$

(ii) for every i, j and every $u \in B_i$,

$$|\{e \in D(G) : \mu(e) \in \{u\} \times B_j, C(e) = c\}| = d_{ij}^{c-},$$

(iii) for every $i \neq j$ and every $u \in B_i$,

$$|\{e \in F(G) : u \in \mu(e), \mu(e) \setminus \{u\} \in B_j, C(e) = c\}| = r_{ij}^c,$$

(iv) for every i and every $u \in B_i$,

$$|\{e \in F(G) : u \in \mu(e), \mu(e) \setminus \{u\} \in B_i, C(e) = c\}| +$$
$$2|\{e \in L(G) : \mu(e) = u, C(e) = c\}| = r^c_{ii}.$$

The collection $r^c_{ij}, d^{c+}_{ij}, d^{c-}_{ij}$ $(i, j = 1, 2, \ldots, k,\ c \in C(D) \cup C(F) \cup C(L))$ is then called the *degree refinement* of G.

As in the case of simple undirected graphs, also for cdm-graphs, the degree partition is unique and can be determined in polynomial time (starting with the partition into vertex-color classes and refining this partition iteratively). The following is a direct corollary of Theorem 3:

Corollary 5. *If a cdm-graph G covers a cdm-graph H then G and H have the same degree refinements. Moreover, if $B_1, \ldots, B_k$ is the degree partition of G and $B'_1, \ldots, B'_k$ the degree partition of H (indexed so that $r^c_{ij} = r'^c_{ij}$, $d^{c+}_{ij} = d'^{c+}_{ij}$ and $d^{c-}_{ij} = d'^{c-}_{ij}$ for all $i, j = 1, 2, \ldots, k$ and $c \in C(D) \cup C(F) \cup C(L)$) then $f(B_i) = B'_i$ for every i and every covering projection $f : G \to H$.*

Proof. If $g : V(G) \to V(H)$ is a vertex-covering projection and $B'_i, i = 1, 2, \ldots, k$ the degree partition of H, define $B_i = \{u \in V(G) : g(u) \in B'_i\}, i = 1, 2, \ldots, k$. It follows from Theorem 3 that $B_i, i = 1, 2, \ldots, k$ is the degree partition of G and has the same degree refinement. The uniqueness of the degree partition of G implies that this partition is the same for every covering projection g.

3 Degree Reductions

In this section we show two reductions that can be performed on both the covering graph and covered graph and for which the existence of a covering projection is an invariant. These reductions enable us to consider graphs without small degrees. Recall that the degree of a vertex in a mixed graph is the number of undirected edges containing that vertex plus twice the number of undirected loops around that vertex plus the number of directed edges leaving and entering that vertex.

A cycle in a mixed graph is a sequence $u_1, e_1, u_2, e_2, \ldots, u_k, e_k$ such that

$$\mu(e_i) = \{u_i, u_{i+1}\} \text{ or } \mu(e_i) = (u_i, u_{i+1}) \text{ or } \mu(e_i) = (u_{i+1}, u_i)$$

for every $i = 1, 2, \ldots, k$ $(u_{k+1} = u_1)$.

3.1 Reduction I - Tree Liquidation

Definition 6. Given a cdm-graph G, denote by $Z(G)$ the maximal subgraph with all degrees greater than 1. Then $Z'(G) = (V(G), E(Z'(G)) = E(G) \setminus E(Z(G)))$ is acyclic, i.e., a disjoint union of trees. Each of these trees intersects the vertex set of $Z(G)$ in exactly one vertex. For every $u \in V(Z(G))$, denote by

T_u the connected component of $Z'(G)$ containing u. Let $\tau(u)$ be the isomorphism type of T_u as a colored tree rooted in u (i.e., $\tau(u) = \tau(v)$ iff there exists a color preserving isomorphism of T_u and T_v mapping u onto v). Redefine the coloring of the vertices of $Z(G)$ by setting $C(u) = \tau(u)$. The graph $Z(G)$ with this new coloring will be called the *dearborization of* G and denoted by $T(G)$ (edges of $T(G)$ are colored as in G).

Given a cdm-graph G, its dearborization can be found in polynomial time. One may want to see an argument why $Z'(G)$ is acyclic: Since G is connected, any cycle Q in $Z'(G)$ would be connected to $Z(G)$ by some path, say P. Then $Z(G)\cup Q\cup P$ would have all degrees ≥ 2 and would be larger than $Z(G)$. Similarly, one may argue that each component of $Z'(G)$ intersects $Z(G)$ in exactly one vertex: Since G is connected, every component does intersect $Z(G)$. On the other hand, if a component Q intersected $Z(G)$ in two vertices, say u, v, then Q would contain a path connecting u and v, say P. Then $Z(G) \cup P$ would have all degrees ≥ 2 and would be larger than $Z(G)$.

Theorem 7. *For any two cdm-graphs G and H, G covers H if and only if $T(G)$ covers $T(H)$.*

Proof. Let $f : G \to H$ be a covering projection of G onto H. Since a connected graph covers a tree only if it is isomorphic to the tree itself, for every vertex $u \in Z(H)$ and every $v \in V(G)$ such that $f(v) = u$, we observe that $v \in Z(G)$ and $T_u \cong T_v$. Thus $\tau(u) = \tau(v)$ and the restriction of f to $Z(G)$ is a covering projection of $T(G)$ onto $T(H)$.

On the other hand, suppose that $f : T(G) \to T(H)$ is a covering projection. Then $T_u \cong T_{f(u)}$ for every $u \in T(G)$, and let $\phi_u : T_u \to T_{f(u)}$ be an isomorphism. Then $g : G \to H$ defined by

$$g(x) = \phi_u(x) \text{ for every } x \in T_u \text{ and every } u \in T(G)$$

is a vertex-covering projection of G onto H. By Theorem 3, G covers H.

3.2 Reduction II - Dumping Small Degrees

A path in a mixed graph is a sequence $u_1, e_1, u_2, e_2, \ldots, u_k$ such that

$$\mu(e_i) = \{u_i, u_{i+1}\} \text{ or } \mu(e_i) = (u_i, u_{i+1}) \text{ or } \mu(e_i) = (u_{i+1}, u_i)$$

for every $i = 1, 2, \ldots, k-1$.

Definition 8. Given a cdm-graph G of minimum degree > 1, denote by $W(G)$ the subgraph induced by the vertices of degrees greater than 2. Let $W'(G)$ be the subgraph induced by $V(G) \setminus W(G)$. Then $W'(G)$ has all degrees ≤ 2, i.e., it is a disjoint union of paths. Since G is connected, the end vertices of each of these paths are connected to $W(G)$ by one edge each. Replace each such extended path P leading from a vertex u to a vertex v (it may be $u = v$) by and edge e_P with $\mu(e_P) = \{u, v\}$ (if P is symetric), or with $\mu(e_P) = (u, v)$ if P is not symetric (in the latter case we decide ad hoc a generic orientation of the edge obtained from a non-symetric path). We denote by $\pi(e_P)$ the isomorphism type of P. Denote the resulting cdm-graph by $S(G)$, again we assume that the newly added edges are colored via π, the original edges of G that remain as edges of $S(G)$ retain their original colors. Note that $S(G)$ can be found in polynomial time.

Theorem 9. *For any two cdm-graphs G and H, G covers H if and only if $S(G)$ covers $S(H)$.*

Proof. Let $f : G \to H$ be a covering projection of G onto H. Since a connected graph covers a path if and only if it is isomorphic to the path itself, every path P connecting vertices u and v in G maps onto a path P' connecting vertices $f(u)$ and $f(v)$ in H, and P and P' are isomorphic. Thus $\pi(e_P) = \pi(e_{P'})$ and the restriction of f to $W(G)$ is a vertex-covering projection of $S(G)$ onto $S(H)$.

On the other hand, suppose that $f : S(G) \to S(H)$ is a covering projection. Consider an edge e_P for a path $P = u, \ldots, v$ of G. Let P' be a path in H such that $f(e_P) = e_{P'}$ ($f(e_P) \notin E(W(H))$ because of its color). Then $e_{P'}$ is connecting vertices $f(u)$ and $f(v)$, and since $\pi(e_P) = \pi(e_{P'})$, $P \cong P'$. Let ϕ_P be an isomorphism of P and P'. Then $g : V(G) \to V(H)$ defined by

$$g(x) = \phi_P(x) \text{ iff } x \in P$$

is a vertex-covering projection of G onto H and G covers H.

3.3 Simple Graphs Versus cdm-graphs

It follows from Theorems 7 and 9 that in order to give a complete characterization of the computational complexity of the H-cover problem for cdm-graphs, it suffices to consider graphs H of minimum degree ≥ 3. On the other hand, if we are given a cdm-graph G as an input graph for a question if G covers a fixed cdm-graph H, we may encode the colors and directions of egdes by simple subgraphs: We assign a different tree T_c to every vertex color c, and both in G and H pend an isomorphic copy of T_c to each vertex u such that $C(u) = c$. Then we assign a different number $n_c > 1$ to every edge color c, and we replace every undirected edge of color c by a path of length n_c. Colors coresponding to directed edges will have assigned $n_c > 2$, and we pend a new tree on the second vertex of each path of length n_c (second from the tail determined by the orientation of the edge). In this way we obtain simple undirected graphs $U(G)$ and $U(H)$ such that G covers H if and only if $U(G) \cong U(H)$. See Figure 1 for an example. Thus we may conclude:

Corollary 10. *To achieve a complete characterization of the complexity of graph covering problems for simple undirected graphs, it is necessary and sufficient to give a complete characterization of the H-cover problem for colored directed multigraphs H of minimum degree ≥ 3.*

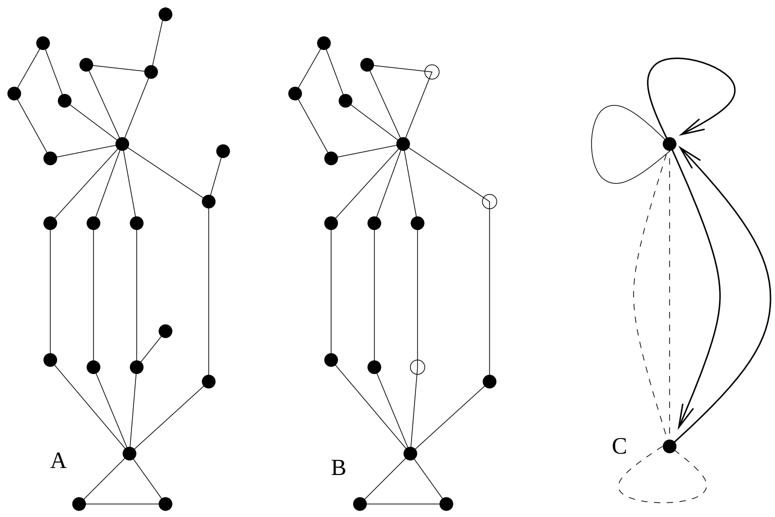

Fig. 1. A 21-vertex simple graph A encoded as a 2-vertex cdm-graph C, and vice-versa. Pending trees in graph A correspond to colored vertices in graph B. Paths of degree two in graph B correspond to colored directed edges and loops in graph C.

4 Two-vertex Graphs

To illustrate the methodology, we will now give a complete characterization of the complexity of the H-cover problem for cdm-graphs H with two vertices. (The case of H having one vertex only is straightforward - in such a case G covers H if and only if it has the same degree refinement as H.) Suppose H has two vertices, say L and R. For any edge color c, let l^c (r^c) be the number of loops of color c around the vertex L (R, respectively), let m^c be the number of edges of color c between L and R when edges of color c are undirected and let m_l^c (m_r^c) be the number of directed edges of color c starting in L and ending in R (starting in R and ending in L) when edges of color c are directed. (Thus $deg^c L = 2l^c + m^c$, $deg^c R = 2r^c + m^c$ in case of edges of color c being undirected, and $deg^{c-} L = l^c + m_l^c$, $deg^{c-} R = r^c + m_r^c$, $deg^{c+} L = l^c + m_r^c$, $deg^{c+} R = r^c + m_l^c$ in case of edges of color c being directed.) We denote by H^c the subgraph induced by the edges and loops of color c.

Theorem 11. *If*
(a) $C(L) \neq C(R)$, *or*
(b) $l^c \neq r^c$ *or* $m_l^c \neq m_r^c$ *for some color* c, *or*
(c) $m^c = 0$ ($m_l^c = m_r^c = 0$) *or* $l^c = r^c = 0$ *or* $l^c = r^c = m_l^c = m_r^c = 1$ *for every color* c
then the H*-cover problem is polynomially solvable. It is NP-complete in all other cases.*

In other words, assuming $P \neq NP$, the H-cover problem is polynomial iff H^c is non-regular for some color c or each H_c is either bipartite, or disconnected, or is regular of indegree 2 and outdegree 2. Rewording once more, the H-cover problem is NP-complete iff H^c is regular for every c and there is a color c for which H^c is connected nonbipartite and of degree at least 3 (resp. both indegree and outdegree at least 3). We will prove this result in several Lemmas. Note that it also follows that in the case of two-vertex cdm-graphs H, the H-cover problem is NP-complete if and only if H^c-cover is NP-complete for at least one edge-color c.

4.1 The Polynomial Cases

Lemma 12. *If*
(a) $C(L) \neq C(R)$, *or*
(b) $l^c \neq r^c$ *or* $m_l^c \neq m_r^c$ *for some color* c,
then the H*-cover problem is polynomially solvable.*

Proof. In both cases the vertices L and R are distinguishable in the degree partition of H. (This is trivial in case (a) as then they are distinguished by their colors. For case (b), note that $deg^c L = 2l^c + m^c = 2r^c + m^c = deg^c R$ implies $l^c = r^c$ if c is a color of undirected edges, while $deg^{c-} L = l^c + m_l^c = r^c + m_r^c = deg^{c-} R$ and $deg^{c+} L = l^c + m_r^c = r^c + m_l^c = deg^{c+} R$ imply $m_l^c = m_r^c$ and consequently $l^c = r^c$ if c is a color of directed edges.) It follows that G covers H if and only if G has the same degree refinement as H, and this can be decided in polynomial time.

The following lemma is a special case of a more general theorem [10]. We include a brief sketch of the proof for the sake of completeness.

Lemma 13. *If*
(i) $C(L) = C(R)$, *and*
for every color c,
(ii) $l^c = r^c$ *and* $m_l^c = m_r^c$, *and*
(iii) $m^c = 0$ $(m_l^c = m_r^c = 0)$ *or* $l^c = r^c = 0$ *or* $l^c = r^c = m_l^c = m_r^c = 1$
then the H*-cover problem is polynomially solvable.*

Proof. In this case H is symmetric and the degree partitions of G and H have each only one block. In particular, for any covering projection of G onto H, the mapping that interchanges the target vertices L and R is again a covering projection.

We will show how to reduce the H-cover problem to 2-SAT (which is well known to be solvable in polynomial time). Given a graph G which has the same degree refinement as H, we introduce a variable x_u for every vertex $u \in V(G)$. We then construct a formula $\Phi(G)$ over these variables so that G covers H if and only if $\Phi(G)$ is satisfiable, and in particular a covering projection $f : V(G) \to V(H)$ would correspond to a satisfying truth assignment so that x_u is true iff $f(u) = L$.

Our $\Phi(G)$ will be a conjuction of subformulas $\Phi(G) = \Phi_1 \wedge \Phi_2 \wedge \Phi_3$ defined as follows:

For every edge color c such that $m^c = 0$ (or $m_l^c = m_r^c = 0$ in case of directed color), Φ_1 will contain clauses

$$(x_u \vee \neg x_v) \wedge (\neg x_u \vee x_v)$$

for any pair of vertices $u, v \in V(G)$ connected by an edge $e \in F(G) \cup L(G)$ (or $e \in D(G)$) of color c (i.e., $\mu(e) = \{u, v\}$ resp. $\mu(e) = (u, v)$). Indeed, these two clauses guarantee that f maps u and v onto the same vertex of H.

For every edge color c such that $l^c = r^c = 0$, Φ_2 will contain clauses

$$(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$$

for any pair of vertices $u, v \in V(G)$ connected by an edge $e \in F(G) \cup L(G)$ (or $e \in D(G)$) of color c (i.e., $\mu(e) = \{u, v\}$ resp. $\mu(e) = (u, v)$). These clauses guarantee that f maps u and v onto distinct vertices of H.

Finally, for every directed edge color c such that $l^c = r^c = m_l^c = m_r^c = 1$, Φ_2 will contain clauses

$$(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$$

for any pair of vertices $u, v \in V(G)$ connected by directed edges $e, e' \in D(G)$ of color c to a common neighbor z so that $\mu(e) = (u, z)$ and $\mu(e') = (v, z)$ (or $\mu(e) = (z, u)$ and $\mu(e') = (z, v)$). These clauses guarantee that f maps u and v onto distinct vertices of H.

If G covers H then Φ is obviously satisfiable. On the other hand, a satisfying truth assignment yields a vertex-covering projection of G onto H. It follows from Theorem 3 that G covers H in such a case.

4.2 The NP-complete Cases

For the NP-complete cases, we assume that H is symmetric, i.e., $C(L) = C(R)$, $l^c = r^c$ and $m_l^c = m_r^c$ for every edge color c (the last equality is required for directed edge colors only). The impact of the first lemma is that we may study color-induced subgraphs separately.

Lemma 14. *The problem H-cover for two-vertex cdm-graphs is NP-complete provided H^c-cover is NP-complete for some edge color c.*

Proof. Given a graph G subject to the question of the existence of a covering projection from G to H^c, we construct $\widetilde{G}$ from two copies of G (say G_1 and G_2 with vertices named u_1 resp. u_2 for $u \in V(G)$). For every edge color $\varepsilon \neq c$, we add $l^\varepsilon = r^\varepsilon$ loops of color ε to every vertex of $\widetilde{G}$, and for every vertex $u \in V(G)$, we add m^ε undirected edges of color ε joining u_1 and u_2 (in case of an undirected edge color ε) and we add $m_l^\varepsilon = m_r^\varepsilon$ directed edges of color ε from u_1 to u_2 and the same number from u_2 to u_1 (in case of a directed edge color ε). Since H is symmetric, $f_2 : V(G_2) \to V(H)$ defined by $f_2(u_2) \neq f_1(u_1)$ is a covering projection of G_2 onto H^c whenever $f_1 : V(G_1) \to V(H)$ is a covering projection from G_1. It follows that $\widetilde{G}$ covers H if and only if G covers H^c.

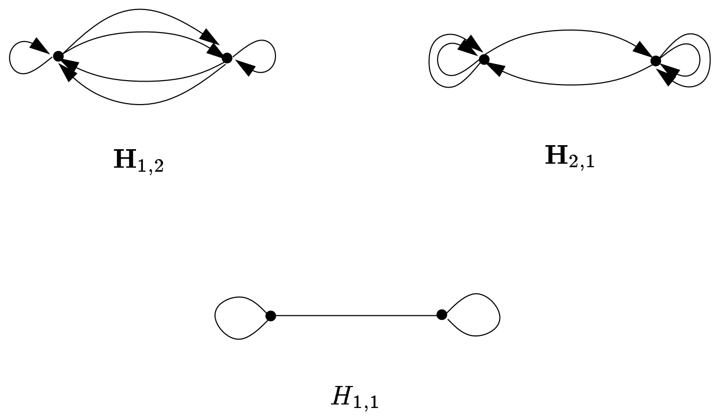

Fig. 2. Base NP-complete cdm-graphs.

In view of Lemmas 12, 13 and 14, it suffices to show that H-cover is NP-complete for monochromatic H such that

(1) $l = r \geq 1$ and $m \geq 1$ (in case of undirected graph H), or

(2) $l = r \geq 1$ and $m_l = m_r \geq 1$ and $l + m_l \geq 3$ (in case of directed H). We will show the NP-completeness by induction, starting with the graphs depicted in Fig. 2. For the sake of simplicity, we introduce notation $H(l, m)$ for an undirected graph with l loops around L and R and m edges joining L and R, and $\mathbf{H}(l, m)$ for a directed graph with l directed loops around L and R and m directed edges from L to R and m directed edges from R to L.

Proposition 15. [1] *The $H(1, 1)$-cover problem is NP-complete.*

Though the NP-completeness result shown in [1] concerns multigraphs on the input, it is not difficult to show that the problem is NP-complete even if the input graph is simple, i.e., it is NP-complete to decide if the vertices of a simple cubic graph may be colored by two colors so that every vertex has two neighbors of the same color and one neighbor of the opposite color. This modification will be used in the sequel.

Lemma 16. *The $H(1, 2)$-cover problem is NP-complete.*

Proof. Let G be a cubic graph subject to the question if G covers $H(1, 1)$. Take two copies of G, say G_1 and G_2 (with vertices u_1 resp. u_2 for every $u \in V(G)$) and construct $\widetilde{G}$ from $G_1 \cup G_2$ by connecting each pair u_1, u_2 by a copy of the connector graph depicted in Fig. 3. This figure also shows a coloring of the connector graph such that every vertex has two black and two white neighbors. If $f : G \to H(1, 1)$ is a covering projection, define $\widetilde{f} : \widetilde{G} \to H(1, 2)$ so that $\widetilde{f}(u_1) = \widetilde{f}(u_2) = f(u)$ for $u \in V(G)$ and $\widetilde{f}(x) = f(u)$ if x is an inner vertex of the connector connecting u_1 and u_2 marked black in Fig. 3. This $\widetilde{f}$ is then a covering projection of $\widetilde{G}$ onto $H(1,2)$.

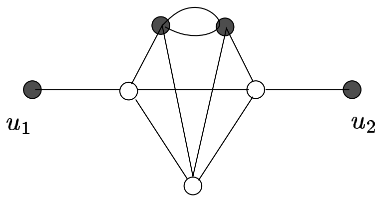

Fig. 3. Connector gadget for $H(1,2)$-cover.

Suppose on the other hand that $\widetilde{f}:\widetilde{G}\to H(1,2)$ is a covering projection. One can easily check that (upto the color reversal) the coloring depicted in Fig. 3 is the only coloring such that every inner vertex has two black and two white neighbors. Hence every vertex u_1 has two neighbors of its own color and one neighbor of the opposite color in G_1, and $f:G\to H(1,1)$ defined by $f(u)=\widetilde{f}(u_1)$ is a covering projection of G onto $H(1,1)$.

Lemma 17. *The $H(1,m)$-cover problem is NP-complete for every $m\geq 3$.*

Proof. Let G be a cubic graph subject to the question if G covers $H(1,1)$. Take two copies of G, say G_1 and G_2 (with vertices u_1 resp. u_2 for every $u\in V(G)$) and construct $\widetilde{G}$ from $G_1\cup G_2$ by connecting each pair u_1,u_2 by $m-1$ parallel edges.

If $f:G\to H(1,1)$ is a covering projection, define $\widetilde{f}:\widetilde{G}\to H(1,m)$ so that $\widetilde{f}(u_1)=f(u)$, $\widetilde{f}(u_2)\neq f(u)$ for $u\in V(G)$. Since interchanging the values in a covering projection onto $H(1,1)$ results again in a covering projection, the restrictions of $\widetilde{f}$ to G_1 and G_2 are both covering projections onto $H(1,1)$. Each u_1 (u_2) has other $m-1$ neighbors of the opposite color in G_2 (resp. G_1). Hence this $\widetilde{f}$ is a covering projection of $\widetilde{G}$ onto $H(1,m)$.

Suppose on the other hand that $\widetilde{f}:\widetilde{G}\to H(1,m)$ is a covering projection. The pairs u_1,u_2 are the only pairs of vertices joined by parallel edges, and so $\widetilde{f}(u_1)\neq\widetilde{f}(u_2)$ for every $u\in V(G)$. Hence every vertex u_1 has two neighbors of its own color and one neighbor of the opposite color in G_1, and $f:G\to H(1,1)$ defined by $f(u)=\widetilde{f}(u_1)$ is a covering projection of G onto $H(1,1)$.

Proposition 18. *The $H(l,m)$-cover problem is NP-complete for every $l\geq 1, m\geq 1$.*

Proof. Let G be a multigraph subject to the question if G covers $H(1,m)$. Construct $\widetilde{G}$ from G by adding $l-1$ loops to each vertex of G. Any $f:V(G)\to\{L,R\}$

is a covering projection of G onto $H(1,m)$ if and only if it is also a covering projection of $\widetilde{G}$ onto $H(l,m)$.

This concludes the case of undirected graphs. For the rest of the section, we assume that H is directed.

Proposition 19. [7] *It is NP-complete to decide if the vertices of a simple cubic graph may be colored by two colors so that every vertex has exactly one neighbor of its own color.*

Lemma 20. *The* $\mathbf{H}(1,2)$*-cover problem is NP-complete.*

Proof. Let G be a simple cubic graph and let $\widetilde{G}$ be the symmetric orientation of G (i.e., every edge e of G is replaced by two directed edges joining the endpoints of e in opposite directions). Obviously, $\widetilde{G}$ covers $\mathbf{H}(1,2)$ if and only if G allows a coloring described in Proposition 19.

Lemma 21. *The* $\mathbf{H}(1,3)$*-cover problem is NP-complete.*

Proof. Take 6 copies of a cubic graph G, say $G_i, i = 1,2,\ldots,6$ (with vertices named $u_i \in V(G_i), i = 1,2,\ldots,6$ for $u \in V(G)$). Construct a 4-regular graph $\widetilde{G}$ from their disjoint union by adding connector graphs depicted in Fig. 4. We claim that the vertices of $\widetilde{G}$ can be colored by two colors so that each vertex has exactly one neighbor of its own color and three neighbors of the opposite color if and only if the vertices of G can be colored by two colors so that each vertex has exactly one neighbor of its own color.

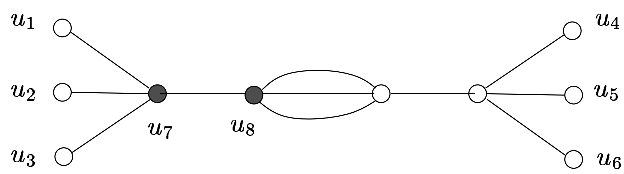

Fig. 4. Connector gadget for $\mathbf{H}(1,3)$-cover.

Indeed, suppose the vertices of $\widetilde{G}$ are colored so that each vertex has 3 neighbors of the opposite color. Then the middle two vertices of the connector graph have to get different colors. It follows that u_7 has the same color as u_8, and hence u_7 gets different color then u_1. Therefore u_1 has exactly one neighbor of its own color and 2 neighbors of the opposite color in G_1. Thus the restriction of this coloring to G_1 yields a coloring of G that satisfies Proposition 19.

On the other hand, if G admits such a coloring, we color the vertices of G_1, G_2, G_3 accordingly and the vertices of G_4, G_5 and G_6 with the colors interchanged. It is seen from the coloring depicted in Fig. 4 that this coloring of $\bigcup_{i=1}^{6} G_i$ can be extended to a coloring of $\widetilde{G}$ in which every vertex has one neighbor of its own color and 3 neighbors of the opposite color.

Finally, we let G' be the symmetric orientation of $\widetilde{G}$. It follows that G' covers $\mathbf{H}(1,3)$ if and only if G allows coloring satisfying Proposition 19, and it follows from this proposition that $\mathbf{H}(1,3)$-cover is NP-complete.

Lemma 22. *The $\mathbf{H}(1,m)$-cover problem is NP-complete for every $m \geq 4$.*

Proof. Let G be a directed graph subject to the question if G covers $\mathbf{H}(1,2)$. Take two copies of G, say G_1 and G_2 (with vertices u_1 resp. u_2 for every $u \in V(G)$) and construct $\widetilde{G}$ from $G_1 \cup G_2$ by connecting each pair u_1, u_2 by $m-2$ parallel edges directed from u_1 to u_2 and $m-2$ edges directed from u_2 to u_1.

If $f : G \to \mathbf{H}(1,2)$ is a covering projection, define $\widetilde{f} : \widetilde{G} \to \mathbf{H}(1,m)$ so that $\widetilde{f}(u_1) = f(u)$, $\widetilde{f}(u_2) \neq f(u)$ for $u \in V(G)$. Since interchanging the values in a covering projection onto $\mathbf{H}(1,2)$ results again in a covering projection, the restrictions of $\widetilde{f}$ to G_1 and G_2 are both covering projections onto $\mathbf{H}(1,2)$. Each u_1 (u_2) has other $m-2$ neighbors of the opposite color in G_2 (resp. G_1). Hence this $\widetilde{f}$ is a covering projection of $\widetilde{G}$ onto $\mathbf{H}(1,m)$.

Suppose on the other hand that $\widetilde{f} : \widetilde{G} \to \mathbf{H}(1,m)$ is a covering projection. The pairs u_1, u_2 are the only pairs of vertices joined by parallel edges, and so $\widetilde{f}(u_1) \neq \widetilde{f}(u_2)$ for every $u \in V(G)$. Hence every vertex u_1 has one neighbor of its own color and two neighbors of the opposite color in G_1, and $f : G \to \mathbf{H}(1,2)$ defined by $f(u) = \widetilde{f}(u_1)$ is a covering projection of G onto $\mathbf{H}(1,2)$.

Lemma 23. *The $\mathbf{H}(2,1)$-cover problem is NP-complete.*

Proof. Let G be a simple cubic graph and let $\widetilde{G}$ be the symmetric orientation of G. Then $\widetilde{G}$ covers $\mathbf{H}(2,1)$ if and only if G allows a coloring such that every vertex has two neighbors of its own color and one neighbor of the opposite color, i.e., if G covers $H(1,1)$. Thus the statement follows from Propostion 15.

Proposition 24. *The $\mathbf{H}(l,m)$-cover problem is NP-complete for every $l \geq 1, m \geq 1$ such that $l + m \geq 3$.*

Proof. Consider first the case $m > 1$. Let G be a directed multigraph subject to the question if G covers $\mathbf{H}(1,m)$. Construct $\widetilde{G}$ from G by adding $l-1$ loops to each vertex of G. Any $f : V(G) \to \{L, R\}$ is a covering projection of G onto $\mathbf{H}(1,m)$ if and only if it is also a covering projection of $\widetilde{G}$ onto $\mathbf{H}(l,m)$. The statement then follows from Lemmas 20, 21 and 22.

If $m = 1$, we have $l \geq 2$. Again, let $\widetilde{G}$ be obtained from G by adding $l-2$ loops to each vertex of G. Any $f : V(G) \to \{L, R\}$ is then a covering projection of G onto $\mathbf{H}(2,1)$ if and only if it is also a covering projection of $\widetilde{G}$ onto $\mathbf{H}(l,1)$. The statement then follows from Lemma 23.

5 Conclusion and Further Research

The primary purpose of this paper was to introduce covers of colored directed multigraphs, and to justify their introduction by showing that the discussion of their complexity is necessary for a complete solution of the complexity of covers of simple undirected graphs. We then illustrated our methodology by showing a complete discussion of the complexity of cdm-graph covers for two-vertex graphs (note that this classification contains as a proper substatement the classification of all simple graphs with exactly two vertices of degree higher than 2). Several of the lemmas used in the proof of this classification theorem can be actually stated in more general form, we have decided to state the simplified versions in order to keep the length of the paper reasonable. A full version of Lemma 13 will appear in [10]. It is rather a coincidence that all NP-complete 2-vertex graphs are symmetric. This is not the case for 3-vertex graphs. There the classification is more involved, as shown in [11]. In particular, it is no more true for 3-vertex cdm-graphs that H-cover is NP-complete if and only if H^c-cover is NP-complete for some edge color c.

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