

The Perfect Matching Cut Problem Revisited*

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Abstract

In a graph, a perfect matching cut is an edge cut that is a perfect matching. PERFECT MATCHING CUT (PMC) is the problem of deciding whether a given graph has a perfect matching cut, and is known to be NP-complete. We revisit the problem and show that PMC remains NP-complete when restricted to bipartite graphs of maximum degree 3 and arbitrarily large girth. Complementing this hardness result, we give two graph classes in which PMC is polynomial-time solvable. The first one includes claw-free graphs and graphs without an induced path on five vertices, the second one properly contains all chordal graphs. Assuming the Exponential Time Hypothesis, we show there is no $O^*(2^{o(n)})$ -time algorithm for PMC even when restricted to n -vertex bipartite graphs, and also show that PMC can be solved in $O^*(1.2721^n)$ time by means of an exact branching algorithm.

Keywords: Matching cut; Perfect matching cut; Computational complexity; Exact branching algorithm; Graph algorithm.

1. Introduction

In a graph $G = (V, E)$, a *cut* (X, Y) is a partition $V = X \cup Y$ of the vertex set into disjoint, non-empty sets X and Y . The set of all edges in G having one endvertex in X and the other endvertex in Y , written $E(X, Y)$, is called the *edge cut* of the cut (X, Y) . A *matching cut* is an edge cut that is a (possibly empty) matching. Another way to define matching cuts is as follows: a cut (X, Y) is a matching cut if and only if each vertex in X has at most one neighbor in Y and each vertex in Y has at most one neighbor in X [8, 12]. MATCHING CUT

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(MC) is the problem of deciding if a given graph admits a matching cut and this problem has received much attention lately [7, 10].

An interesting special case, where the edge cut $E(X, Y)$ is a *perfect matching*, see Figure 9 for an example, was considered by Heggenes and Telle [13], who proved that PERFECT MATCHING CUT (PMC), the problem of deciding if a given graph admits an edge cut that is a perfect matching, is NP-complete. A perfect matching cut (X, Y) can be described as a (σ, ρ) 2-partitioning problem [23], as every vertex in X must have exactly one neighbor in Y and every vertex in Y must have exactly one neighbor in X . By results of Bui et al [6, 23, 24] it can therefore be solved in FPT time when parameterized by treewidth or cliquewidth (to mention only the two most famous width parameters) and in XP time when parameterized by mim-width (maximum induced matching-width) of a given decomposition of the graph. For several classes of graphs, like interval and permutation graphs, a decomposition of bounded mim-width can be computed in polynomial-time [3], thus the problem is polynomial-time solvable on such classes.

In this paper, we revisit the PMC problem. Our results are:

- While MC is polynomial-time solvable when restricted to graphs of maximum degree 3 and its computational complexity is still open for graphs with large girth, we prove that PMC is NP-complete in the class of bipartite graphs of maximum degree 3 and arbitrarily large girth. Further, we show that PMC cannot be solved in $O^*(2^{o(n)})$ time for n -vertex bipartite graphs and cannot be solved in $O^*(2^{o(\sqrt{n})})$ time for bipartite graphs with maximum degree 3 and arbitrarily large girth.
- We provide the first exact algorithm to solve PMC on n -vertex graphs, of runtime $O^*(1.2721^n)$. Note that the fastest algorithm for MC has runtime $O^*(1.3280^n)$ and is based on the current-fastest algorithm for 3-SAT [17].
- We give two graph classes of unbounded mim-width in which PMC is solvable in polynomial time. The first class contains all claw-free graphs and graphs without an induced path on 5 vertices, the second class contains all chordal graphs.

Related work. The computational complexity of MC was first considered by Chvátal [8], who proved that MC is NP-complete for graphs with maximum degree 4 and polynomial-time solvable for graphs with maximum degree at most 3. Hardness results were obtained for further restricted graph classes such as bipartite graphs, planar graphs and graphs of bounded diameter [4, 19, 22]. Further graph classes in which MC is polynomial-time solvable were identified, such as graphs of bounded tree-width, claw-free, hole-free and Ore-graphs [4, 7, 22]. FPT algorithms and kernelization for MC with respect to various parameters have been discussed by Aravind et al [1, 2, 10, 11, 17, 18]. The current-best exact algorithm solving MC is based on 3-SAT and has a running time of $O^*(1.3280^n)$ where n is the vertex number of the input graph [17]. Faster exact algorithms can be obtained for the case when the minimum degree is large [7]. The recent paper of Golovach et al [10] addresses enumeration aspects

of matching cuts.

Very recently, a related notion has been discussed by Bouquet and Picouleau [5], where they consider perfect matchings $M \subseteq E$ of a graph $G = (V, E)$ such that $G \setminus M = (V, E \setminus M)$ is disconnected, which they call perfect matching-cuts. To avoid confusion, we call such a perfect matching a *disconnecting perfect matching*. Note that, by definition, every perfect matching cut is a disconnecting perfect matching but a disconnecting perfect matching need not be a perfect matching cut. Indeed, all perfect matchings of the cycle on $4k + 2$ vertices are disconnecting perfect matchings and none of them is a perfect matching cut. In the paper by Bouquet and Picouleau [5] it was shown that recognizing graphs having a disconnecting perfect matching is NP-complete even when restricted to graphs with maximum degree 4, and it left open the case of maximum degree 3. It is not clear whether our hardness result on degree-3 graphs can be modified to obtain a hardness result of recognizing degree-3 graphs having a disconnecting perfect matching.

Notation and terminology. Let $G = (V, E)$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The neighborhood of a vertex v in G , denoted by $N_G(v)$, is the set of all vertices in G adjacent to v ; if the context is clear, we simply write $N(v)$. Let $\deg(v) := |N(v)|$ be the degree of the vertex v , and $N[v] := N(v) \cup \{v\}$ be the closed neighborhood of v . For a subset $F \subseteq V$, $G[F]$ is the subgraph of G induced by F , and $G - F$ stands for $G[V \setminus F]$. We write $N_F(v)$ and $N_F[v]$ for $N(v) \cap F$ and $N[v] \cap F$, respectively, and call the vertices in $N(v) \cap F$ the *F-neighbors* of v . A graph is 2-connected if it is connected and stays connected after the removal of any single vertex. A subgraph of G is maximal 2-connected if it is 2-connected and the addition of any edge or vertex from G would make it not 2-connected. The *girth* of G is the length of a shortest cycle in G , assuming G contains a cycle. The path on n vertices is denoted by P_n , the complete bipartite graph with one color class of size p and the other of size q and every pair of vertices in the two sets adjacent is denoted by $K_{p,q}$; $K_{1,3}$ is also called a *claw*.

When an algorithm branches on the current instance of size n into r subproblems of sizes at most $n - t_1, n - t_2, \dots, n - t_r$, then (t_1, t_2, \dots, t_r) is called the *branching vector* of this branching, and the unique positive root of $x^n - x^{n-t_1} - x^{n-t_2} - \dots - x^{n-t_r} = 0$, denoted by $\tau(t_1, t_2, \dots, t_r)$, is called its *branching factor*. The running time of a branching algorithm is $O^*(\alpha^n)$, where $\alpha = \max_{1 \leq i \leq r} \alpha_i$ and α_i is the branching factor of branching rule i , and the maximum is taken over all branching rules. Throughout the paper we use the O^* notation which suppresses polynomial factors. We refer to [9] for more details on exact branching algorithms.

Algorithmic lower bounds in this paper are conditional, based on the Exponential Time Hypothesis (ETH) [14]. The ETH implies that there is no $O^*(2^{o(n)})$ -time algorithm for 3-SAT where n is the variable number of the input 3-CNF formula. It is known that the hard case for 3-SAT already consists of formulas with $O(n)$ clauses [15]. Thus, assuming ETH, there is no $O^*(2^{o(m)})$ -time algorithm for 3-SAT where m is the clause number of the input formula.

Observe that a graph has a perfect matching cut if and only if each of its connected components has a perfect matching cut. Thus, we may assume that all graphs in this paper are connected.

2. Hardness results

In this section, we give two polynomial-time reductions from POSITIVE NAE 3-SAT to PMC. Recall that an instance for POSITIVE NAE 3-SAT is a 3-CNF formula $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$ over n variables x_1, x_2, \dots, x_n , in which each clause C_j consists of three distinct variables, all positive. The problem asks whether there is a truth assignment of the variables such that every clause in F has at least one true and at least one false variable. Such an assignment is called an *NAE assignment*, i.e. a not-all-equal assignment. Note that in contrast to 3SAT there are no negated variables and it does not allow a truth assignment that for any clause assigns all 3 variables of that clause the value True.

It is well-known [21] that there is a polynomial reduction from 3-SAT to POSITIVE NAE 3-SAT where the variable and clause number of the reduced formula is linear in the clause number of the original formula. Hence, ETH implies that there is no subexponential time algorithm for POSITIVE NAE 3-SAT in the number of variables.

Theorem 1. *Assuming ETH, PMC cannot be solved in subexponential time in the vertex number, even when restricted to bipartite graphs.*

Proof. We give a polynomial reduction from POSITIVE NAE 3-SAT to PMC restricted to bipartite graphs. Given a 3-CNF formula F , construct a graph G as follows.

For each clause $C_j = \{c_{j1}, c_{j2}, c_{j3}\}$, let $G(C_j)$ be the cube with *clause vertices* labeled c_{j1}, c_{j2}, c_{j3} , respectively, as depicted in Fig. 1. For each variable x_i , we introduce a *variable vertex* x_i and a dummy vertex x'_i adjacent only to x_i . Finally, we connect a variable vertex x_i to a clause vertex in $G(C_j)$ if and only if C_j contains the variable x_i , i.e., $x_i = c_{jk}$ for some $k \in \{1, 2, 3\}$.

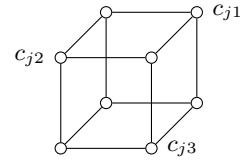


Figure 1: The graph $G(C_j)$.

Observe that G is bipartite, with one color class consisting of all clause vertices and dummy vertices, and it has the following property: no perfect matching M of G (in particular, no perfect matching cut) contains an edge between a clause vertex and a variable vertex, as it must contain the edges incident to dummy vertices. Thus, for every perfect matching cut $M = E(X, Y)$ of G , and every clause C_j the edges of M that belong to the cube $G(C_j)$ is a perfect matching cut of $G(C_j)$. Moreover, $G(C_j)$ has the following property: it has exactly three perfect matching cuts, and in any perfect matching cut of $G(C_j)$ not all clause vertices belong to the same side of the cut. Conversely, any partition into 2 non-empty parts of the 3 clause vertices of $G(C_j)$ can be extended (in a unique way) to a perfect matching cut M_j of $G(C_j)$. See also Fig. 2.

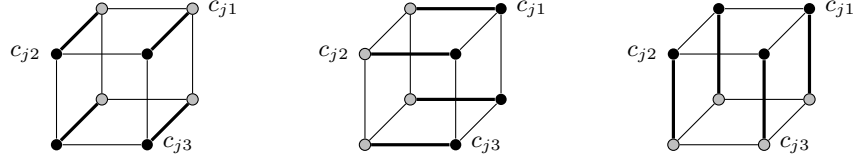


Figure 2: The three perfect matching cuts of $G(C_j)$; black vertices in X , gray vertices in Y .

We are now ready to see that F has an NAE assignment if and only if G has a perfect matching cut: First, if there is an NAE assignment for F then put all true variable vertices into X , all false variable vertices into Y , and extend X and Y (in a unique way) to a perfect matching cut of G ; note that x'_i and x_i have to belong to different parts. Second, if (X, Y) is a perfect matching cut of G then defining x_i to be **true** if $x_i \in X$ and **false** if $x_i \in Y$ we obtain an NAE assignment for F .

Observe that G is bipartite and has $N = O(n + m)$ vertices. Hence if PMC restricted to bipartite graphs had a subexponential time algorithm in vertex number N then we would also have a subexponential time algorithm for POSITIVE NAE 3-SAT in the number of variables, which by the reduction of Moret [21] would violate ETH. \square

We now describe how to avoid vertices of degree 4 and larger (the clause and variable vertices) in the previous reduction to obtain a bipartite graph with maximum degree 3 and large girth.

Theorem 2. *Let $g > 0$ be a given integer. PMC remains NP-complete when restricted to bipartite graphs of maximum degree three and girth at least g .*

Proof. We modify the gadgets used in the proof of Theorem 1. Let $h \geq 0$ be a fixed integer, which will be more concrete later.

Clause gadget: we subdivide every edge of the cube with $4h + 4$ new vertices, fix a vertex c_j of degree 3 and label the three neighbors of c_j with c_{j1} , c_{j2} and c_{j3} , respectively. We denote the obtained graph again by $G(C_j)$ and call the labeled vertices the *clause vertices*. The case $h = 0$ is shown in Fig. 3. Observe, $G(C_j)$ has the same properties of the cube used in the previous reduction: it has exactly three perfect matching cuts, and in any perfect matching cut of $G(C_j)$ not all clause vertices belong to the same part. Moreover, any bipartition of C_j can be extended (in a unique way) to a perfect matching cut M_j of $G(C_j)$. See also Fig. 4.

Variable gadget: for each variable x_i we introduce m variable vertices x_i^j , one for each clause C_j , $1 \leq j \leq m$, as follows. (We assume that the formula F consists of $m \geq 3$ clauses.) First, take a cycle with m vertices $x_i^1, x_i^2, \dots, x_i^m$ and edges $x_i^1 x_i^2, x_i^2 x_i^3, \dots, x_i^{m-1} x_i^m$ and $x_i^m x_i^1$. Then subdivide every edge with $4h + 3$ new vertices to obtain the graph $G(x_i)$. Thus, $G(x_i)$ is a cycle on $4m(h + 1)$ vertices. The case $m = 3, h = 0$ is shown in Fig. 3. The following property of $G(x_i)$ can be verified immediately: in any perfect matching cut of $G(x_i)$, all variable vertices x_i^j , $1 \leq j \leq m$, belong to the same part.

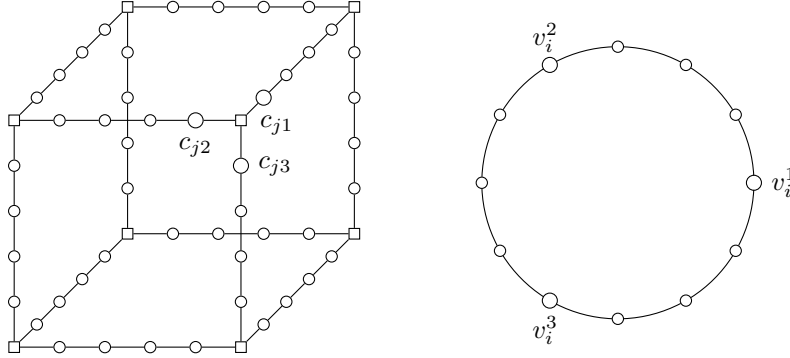


Figure 3: The clause gadget $G(C_j)$ (left) and the variable gadget $G(x_i)$ (right) in case $m = 3$ and $h = 0$.

Finally, the graph G is obtained by connecting the variable vertex x_i^j in $G(x_i)$ to a clause vertex in $G(C_j)$ by an edge whenever x_i appears in clause C_j , i.e., $x_i = c_{jk}$ for some $k \in \{1, 2, 3\}$. It follows from construction, that

- G has maximum degree 3.
- G is bipartite. This can be seen as follows. The bipartite subgraph formed by all $G(C_j)$ has a bipartition into independent sets A and B such that all clause vertices c_{jk} are in A . The bipartite subgraph formed by all $G(x_i)$ has a bipartition into independent sets C and D such that all variable vertices x_i^j are in C . Since the edges in G between these two subgraphs connect clause vertices and variable vertices, therefore the vertex set of G can be partitioned into independent sets $A \cup D$ and $B \cup C$.
- G has girth at least $\min\{4m(h+1), 8(h+2)\}$. This can be seen as follows. There are 3 types of cycles in G . Any of the cycles $G(x_i)$ has length $4m(h+1)$. A shortest cycle in any $G(C_j)$ is a subdivision of a 4-cycle and has length $4(4h+5)$. The cycles of the last type go through some $G(x_i)$ s and some $G(C_j)$ s; the length of a shortest one among them is at least $4 + (4h+4) + 4 + (4h+4) = 8(h+2)$, such a cycle, if any, starts at x_i^j in $G(x_i)$ then $G(C_j)$ then $G(x_k^j)$ then $G(x^k)$ then $G(C_p)$ then $G(x_i)$ at x_i^p and goes back to x_i^j along the shortest path of the cycle $G(x_i)$.
- No perfect matching M of G (in particular, no perfect matching cut) contains an edge between a clause vertex and a variable vertex. As with the previous reduction, this is because of the unique way of extending a perfect matching cut to the clause gadget, see Figure 4. Suppose an edge between a clause vertex c_{j1} and a variable vertex belongs to a perfect matching cut $M = (X, Y)$ of G . Then c_{j1} and its two neighbors in the clause gadget $G(C_j)$ must belong to the same part of the perfect matching cut, say X . This implies that, by symmetry of $G(C_j)$, $c_{j2} \in X$ and $c_{j3} \in Y$. When trying to extend this to the rest of $G(C_j)$ we have no choices and are forced into a contradiction.

Thus, for every perfect matching cut $M = E(X, Y)$ of G , the restrictions of M to $G(C_j)$ and to $G(x_i)$ are perfect matching cuts of $G(C_j)$ and of $G(x_i)$, respectively.

Now, as in the proof of Theorem 1, we can argue that F has an NAE assignment if and only if G has a perfect matching cut. First, if there is an NAE assignment for F then put all true variable vertices and true clause vertices into X , all false variable vertices and false clause vertices into Y , and extend X and Y (in a unique way) to a perfect matching cut of G . See Figure 4 for an extension in $G(C_j)$. Second, if (X, Y) is a perfect matching cut of G then defining x_i as **true** if $x_i \in X$ and **false** if $x_i \in Y$ we obtain an NAE assignment for F .

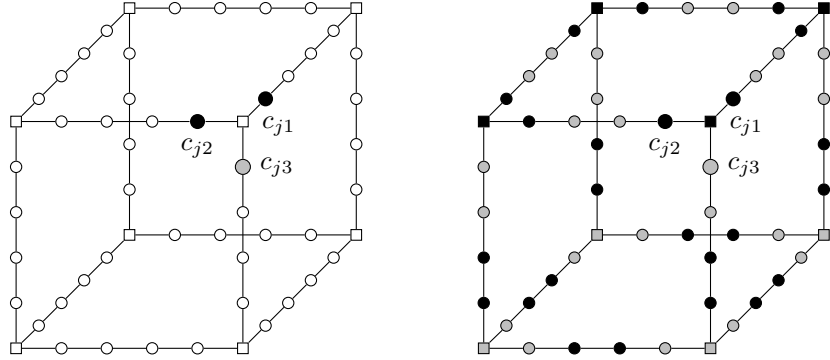


Figure 4: How to extend X (black) and Y (gray) on the left-hand side (uniquely) to a perfect matching cut in $G(C_j)$ on the right-hand side.

Finally, given $g > 0$, let $h \geq 0$ be an integer at least $\max\{\frac{g}{4m} - 1, \frac{g}{8} - 2\}$. Then G has girth at least $\min\{4m(h+1), 8(h+2)\} \geq g$. This completes the proof. \square

Note that the graph G in the proof of Theorem 2 has $N = O(m + nm)$ vertices, where n and m are the variable number and clause number, respectively, of the formula F . By the Sparsification lemma [15] the hard cases of 3SAT consist of sparse formulas, and the reduction of Moret [21] is linear, so we may assume that F has $m = O(n)$ clauses, and thus G has $N = O(n^2)$ vertices and hence if we could solve this problem in time $O^*(2^{o(\sqrt{n})})$ then we would violate ETH. Hence we obtain the following.

Theorem 3. *Assuming ETH, there is no $O^*(2^{o(\sqrt{n})})$ -time algorithm for PMC even when restricted to n -vertex bipartite graphs with maximum degree 3 and arbitrary large girth.*

Observe that PMC is trivial for graphs with maximum degree 2: a (connected) graph with maximum degree 2 has a perfect matching cut if and only if it is a path with even number of vertices or a cycle with $4k$ vertices. Thus, the maximum degree constraint in Theorems 2 and 3 is optimal.

3. An exact exponential algorithm

The main result in this section is an algorithm solving PMC in $O^*(1.2721^n)$ time.

Recall that all graphs considered are connected. Our algorithm follows the idea of known branching algorithms for MC [7, 17, 18]. We adapt basic reduction rules, that allow to simplify an instance to a simpler instance, for matching cuts to perfect matching cuts, and add new reduction and branching rules for perfect matching cuts.

If the input graph $G = (V, E)$ has a perfect matching cut (X, Y) , then some edge has one endvertex a in X and the other endvertex b in Y . The branching algorithm will be executed once for each edge $ab \in E$, hence $O(|E|) = O(m)$ times. To do this, set $A := \{a\}$, $B := \{b\}$, and $F := V \setminus \{a, b\}$ at the start of the branching algorithm. We say that a perfect matching cut (X, Y) with $A \subseteq X$ and $B \subseteq Y$ *separates* A and B . At each stage of the algorithm, A and B will be extended or it will be determined that there is no perfect matching cut separating A and B . We describe our algorithm by a list of reduction and branching rules given in preference order, i.e., in an execution of the algorithm on any instance of a subproblem one always applies the first rule applicable to the instance, which could be a reduction or a branching rule. A reduction rule may solve the problem directly or produce one subproblem while a branching rule results in at least two subproblems, with new value for A and B . Such rules must be shown safe, i.e. that the original problem has a solution if and only if (one of) the subproblem(s) has a solution. Note that G has a perfect matching cut that separates A from B if and only if in at least one recursive branch, extensions A' of A and B' of B are obtained such that G has a perfect matching cut that separates A' from B' . Typically a rule assigns one or more free vertices, vertices of F , either to A or to B and removes them from F , that is, we always have $F = V \setminus (A \cup B)$.

3.1. Some old reduction rules

Reduction Rules 1 (except the last two items), 2 (except the last item), 3 and 4 below are given in [18] for matching cuts. Note that a perfect matching cut is also a matching cut, and that the old reductions alter only the sets A, B but do not alter the graph, so that if the original instance had a matching cut but no perfect matching cut, this will remain the case also for the reduced instance. Thus we will not repeat the arguments for the old rules here.

Reduction Rule 1.

- a) If there is a vertex in A with two B -neighbors, or a vertex in B with two A -neighbors then STOP: “ G has no matching cut separating A, B ”.
- b) If there is a vertex $v \in F$, $|N(v) \cap A| \geq 2$ and $|N(v) \cap B| \geq 2$ then STOP: “ G has no matching cut separating A, B ”.
- c) If there is an edge xy in G such that $x \in A$ and $y \in B$ and $N(x) \cap N(y) \cap F \neq \emptyset$ then STOP: “ G has no matching cut separating A, B ”.
- d) If a vertex in A and a vertex in B have three or more common neighbors in F then STOP: “ G has no matching cut separating A and B ”. (Proof: 2 of those neighbors would have to belong to the same side, say X , but then a vertex in Y would have 2 neighbors in X)

- e) If there is a vertex in A (respectively, in B) that has no neighbor in $B \cup F$ (respectively, in $A \cup F$) then STOP: “ G has no perfect matching cut separating A, B ”. (Proof: Trivial)

Reduction Rule 2.

- a) If $v \in F$ has at least 2 A -neighbors (respectively, B -neighbors) then $A := A \cup \{v\}$ (respectively, $B := B \cup \{v\}$).
- b) If $v \in F$ with $|N(v) \cap N(z) \cap F| \geq 3$ for some $z \in A$ (respectively, $z \in B$) then $A := A \cup \{v\} \cup (N(v) \cap N(z) \cap F)$ (respectively, $B := B \cup \{v\} \cup (N(v) \cap N(z) \cap F)$). (Proof: Assume wlog $|N(v) \cap N(z) \cap F| = k$ and $z \in A$. At least $k-1$ of the common neighbors must be in A , otherwise z would have more than 1 neighbor in B . Call the k th neighbor w . If B contains any non-empty subset of $\{v, w\}$ then again z would have more than 1 neighbor in B .)

Reduction Rule 3. If $x \in A$ (respectively, $y \in B$) has two adjacent F -neighbors u, v then $A := A \cup \{u, v\}$ (respectively, $B := B \cup \{u, v\}$).

Reduction Rule 4. If there is an edge xy in G such that $x \in A$ and $y \in B$ then add $N(x) \cap F$ to A , and add $N(y) \cap F$ to B .

If none of these reduction rules can be applied then the following facts hold:

- The edge cut $E(A, B)$ is a (not necessary perfect) matching cut of $G[A \cup B] = G - F$ due to Reduction Rule 1a). Moreover, by Reduction Rule 1b) any vertex in A and any vertex in B have at most two common neighbors in F .
- Every vertex in F is adjacent to at most one vertex in A and at most one vertex in B due to Reduction Rule 2a).
- The neighbors in F of any vertex in A and the neighbors in F of any vertex in B form an independent set due to Reduction Rule 3, and
- Every vertex in A adjacent to a vertex in B has no neighbor in F and every vertex in B adjacent to a vertex in A has no neighbor in F due to Reduction Rule 4.

Reduction Rule 5 below is given in [17] and remains correct for perfect matching cuts.

Reduction Rule 5. If there are vertices $u, v \in F$ such that $N(u) = N(v) = \{x, y\}$ with $x \in A, y \in B$, then $A := A \cup \{u\}$, $B := B \cup \{v\}$.

3.2. New reduction rules

The remaining reduction rules work for perfect matching cuts but not for matching cuts.

Reduction Rule 6. If $x \in A$ (respectively, $y \in B$) has $N(x) \cap F = \{v\}$ then $B := B \cup \{v\}$ (respectively, $A := A \cup \{v\}$).

Proof. [of safeness] Let $x \in A$ with $N(x) \cap F = \{v\}$. By Reduction Rule 4, $N(x) \cap B = \emptyset$. If (X, Y) is a perfect matching separating A and B , then $N(x) \setminus \{v\} \subseteq X$, hence the neighbor v of x must belong to Y . The case $y \in B$ is symmetric. \square

Reduction Rule 7. Let $z \in A$ (respectively, $z \in B$) and let $v \in N(z) \cap F$.

- a) If $\deg(v) = 1$ then $B := B \cup \{v\}$ (respectively, $A := A \cup \{v\}$).
- b) If $\deg(v) = 2$ and $w \in F$ is other neighbor of v then $B := B \cup \{w\}$ (respectively, $A := A \cup \{w\}$).

Proof. [of safeness] Let $z \in A$ and $v \in N(z) \cap F$. Let (X, Y) be a perfect matching of G separating A and B . If z is the only neighbor of v , then, as $z \in X$, v must belong to Y . If $N(v) = \{z, w\}$ with $w \in F$, then w must belong to Y , otherwise both neighbors of v would be in X and (X, Y) would not be a perfect matching cut. The case $z \in B$ is symmetric. \square

Reduction Rule 8. Let $x \in A$ and $y \in B$ with $|N(x) \cap N(y) \cap F| = 2$. If $|N(x) \cap F| \geq 3$ or $|N(y) \cap F| \geq 3$ then $A := A \cup N(x) \setminus N(y)$, $B := B \cup N(y) \setminus N(x)$.

Proof. [of safeness] Assume that (X, Y) is a perfect matching cut of G separating A and B . Then $N(x) \cap N(y) \cap F$ must contain one vertex in X and one vertex in Y . Hence $N(x) \setminus N(y) \subseteq X$ and $N(y) \setminus N(x) \subseteq Y$. \square

3.3. Branching rules

We now describe the branching rules; see also Fig. 5, 6 and 7. The correctness of all branching rules follows from the definition of perfect matchings, namely that, in any perfect matching cut (X, Y) separating A and B , every vertex in X has in Y exactly one neighbor and every vertex in Y has in X exactly one neighbor. Thus, if some vertex in A has no neighbor in B , it must have a neighbor in F that must go to Y , and if some vertex in B has no neighbor in A , it must have a neighbor in F that must go to X . Note that by Reduction Rule 6, every vertex in $A \cup B$ has none or at least two neighbors in F . By Reduction Rule 1d), every two vertices $x \in A$ and $y \in B$ have at most two common neighbors in F .

To determine the branching vectors which correspond to our branching rules, we set the size of an instance (G, A, B) as its number of free vertices, i.e., $|V(G)| - |A| - |B|$. Vertices in $A \cup B$ having exactly two neighbors in F will be covered by the first three branching rules, while the fourth rule covers vertices with two or more neighbors.

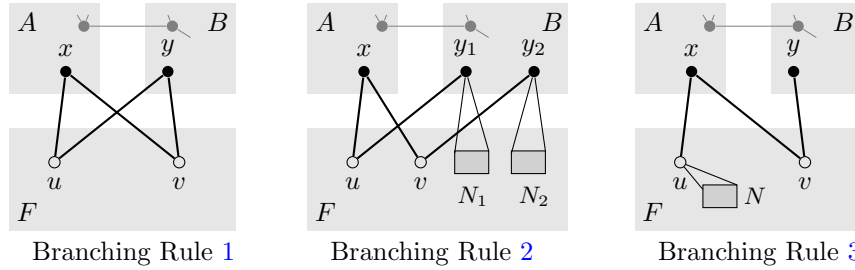


Figure 5: When Branching Rules 1, 2 and 3 are applicable.

Branching Rule 1. Let $x \in A$ and $y \in B$ with $N(x) \cap N(y) \cap F = \{u, v\}$. By Reduction Rule 8, $N(x) \cap F = N(y) \cap F = \{u, v\}$. We branch into two subproblems.

- First, add $N[u] \cap F$ to A . Then $N[v] \cap F$ has to be added to B .
- Second, add $N[u] \cap F$ to B . Then $N[v] \cap F$ has to be added to A .

The branching vector of Branching Rule 1 is

$$(|(N[u] \cup N[v]) \cap F|, |(N[u] \cup N[v]) \cap F|).$$

as the size of the original instance is $|F|$, and for vector (i, j) the two subproblems are of size $|F| - i$ and $|F| - j$. By Reduction Rule 2a) u and v have no other A or B -neighbors. By Reduction Rule 3 $uv \notin E$, and by Reduction Rule 5 either u or v must have degree at least 2, thus we must have $|(N[u] \cup N[v]) \cap F| \geq 3$, hence the branching factor of Branching Rule 1 is at most $\tau(3, 3) = \sqrt[3]{2} < 1.2560$, see e.g. the well-known tables of such $\tau(i, j)$ values.

Branching Rule 2. Let $x \in A$ with $N(x) \cap F = \{u, v\}$ and $N(u) \cap B = \{y_1\}$, $N(v) \cap B = \{y_2\}$ and $y_1 \neq y_2$. We branch into 2 subproblems.

- First, add u to B . Then v has to be added to A and $N_2 := N(y_2) \cap F \setminus \{v\}$ has to be added to B .
- Second, add v to B . Then u has to be added to A and $N_1 := N(y_1) \cap F \setminus \{u\}$ has to be added to B .

Symmetrically for $y \in B$ with $N(y) \cap F = \{u, v\}$ and $N(u) \cap A = \{x_1\}$, $N(v) \cap A = \{x_2\}$.

By Branching Rule 1, $v \notin N_1$, $u \notin N_2$. Hence, the branching vector of Branching Rule 2 is

$$(2 + |N_1|, 2 + |N_2|).$$

By Reduction Rule 6, $|N_1| \geq 1$, $|N_2| \geq 1$. Hence the branching factor is at most $\tau(3, 3) = \sqrt[3]{2} < 1.2560$.

Branching Rule 3. Let $x \in A$ with $N(x) \cap F = \{u, v\}$ and $N(u) \cap B = \emptyset$, $N(v) \cap B = \{y\}$. We branch into two subproblems.

- First, add u to B . Then v has to be added to A and $N := N(y) \cap F$ has to be added to B .
- Second, add v to B . Then u has to be added to A .

There is a symmetric rule for $y \in B$ with $N(y) \cap F = \{u, v\}$ and $N(u) \cap A = \emptyset$, and $N(v) \cap A = \{x\}$.

The branching vector of Branching Rule 3 is

$$(2 + |N|, 2).$$

By Reduction Rule 7, $|N| \geq 2$, hence the branching factor of Branching Rule 3 is at most $\tau(4, 2) < 1.2721$.

Branching Rule 4. Let $x \in A$ with $N(x) \cap F = \{u_1, u_2, \dots, u_r\}$, $r \geq 2$, and $N(u_i) \cap B = \emptyset$, $1 \leq i \leq r$. We branch into r subproblems. For each $1 \leq i \leq r$, the instance of the i -th subproblem is obtained by adding u_i to B . Then $N(x) \cap F \setminus \{u_i\}$ has to be added to A and $N_i := N(u_i) \cap F$ has to be added to B . There is a symmetric rule for $y \in B$ with $N(y) \cap F = \{v_1, v_2, \dots, v_r\}$ and v_i has no neighbor in A , $1 \leq i \leq r$.

The branching vector of Branching Rule 4 is

$$(r + |N_1|, r + |N_2|, \dots, r + |N_r|).$$

By Reduction Rule 7, $|N_i| \geq 2$, hence the branching factor of Branching Rule 4 is at most $\tau(r + 2, r + 2, \dots, r + 2) = \sqrt[r+2]{r} < 1.2600$.

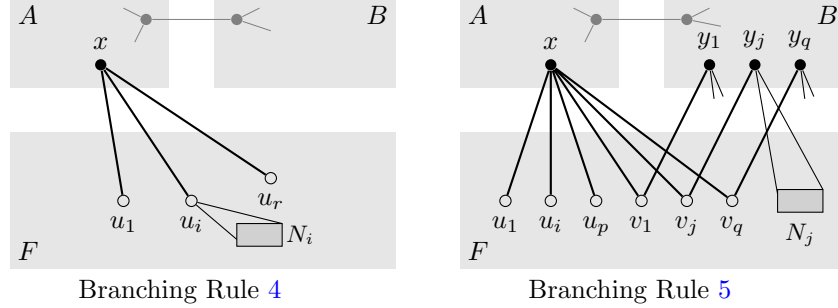


Figure 6: When Branching Rules 4 and 5 are applicable.

Branching Rules 1 and 4 together with the remaining branching rules cover vertices in $A \cup B$ having at least three neighbors in F . Branching Rule 5 deals with the case $z \in A$ (respectively, $z \in B$) in which at least two vertices in $N(z) \cap F$ have neighbors in B (respectively, in A).

Branching Rule 5. Let $x \in A$ with $N(x) \cap F = \{u_1, \dots, u_p, v_1, v_2, \dots, v_q\}$, $p \geq 0$, $q \geq 2$, such that $N(u_i) \cap B = \emptyset$, $1 \leq i \leq p$ and $N(v_j) \cap B = \{y_j\}$, $1 \leq j \leq q$. We branch into $r = p + q$ subproblems.

- For each $1 \leq i \leq p$, the instance of the i -th subproblem is obtained by adding u_i to B . Then $N(x) \cap F \setminus \{u_i\}$ has to be added to A and all $N_j := N(y_j) \cap F \setminus \{v_j\}$, $1 \leq j \leq q$, have to be added to B .
- For each $1 \leq j \leq q$, the instance of the $(p + j)$ -th subproblem is obtained by adding v_j to B . Then $N(x) \cap F \setminus \{v_j\}$ has to be added to A and all $N_k := N(y_j) \cap F \setminus \{v_j\}$, $1 \leq k \leq q$, $k \neq j$, have to be added to B .

There is a symmetric rule for $y \in B$ with $N(y) \cap F = \{u_1, \dots, u_p, v_1, v_2, \dots, v_q\}$, $p \geq 0$, $q \geq 2$ such that $N(u_i) \cap A = \emptyset$, $1 \leq i \leq p$ and $N(v_j) \cap A = \{x_j\}$, $1 \leq j \leq q$.

By Branching Rule 1 and Reduction Rule 2, N_j and $N_{j'}$ are pairwise disjoint if $j \neq j'$ and $N_j \cap \{v_1, \dots, v_q\} = \emptyset$. Hence, the branching vector of Branching Rule 5 is

$$\left(r + \sum_j |N_j|, \dots, r + \sum_j |N_j|, r + \sum_{k \neq 1} |N_k|, \dots, r + \sum_{k \neq q} |N_k| \right).$$

Due to Branching Rules 1–4, each y_j has at least three neighbors in F . Hence $|N_j| \geq 2$, $1 \leq j \leq q$. Thus, the branching factor is at most $\tau(r + 2q, \dots, r + 2q, r +$

$2(q-1), \dots, r+2(q-1)) \leq \tau(r+4, \dots, r+4, r+2, \dots, r+2) < \tau(r+2, \dots, r+2) = \sqrt[r+2]{r} < 1.2600$ for any integer r .

The last two branching rules deal with the case $z \in A$ (respectively, $z \in B$) in which exactly one vertex in $N(z) \cap F$ has a unique neighbor in B (respectively, in A).

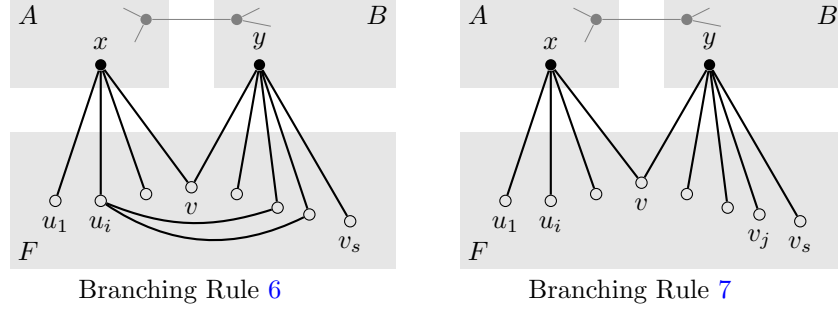


Figure 7: When Branching Rules 6 and 7 are applicable.

Branching Rule 6. Let $x \in A$ with $N(x) \cap F = \{u_1, u_2, \dots, u_r, v\}$, $r \geq 2$, such that $N(u_i) \cap B = \emptyset$, $1 \leq i \leq r$, and $N(v) \cap B = \{y\}$. Let $N(y) \cap F \setminus \{v\} = \{v_1, \dots, v_s\}$, $s \geq 2$, where some u_i has two neighbors in $\{v_1, \dots, v_s\}$. We branch into 2 subproblems.

- First, add v to A . Then $\{v_1, \dots, v_s\}$ and u_i have to be added to B , and $\{u_1, \dots, u_r\} \setminus \{u_i\}$ has to be added to A .
- Second, add v to B . Then $\{u_1, \dots, u_r\}$ has to be added to A .

There is a symmetric rule for $y \in B$ with $N(y) \cap F = \{u_1, u_2, \dots, u_r, v\}$ such that $N(u_i) \cap A = \emptyset$, $1 \leq i \leq r$, and $N(v) \cap A = \{x\}$ and some u_i has two neighbors in $N(x) \cap F \setminus \{v\}$.

The branching vector of Branching Rule 6 is

$$(r + s + 1, r + 1).$$

Since $r \geq 2$ and $s \geq 2$, we have $\tau(r + s + 1, r + 1) \leq \tau(5, 3) < 1.1939$.

Branching Rule 7. Let $x \in A$ with $N(x) \cap F = \{u_1, u_2, \dots, u_r, v\}$, $r \geq 2$, such that $N(u_i) \cap B = \emptyset$, $1 \leq i \leq r$, and $N(v) \cap B = \{y\}$. Let $N(y) \cap F \setminus \{v\} = \{v_1, \dots, v_s\}$, $s \geq 2$. We branch into $r + s$ subproblems.

- For each $1 \leq i \leq r$, the instance of the i -th subproblem is obtained by adding u_i to B . Then $\{u_1, \dots, u_r\} \setminus \{u_i\}$ and v have to be added to A , $N_i := N(u_i) \cap F$ and $\{v_1, \dots, v_s\}$ have to be added to B .
- For each $1 \leq j \leq s$, the instance of the $(r + j)$ -th subproblem is obtained by adding v_j to A . Then $\{v_1, \dots, v_s\} \setminus \{v_j\}$ and v have to be added to B , $M_j := N(v_j) \cap F$ and $\{u_1, \dots, u_r\}$ have to be added to A .

There is a symmetric rule for $y \in B$ with $N(y) \cap F = \{u_1, u_2, \dots, u_r, v\}$ such that $N(u_i) \cap A = \emptyset$, $1 \leq i \leq r$, and $N(v) \cap A = \{x\}$.

Let $\alpha_i = |N_i \cap \{v_1, \dots, v_s\}|$, $1 \leq i \leq r$, and $\beta_j = |M_j \cap \{u_1, \dots, u_r\}|$, $1 \leq j \leq s$. The branching vector of Branching Rule 7 is

$$(r+s+1+|N_1|-\alpha_1, \dots, r+s+1+|N_r|-\alpha_r, r+s+1+|M_1|-\beta_1, \dots, r+s+1+|M_s|-\beta_s)$$

By Reduction Rule 7, $|N_i| \geq 2$. By Branching Rule 5, v_j has no neighbor in A , hence, by Reduction Rule 7, $|M_j| \geq 2$. By Branching Rule 6, $\alpha_i \leq 1$, $\beta_j \leq 1$. Hence the branching factor is at most $\tau(r+s+2, \dots, r+s+2) = \sqrt[r+s+2]{r+s+2} < 1.2600$.

The description of all branching rules is completed. Among all branching rules, Branching Rule 3 has the largest branching factor of 1.2721. Consequently, the running time of our algorithm is $O^*(1.2721^n)$.

It remains to show that if none of the reduction rules and none of the branching rules is applicable to an instance (G, A, B) then the graph G has a perfect matching cut (X, Y) such that $A \subseteq X$ and $B \subseteq Y$ if and only if (A, B) is a perfect matching cut of G .

Lemma 1. *If no reduction or branching rule is applicable, then no vertex in $A \cup B$ has a neighbor in F .*

Proof. Consider a vertex $v \in A$. We do a case analysis based on the number of neighbors F_v that v has in F , and show that in each case some rule is applicable that will lower the value of F_v .

- $F_v = 1$: Reduction Rule 6 applies
- $F_v = 2$
 - no F -neighbor of v has a neighbor in B : Branching Rule 4 applies
 - both F -neighbors of v have a neighbor in B : Branching Rules 1 and 2 apply.
 - only one F -neighbor of v has a neighbor in B : Branching Rule 3 applies
- $F_v \geq 3$
 - no F -neighbor of v has a neighbor in B : Branching Rule 4 applies
 - at least 2 F -neighbors of v have a neighbor in B : Branching Rule 5 applies
 - only one F -neighbor of v has a neighbor in B : Branching Rules 6 and 7 apply.

The argument for a vertex in B is analogous. \square

Hence, by connectedness of G , $F = \emptyset$. Therefore, G has a perfect matching cut separating A and B if and only if (A, B) is a perfect matching cut. In summary, we obtain:

Theorem 4. *There is an algorithm for PMC running in $O^*(1.2721^n)$ time.*

4. Two polynomial-time solvable graph classes

In this section, we provide two graph classes in which PMC is solvable in polynomial-time. Both classes are well motivated by the hardness results.

4.1. Excluding a (small) tree of maximum degree three

Let H be a fixed graph. A graph G is H -free if G contains no induced subgraph isomorphic to H . Since by Theorem 2 PMC remains NP-complete on the class of graphs of maximum degree three and arbitrarily high girth, it is also NP-complete on H -free graphs whenever H is outside this class, e.g. if H has a vertex of degree larger than three or has a (fixed-size) cycle. This suggests studying the computational complexity of PMC restricted to H -free graphs for a fixed forest H with maximum degree at most three.

As the first step in this direction, we show that PMC is solvable in polynomial time for H -free graphs, where H is the tree T with 6 vertices obtained from the claw $K_{1,3}$ by subdividing two edges each with one new vertex; see Fig. 8. In particular, PMC is polynomial-time solvable for $K_{1,3}$ -free graphs but hard for $K_{1,4}$ -free graphs (by Theorem 2).

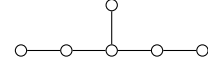


Figure 8: The tree T .

Given a connected T -free graph $G = (V, E)$, our algorithm works as follows. Fix an edge $ab \in E$ and decide if G has a perfect matching cut $M = E(X, Y)$ separating $A = \{a\}$ and $B = \{b\}$. We use the notations and reduction rules from Section 3. In addition, we need one new reduction rule; recall that $F = V \setminus (A \cup B)$. This additional reduction rule is correct for matching cuts in general and is already used in [7]. For completeness, we give a correctness proof for perfect matching cuts.

Reduction Rule 9. *If there are vertices $u, v \in F$ with a single common neighbor in A (respectively, in B) and $|N(u) \cap N(v) \cap F| \geq 2$, then $A := A \cup \{u, v\}$ (respectively, $B := B \cup \{u, v\}$).*

Proof. [of safeness] Let $u, v \in F$ with $N(u) \cap N(v) \cap A = \{x\}$ and $|N(u) \cap N(v) \cap F| \geq 2$. We show that G has a perfect matching cut separating A, B if and only if G has a perfect matching cut separating $A \cup \{u, v\}$ and B . First, let (X, Y) be a perfect matching cut of G with $A \subseteq X$ and $B \subseteq Y$. If $u \in Y$ then, as $x \in A$, $N(u) \cap F$ must belong to Y and v must belong to X . But then, as $|N(v) \cap N(u) \cap F| \geq 2$, v has two neighbors in Y , a contradiction. Thus, $u \in X$, and similarly, $v \in X$. That is (X, Y) separates $A \cup \{u, v\}$ and B . The other direction is obvious: any perfect matching cut separating $A \cup \{u, v\}$ and B separates A and B .

The second case is symmetric. □

Now, we apply the Reduction Rules 1–9 exhaustively. Note that this part takes polynomial time. If $F = V \setminus (A \cup B)$ is empty, then G has a perfect matching cut separating A and B if and only if (A, B) is a perfect matching cut of G . Verifying whether (A, B) is a perfect matching cut also takes polynomial time. So, let us assume that $F \neq \emptyset$. Then due to the reduction rules (recall that G is connected),

- any vertex in A (respectively, in B) having no neighbor in B (respectively, in A) has at least two neighbors in F , and
- any vertex in A (respectively, in B) having a neighbor in F has no neighbor

in B (respectively, in A).

At this point, we will explicitly give an induced subgraph in G isomorphic to the tree T or correctly decide that G has no perfect matching cut separating A and B . Let

$$A^* = \{x \in A \mid N(x) \cap B \neq \emptyset\}, B^* = \{y \in B \mid N(y) \cap A \neq \emptyset\}.$$

Recall that $A^* \neq \emptyset$ and $B^* \neq \emptyset$, and there are no edges between $A^* \cup B^*$ and F , no edges between $A \setminus A^*$ and $B \setminus B^*$.

Thus, as G is connected and $F \neq \emptyset$, there is a vertex in $A \setminus A^*$ adjacent to a vertex in A^* , or there is a vertex in $B \setminus B^*$ adjacent to a vertex in B^* . By symmetry, let us assume that there is a vertex $x \in A \setminus A^*$ adjacent to a vertex $x^* \in A^*$. Let $y^* \in B^*$ be the unique neighbor of x in B^* . Recall that, every vertex in $(A \setminus A^*) \cup (B \setminus B^*)$ has at least two neighbors in F .

First, suppose that there is a vertex $y \in B$ with $|N(x) \cap N(y) \cap F| \geq 2$. Let $u, v \in N(x) \cap N(y) \cap F$. If (X, Y) is a perfect matching cut with $A \subseteq X$ and $B \subseteq Y$, then u and v must belong to different parts, say $u \in X, v \in Y$. Now, if there were some vertex $w \in N(u) \cap N(v) \cap F$, then u would have two neighbors in Y (if $w \in Y$) or v would have two neighbors in X (if $w \in X$). So, let us assume that $N(u) \cap N(v) \cap F = \emptyset$. Then due to Reduction Rule 5, there exists a vertex $w \in N(u) \cap F \setminus N(v)$. Due to Reduction Rule 3, $N(x) \cap F$ is an independent set, hence w, u, x, x^*, y^* and v induce the tree T in G . Thus, we may assume that

$$\text{for any vertex } y \in B, |N(x) \cap N(y)| \leq 1. \quad (1)$$

Next, we claim that

$$\text{every vertex in } B \setminus B^* \text{ adjacent to a vertex in } N(x) \text{ is adjacent to } y^*. \quad (2)$$

This can be seen as follows: Let $z \in B \setminus B^*$ be adjacent to some $u \in N(x)$. Then $u \in F$. By (1), z is non-adjacent to all vertices in $N(x) \cap F \setminus \{u\}$. Recall that some vertex $v \in N(x) \cap F \setminus \{u\}$ exists. So, if z is not adjacent to y^* , then z, u, x, x^*, y^* and v induce the tree T in G .

Now, fix two vertices $u, v \in N(x) \cap F$. Suppose that $N(u) \cap B = \emptyset$. Then, due to Reduction Rules 7 and 9, there exists a vertex $w \in N(u) \cap F \setminus N(v)$, and as above, w, u, v, x, x^* and y^* induce the tree T in G . Thus, we may assume that $N(u) \cap B \neq \emptyset$ and, by symmetry, $N(v) \cap B \neq \emptyset$.

Let $y_1, y_2 \in B \setminus B^*$ be the unique neighbors of u and v in B , respectively. By (1), y_1 is non-adjacent to v , and y_2 is non-adjacent to u . By (2), y_1 and y_2 are adjacent to y^* . If y_1 and y_2 are non-adjacent, then u, y_1, y^*, y_2, v and x^* induce the tree T . So, let us assume that y_1 and y_2 are adjacent.

Let $u' \neq u$ be a second neighbor of y_1 in F , and $v' \neq v$ be a second neighbor of y_2 in F . By (1), x is non-adjacent to u' and v' . Now, consider two cases:

- assume that u and v' are adjacent. Then v', u, x, x^*, y^* and v induce the tree T in G , and

- assume that u and v' are non-adjacent. Then v', y_2, y_1, u, x and u' (if u' and v' are non-adjacent), or else u', v', y_2, y^*, x^* and v (if u' and v' are adjacent) induce the tree T in G .

In each case, we reach a contradiction.

Thus, we have seen that, in case $F \neq \emptyset$, G has no perfect matching cut separating A and B , or G contains the tree T as an induced subgraph. So, after at most $|E|$ rounds, each for a candidate $ab \in E$ and in polynomial time, our algorithm will find out whether G has a perfect matching cut at all. In summary, we obtain:

Theorem 5. *PMC is solvable in polynomial time for T -free graphs.*

4.2. Interval, chordal and pseudo-chordal graphs

Recall that a graph has girth at least g if and only if it has no induced cycles of length less than g . Thus, Theorem 2 implies that PMC remains hard when restricted to graphs without short induced cycles. This suggests studying PMC restricted to graphs without long induced cycles, i.e., k -chordal graphs. Here, given an integer $k \geq 3$, a graph is k -chordal if it has no induced cycles of length larger than k ; the 3-chordal graphs are known as chordal graphs.

In this subsection we show that PMC can be solved in polynomial time when restricted to what we call pseudo-chordal graphs, that contain 3-chordal graphs and thus are known to have unbounded mim-width [16].

We begin with a concise characterization of interval graphs having perfect matching cuts, to yield a polynomial-time algorithm deciding if an interval graph has a perfect matching cut which is much simpler than what we get by the mim-width approach [6].

Fact 1. *Let G have a vertex set $U \subseteq V(G)$ such that $G[U]$ is connected with every edge of $G[U]$ belonging to a triangle. Then if (X, Y) is a perfect matching cut of G we must have $U \subseteq X$ or $U \subseteq Y$.*

This must hold since otherwise we have a triangle K and two vertices u, v with $u \in K \cap X$ and $v \in K \cap Y$ having a common neighbor in K so this cannot be a perfect matching cut.

If an interval graph G has a cycle, then it has a 3-clique. By Fact 1 these 3 vertices would have to belong to the same side of the cut, and each would need to have a unique neighbor on the other side of the cut. But then those 3 neighbors would form an asteroidal triple, contradicting that G was an interval graph. Thus an interval graph which is not a tree does not have a perfect matching cut. A tree T is an interval graph if and only if it does not have the subdivided claw as a subgraph. Thus, T is a caterpillar with basic path x_1, \dots, x_k , where x_1 and x_k does not have a leaf attached, while the other x_i may have any number of leaves attached. (A *caterpillar* is a tree with a (basic) path such that all vertices outside the path have a neighbor on the path.) If some x_i has at least two leaves attached then T does not have a perfect matching cut (X, Y) , as $x_i \in X$ would imply that at least one of those leaves is in X and this leaf would not have a neighbor in Y . Since a leaf vertex and its neighbor must belong to

opposite sides of the cut, it is not hard to verify the following.

Fact 2. *An interval graph has a perfect matching cut if and only if it is a caterpillar with basic path x_1, \dots, x_k such that any x_i for $1 < i < k$ has either zero or one leaf, and any maximal sub-path of x_1, \dots, x_k with zero leaves contains an even number of vertices.*

In particular, caterpillars having a perfect matching cut can be recognized in polynomial time. For an arbitrary tree T we can decide whether T has a perfect matching as follows: Root T at a vertex r and let r_1, \dots, r_k be the children of r . Then T has a perfect matching cut if and only if there exists some $1 \leq i \leq k$ such that each subtree T_j rooted at r_j , $j \neq i$, has a perfect matching cut, and $T_i - r_i$ has a perfect matching cut. This fact implies a bottom-up dynamic programming to decide if T has a perfect matching cut.

A similar idea works for a large graph class that properly contains all chordal graphs. We will show a polynomial-time algorithm for what we call pseudo-chordal graphs. The bridges and maximal 2-connected subgraphs of a graph are called its blocks, and a block is non-trivial if it contains at least 3 vertices.

Definition 1. *A graph is pseudo-chordal if, for every non-trivial block B , every edge of B belongs to a triangle.*

Note that chordal graphs are pseudo-chordal, but pseudo-chordal graphs may contain induced cycles of any length, e.g. take a cycle and for any two neighbors add a new vertex adjacent to both of them.

Theorem 6. *There is a polynomial-time algorithm deciding if a pseudo-chordal graph G has a perfect matching cut.*

Proof. We first compute the blocks of G and let D be the subgraph of G formed by the edges of non-trivial blocks of G . Let D_1, D_2, \dots, D_k be the connected components of D . Note that by collapsing each D_i into a supernode we can treat the graph G as having a tree structure T (related to the block structure) with one node for each $v \in V(G) \setminus V(D)$, and a supernode for each D_i . See Fig. 9.

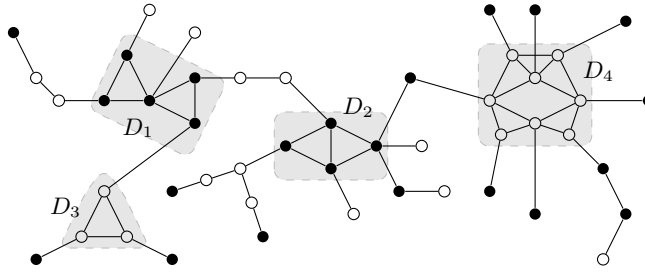


Figure 9: A pseudo-chordal graph and perfect matching cut given by (X, Y) with X being black vertices. Note the tree structure composed of (i) those vertices that do not belong to a clique of size 3 and (ii) the four supernodes D_1, D_2, D_3, D_4 .

Note that since G is pseudo-chordal then by Fact 1 all the vertices in a fixed supernode D_i must be on the same side in any perfect matching cut of G .

Our algorithm will pick a root R of T and proceed by bottom-up dynamic programming on the rooted tree T . Each node S of T will be viewed as the set of vertices it represents in G . If S is not the root of T then we denote by $r(S)$ the unique vertex of S that has a parent in T . For each node S of T we will compute two boolean values that concern the subgraph G_S of G induced by vertices of G contained in the subtree of T rooted at S . These boolean values are defined as follows:

- $pmc(S) = \text{true}$ if and only if G_S has a perfect matching cut
- $m(S) = \text{true}$ if and only if $G_S \setminus r(S)$ has a perfect matching cut where all vertices of $S \setminus r(S)$ are on the same side.

We first initialize $pmc(S)$ and $m(S)$ to **false** for all nodes S of T . For a leaf S of T we set $m(S) = \text{true}$ if $|S| = 1$, i.e. if S is not a supernode.

Consider an inner node S of T , with $S = \{v_1, \dots, v_q\} \subseteq V(G)$. In the rooted tree T , let the set of children of S that has a neighbor of v_i be $C(v_i)$ (note that each child of S in T has a unique vertex that has a neighbor $v_i \in S$ and the neighbor is unique). Assuming the values $pmc(\cdot)$ and $m(\cdot)$ have been computed for all children of S , we do the following:

- set $pmc(S) = \text{true}$ if for each $v_i \in S$ we have $C(v_i) = \{S_1, \dots, S_k\}$ with $k \geq 1$ and we can find a child with $m(S_i) = \text{true}$ such that $pmc(S_j) = \text{true}$ for the other $k - 1$ children $j \neq i$.
- set $m(S) = \text{true}$ if (i) for each $v_i \in S \setminus r(S)$ we have $C(v_i) = \{S_1, \dots, S_k\}$ with $k \geq 1$ and we can find a child with $m(S_i) = \text{true}$ such that $pmc(S_j) = \text{true}$ for the other $k - 1$ children $j \neq i$, and (ii) for every child $S' \in C(r(S))$ we have $pmc(S') = \text{true}$.

For the root R of T we update $pmc(R)$ but not $m(R)$ since $r(R)$ is not defined. When we are done with the bottom-up dynamic programming, then for the root R of T we note that $G = G_R$ so that by the definition of the values G has a perfect matching cut if and only if $pmc(R) = \text{true}$.

The correctness follows by structural induction on the tree T . By definition, $pmc(S) = \text{true}$ (respectively, $m(S) = \text{true}$) if and only if there is a cut of G_S so that every node (respectively, every node except $r(S)$) has a single neighbor, its ‘mate’, on the other side of the cut. The values at leaves are initialized correctly according to this definition. At an inner node S we inductively assume the values at children are correct and end up setting $pmc(S)$ to **true** if and only if G_S has a perfect matching cut, since for each node in S we require a single child neighbor that needs a mate, while all other child neighbors are required to already have a mate. Similarly for $m(S)$ but now all children of $r(S)$ are required to already have a mate. Since each node v of S is a cut vertex of G separating G_S so that each child of S defines its own unique component, we can merge all the cuts in all children while keeping all the nodes of S on the same side of the cut, to satisfy Fact 1 that requires all the nodes in S to be on the same side of the cut. The runtime is clearly polynomial. \square

5. Conclusion

We have shown that, assuming ETH, there is no $O^*(2^{o(n)})$ -time algorithm for PMC even when restricted to n -vertex bipartite graphs, and that PMC remains NP-complete when restricted to bipartite graphs of maximum degree 3 and arbitrarily large girth. This implies that PMC remains NP-complete when restricted to H -free graphs where H is any fixed graph having a vertex of degree at least 4 or a cycle. This suggests the following problem for further research:

Let F be a fixed forest with maximum degree at most 3. What is the computational complexity of PMC restricted to F -free graphs?

We have proved a first polynomial case for this problem where F is a certain 6-vertex tree, including claw-free graphs and graphs without an induced 5-path. Our hardness result also suggests studying PMC restricted to graphs without long induced cycles:

What is the computational complexity of PMC on k -chordal graphs?

It follows from our results that PMC is polynomially solvable for 3-chordal graphs.

We have also given an exact branching algorithm for PMC running in $O^*(1.2721^n)$ time. It is natural to ask whether the running time of the branching algorithm can be improved.

Finally, as for matching cuts, also for perfect matching cuts it would be interesting to determine the computational complexity on planar graphs and to study counting and enumeration as well as FPT and kernelization algorithms.

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