A linear-time algorithm for recognition of catval graphs

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Abstract

We give a linear-time algorithm for recognizing graphs that can be represented as the intersection graph of connected subgraphs of a caterpillar.

1 Motivation and background

The interval graphs and chordal graphs are well-known classes of intersection graphs, arising respectively from connected subpaths of a path, and from connected subtrees of a tree. In a recent paper [7] a class of graphs lying properly between interval and chordal graphs was introduced, the so-called catval graphs. The motivation for studying catval graphs came from memory requirement considerations in dynamic programming algorithms on graphs of bounded treewidth. However, the problem of recognizing catval graphs was not addressed since the characterization given in [7] hints only at a brute force algorithm of complexity $\mathcal{O}(n^6)$. In this short note we give a linear-time algorithm for this problem. In essence the algorithm follows from the well-known linear-time algorithms for recognition of interval graphs and the following characterization which we give in the next section: 'G is a catval graph iff the graph $G \setminus \{v : v \text{ simplicial in } G\}$ is interval'.

2 The main result

We consider only connected, simple graphs with at least one edge. A tree is an acyclic graph. A leaf is a vertex with one neighbor. A path is a tree with two leaves. A caterpillar is a tree consisting of a path (the 'body') with added leaves (the 'hairs'). A simplicial vertex is a vertex whose neighbors are pairwise adjacent.

Definition 1 *We say that a graph G is a interval/catval/chordal graph if G is the intersection graph of a set of connected subgraphs of a path/caterpillar/tree.*

Interval graphs have a characterization in terms of chordal graphs, based on the following definition:

Definition 2 *Three non-adjacent vertices x, y, z of a graph G form an asteroidal triple (AT) if between any two of them there exists a path in G that avoids the neighborhood of the third.*

Theorem 1 *[5] G is an interval graph iff it is chordal and AT-free.*

Similarly, catval graphs have a characterization in terms of chordal graphs:

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Definition 3 *An asteroidal triple x, y, z of a graph G is an extended asteroidal triple (XAT) if there exists 3 distinct vertices x', y', z' with $|\{x, y, z, x', y', z'\}| = 6$ such that $N[x'] \subset N[x], N[y'] \subset N[y], N[z'] \subset N[z]$. (where $\subset$ denotes strict inclusion and $N[x] = N(x) \cup \{x\}$ is the closed neighorhood)*

Theorem 2 *[7] G is a catval graph iff it is chordal and XAT-free.*

We now show a characterization of catval graphs in terms of interval graphs:

Theorem 3 *G is a catval graph iff $G' = G \setminus \{v : v \text{ simplicial in } G\}$ is interval.*

Proof: For the forward direction, let G have vertices $\{v_1, v_2, ..., v_n\}$ represented, respectively, by the connected subgraphs induced by the subsets of vertices $\{A_1, A_2, ..., A_n\}$ of a caterpillar A with body B. Thus each A_i induces a connected subgraph of A and $v_i v_j$ is an edge of G iff $A_i \cap A_j \neq \emptyset$. We show that the connected subgraphs induced by subsets of vertices $\{B_i : B_i = A_i \cap B \wedge B_i \neq \emptyset\}$ of the path B is a representation of a subgraph G'' of G that contains G' as induced subgraph. Firstly, it is clear that G'' is a subgraph of G and that $A_i \cap B = \emptyset$ implies that v_i is simplicial in G so that G' is a subgraph of G''. Moreover, G' is an induced subgraph of G'' since if two adjacent vertices v_i and v_j are non-simplicial in G then since a vertex of $A \setminus B$ has only one neighbor and this neighbor belongs to B we must have $A_i \cap A_j \cap B \neq \emptyset$. Since G' is an induced subgraph of an interval graph we can conclude that G' is itself an interval graph.

For the reverse direction, let G' have vertices $V(G') = \{v_1, v_2, ..., v_m\}$ represented, respectively, by the connected subgraphs induced by the subsets of vertices $\{B_1, B_2, ..., B_m\}$ of a path B, and let $S = \{v_n, v_{n-1}, ..., v_{n-m}\}$ be the set of vertices that are simplicial in G. Let vertex u have neighbors $N(u)$ in G. Let the subgraph of G induced by the simplicial vertices S have components $C_1, C_2, ..., C_t$. Note that if two simplicial vertices u, v are adjacent and $N[u] \neq N[v]$ then some vertex x is adjacent to only one of them, say u, but not to v, and the neighbors of simplicial vertex u would include two non-adjacent vertices v and x, a contradiction. Applying this argument we can show inductively that each component C_i must contain pairwise adjacent vertices that all have the same closed neighborhood in G. For $u \in S$ we must have $N(u) \cap V(G') = \{v_{u_1}, v_{u_2}, ..., v_{u_k}\}$ pairwise adjacent and since a set of subtrees of a tree satisfy the Helly property we must have that the subsets $\{B_{u_1}, B_{u_2}, ..., B_{u_k}\}$ have a nonempty intersection, say for example that vertex $u_{base} \in B$ is in the intersection. We can now describe how to find a representation of G by connected subgraphs of a caterpillar A starting from the given representation of G' by subsets $\{B_1, B_2, ..., B_m\}$ of the path B and augmenting it stepwise. Firstly, initialize $A = B$ and then iterate the following procedure for $i = 1$ to t: for u an arbitrary vertex in component C_i update A to include the new leaf l_i, make this leaf adjacent to u_{base}, add $|C_i|$ subsets $\{l_i\}$ of A as the subgraphs representing each vertex in the component C_i, and add l_i to the subsets representing each $v \in N(u) \setminus S$. The resulting collection of connected subgraphs of the caterpillar A is a representation of G and shows that G is catval. □

To decide if an input graph G is a catval graph we can therefore run the following algorithm:

1. Check that G chordal, else output NO
2. Find all simplicial vertices S of G
3. Output YES if $G \setminus S$ is interval, else output NO

Step 1 is done in linear-time for example by the Maximum Cardinality Search (MCS) algorithm [6]. Step 2 is done in linear-time by a modification of the MCS algorithm that finds all maximal cliques [1] while setting S to be those vertices belonging to only one maximal clique. Step 3 is done in linear-time by one of the standard linear-time interval graph recognition algorithms [2, 4, 3].

Corollary 1 *The recognition problem for catval graphs can be solved in linear time.*

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