The graph formulation of the union-closed sets conjecture

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Abstract

In 1979 Frankl conjectured that in a finite non-trivial union-closed collection of sets there has to be an element that belongs to at least half the sets. We show that this is equivalent to the conjecture that in a finite non-trivial graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. In this graph formulation other special cases become natural. The conjecture is trivially true for non-bipartite graphs and we show that it holds also for the class of chordal bipartite graphs and the class of bipartitioned circular interval graphs.

1 Introduction

A set $\mathcal{X}$ of sets is *union-closed* if $X, Y \in \mathcal{X}$ implies $X \cup Y \in \mathcal{X}$. The following conjecture was formulated by Peter Frankl in 1979 [6].

Union-closed sets conjecture. *Let $\mathcal{X}$ be a finite union-closed set of sets with $\mathcal{X} \neq \{\emptyset\}$. Then there is a $x \in \bigcup_{X\in\mathcal{X}} X$ that lies in at least half of the members of $\mathcal{X}$.*

In spite of a great number of papers, see e.g. the good bibliography of Marković [11] for papers up to 2007, this conjecture is still wide open. Several special cases are known to hold, for example when $|\mathcal{X}|$ is upper bounded, with current best being 46 by Roberts and Simpson [17], or when $|\bigcup_{X\in\mathcal{X}} X|$ is upper bounded, with current best being 11 by Bošnjak and Marković [1], or when certain sets are present in $\mathcal{X}$, such as a set of size 2 as shown by Sarvate and Renaud [19]. Possibly as a reflection of its general difficulty, Gowers [8] suggested that work on this conjecture could fruitfully be done as a collaborative Polymath project.

Various equivalent formulations are known, in particular by Poonen [13] who among other things translates the conjecture into the language of lattice theory. Several subsequent results together with their proofs belong to lattice theory, for example Reinhold [16] who proves this conjecture for lower semimodular

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lattices. A version of the conjecture is also known for hypergraphs, by El-Zahar [5].

In this paper we give a formulation of the conjecture in the language of graph theory. A set of vertices in a graph is *stable* if no two vertices of the set are adjacent. A stable set is *maximal* if it is maximal under inclusion, that is, every vertex outside has a neighbour in the stable set.

Conjecture 1. *Let G be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.*

Note that Conjecture 1 is true for non-bipartite graphs. Indeed, if vertices u and v are adjacent there is no stable set containing them both and so one of them must belong to at most half of the maximal stable sets. An odd cycle will therefore imply the existence of two adjacent vertices each belonging to at most half of the maximal stable sets. The conjecture is for this reason open only for bipartite graphs. Moreover, in a connected bipartite graph, for any two vertices u and v in different bipartition classes we have a path from u to v containing an odd number of edges, so that if u and v each belongs to at most half the maximal stable sets there will be two adjacent vertices each belonging to at most half the maximal stable sets. Conjecture 1 is therefore equivalent to the following.

Conjecture 2. *Let G be a finite bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.*

In this paper we show that Conjectures 1 and 2 are equivalent to the union-closed sets conjecture. The merit of this graph formulation is that other special cases become natural, in particular subclasses of bipartite graphs. We show that the conjecture holds for the class of chordal bipartite graphs and the class of bipartitioned circular interval graphs. Moreover, the reformulation allows to test Frankl's conjecture in a probabilistic sense: Bruhn and Schaudt [2] show that almost every random bipartite graph satisfies Conjecture 2 up to any given $\varepsilon > 0$, that is, almost every such graph contains in each bipartition class a vertex for which the number of maximal stable sets containing it is at most $\frac{1}{2} + \varepsilon$ times the total number of maximal stable sets.

Stable sets are also called independent sets, with the maximal stable sets being exactly the independent dominating sets. A stable set of a graph is a clique of the complement graph and the graph formulation of the conjecture can also be stated in terms of maximal cliques, instead of maximal stable sets. The set of all maximal stable sets of a bipartite graph, or rather maximal complete bipartite cliques (bicliques) of the bipartite complement graph, was studied by Prisner [14] who gave upper bounds on the size of this set, also when excluding certain subgraphs. More recently, Duffus, Frankl and Rödl [4] and Ilinca and Kahn [10] investigate the number of maximal stable sets in certain regular and biregular bipartite graphs. In work related to the graph parameter boolean-width, Rabinovich, Vatshelle and Telle [15] study balanced bipartitions of a graph that bound the number of maximal stable sets. However, we have not found in the graph theory literature any previous work focusing on the number of maximal stable sets that vertices belong to.

2 Equivalence of the conjectures

For a subset S of vertices of a graph we denote by $N(S)$ the set of vertices adjacent to a vertex in S. All our graphs will be finite, and whenever we consider a union-closed set $\mathcal{X}$ of sets, it will be a finite set, all of whose member-sets will be finite as well. As Poonen [13] observed the latter assumption does not restrict generality, while the conjecture becomes false if $\mathcal{X}$ is allowed to have infinitely many sets.

We need two easy lemmas. The proof of the first is trivial.

Lemma 3. *Let G be a bipartite graph with bipartition U, W, and let S be a maximal stable set. Then $S = (U \cap S) \cup (W \setminus N(U \cap S))$.*

Lemma 4. *Let G be a bipartite graph with bipartition U, W, and let S and T be maximal stable sets. Then $(U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is a maximal stable set.*

Proof. Clearly, $R = (U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is stable. Trivially, any vertex in $W \setminus R$ has a neighbour in R. A vertex u in $U \setminus R$ does not lie in S or not in T (perhaps, it is not contained in either), let us say that $u \notin T$. As T is maximal, u has a neighbour $w \in W \cap T$. This neighbour w cannot be adjacent to any vertex in $U \cap S \cap T$ as T is stable. So, w belongs to R as well, which shows that R is a maximal stable set. □

For a fixed graph G let us denote by $\mathcal{A}$ the set of all maximal stable sets, and for any vertex v let us write $\mathcal{A}_v$ for the sets of $\mathcal{A}$ that contain v and $\mathcal{A}_{\overline{v}}$ for the sets of $\mathcal{A}$ that do not contain v. Let us call a vertex v *unstable* if $|\mathcal{A}_v| \leq \frac{1}{2}|\mathcal{A}|$. A vertex that is not unstable is *solid.*

Theorem 5. *Conjecture 2 is equivalent to the union-closed sets conjecture.*

Proof. Let us consider first a union-closed set $\mathcal{X} \neq \{\emptyset\}$, which, without restricting generality, we may assume to include $\emptyset$ as a member. We put $U = \bigcup_{X \in \mathcal{X}} X$ and we define a bipartite graph G with vertex set $U \cup \mathcal{X}$, where we make $X \in \mathcal{X}$ adjacent with all $u \in X$.

Now we claim that $\tau : S \mapsto U \setminus S$ is a bijection between $\mathcal{A}$ and $\mathcal{X}$. First note that indeed $\tau(S) \in \mathcal{X}$ for every maximal stable set: Set $A = U \cap S$ and $\mathcal{B} = \mathcal{X} \cap S$. If $U \subseteq S$ then $U \setminus S = \emptyset \in \mathcal{X}$, by assumption. So, assume $U \not\subseteq S$, which implies $\mathcal{B} \neq \emptyset$. As S is a maximal stable set, it follows that $U \setminus S = U \setminus A = N(\mathcal{B})$. On the other hand, $N(\mathcal{B})$ is just the union of the $X \in S \cap \mathcal{X} = \mathcal{B}$, which is by the union-closed property equal to a set X' in $\mathcal{X}$. To see that τ is injective note that, by Lemma 3, S is determined by $U \cap S$, which in turn determines $U \setminus S$. For surjectivity, consider $X \in \mathcal{X}$. We set $A = U \setminus N(X)$ and observe that $S = A \cup (\mathcal{X} \setminus N(A))$ is a stable set. Moreover, as $X \in \mathcal{X} \setminus N(A)$ every vertex in $U \setminus A$ is a neighbour of $X \in S$, which means that S is maximal.

Now, assuming that Conjecture 2 is true, there is an unstable $u \in U$, that is, it holds that $|\mathcal{A}_u| \leq \frac{1}{2}|\mathcal{A}|$. Clearly $\mathcal{A}$ is the disjoint union of $\mathcal{A}_u$ and of $\mathcal{A}_{\overline{u}}$, so that

$$|\tau(\mathcal{A}_{\overline{u}})| = |\mathcal{A}_{\overline{u}}| \geq \frac{1}{2}|\mathcal{A}| = \frac{1}{2}|\mathcal{X}|.$$

As $u \in \tau(S) \in \mathcal{X}$ for every $S \in \mathcal{A}_{\overline{u}}$, the union-closed sets conjecture follows.

For the other direction, consider a bipartite graph with bipartition U, W and at least one edge. Define $\mathcal{X} := \{U \setminus S : S \in \mathcal{A}\}$, and note that $\mathcal{X} \neq \{\emptyset\}$ as G has at least two distinct maximal stable sets. By Lemma 3, there is a bijection between $\mathcal{X}$ and $\mathcal{A}$. Moreover, it is a direct consequence of Lemma 4 that $\mathcal{X}$ is union-closed. From this, it is straightforward that Conjecture 2 follows from the union-closed sets conjecture. □

3 Application to two graph classes

For a set X of vertices we define $\mathcal{A}_X$ to be the set of maximal stable sets containing all of X. As before, we abbreviate $\mathcal{A}_{\{x\}}$ to $\mathcal{A}_x$.

Lemma 6. *Let x be a vertex of a bipartite graph G. Then there is an injection $\mathcal{A}_{N(x)} \to \mathcal{A}_x$.*

Proof. We define

$$i : \mathcal{A}_{N(x)} \to \mathcal{A}_x,\ S \mapsto S \setminus L_1 \cup \{x\} \cup (L_2 \setminus N(S \cap L_3)),$$

where L_i denotes the set of vertices at distance i to x. That $i(S)$ is stable and maximal is a direct consequence of the definition. Moreover, $i(S) = i(T)$ for $S, T \in \mathcal{A}_{N(x)}$ implies that S and T are identical outside $L_1 \cup L_2$. Moreover, S and T are also identical on $L_1 \cup L_2$: First, $L_1 = N(x)$ shows that L_1 lies in both S and T. Second, since every vertex in L_2 is a neighbour of one in $L_1 \subseteq S \cap T$, no vertex of L_2 can lie in either of S or T. Thus, $S = T$, and we see that i is an injection. □

We denote by $N^2(x) = N(N(x))$ the second neighbourhood of a vertex x. The following lemma generalises the observation that if a union-closed set contains a singleton then it satisfies the union-closed sets conjecture:

Lemma 7. *Let x, y be two adjacent vertices in a bipartite graph G with $N^2(x) \subseteq N(y)$. Then y is unstable.*

Proof. From $N^2(x) \subseteq N(y)$ it follows that every maximal stable set containing y must contain all of $N(x)$. Thus, $\mathcal{A}_y = \mathcal{A}_{N(x)}$, which means by Lemma 6 that $|\mathcal{A}_y| \leq |\mathcal{A}_x|$ and as $|\mathcal{A}_y| + |\mathcal{A}_x| \leq |\mathcal{A}|$ the lemma is proved. □

We now apply the lemma to the class of *chordal bipartite* graphs. This is the class of bipartite graphs in which every cycle with length at least six has a chord.

This graph class was originally defined in 1978 by Golumbic and Gross [7]. It is also known as the class of bipartite weakly chordal graphs.

A vertex v in a bipartite graph is *weakly simplicial* if the neighbourhoods of its neighbours form a chain under inclusion. Hammer, Maffray and Preissmann [9], and also Pelsmajer, Tokaz and West [12] prove the following:

Theorem 8. *A bipartite graph with at least one edge is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Moreover, such a vertex can be found in each of the two bipartition classes.*

Let us say that a bipartite graph *satisfies the union-closed sets conjecture* if each of its bipartition classes contains a unstable vertex.

Theorem 9. *Any chordal bipartite graph with at least one edge satisfies the union-closed sets conjecture.*

Proof. For a given bipartition class, let x be a weakly simplicial vertex in it. Among the neighbours of x denote by y the one whose neighbourhood includes the neighbourhoods of all other neighbours of x. Then y is unstable, by Lemma 7. □

Going beyond chordal bipartite graphs, we quickly encounter graphs that cannot be handled anymore by Lemma 7: No vertex in an even cycle of length at least six can be proved to be unstable by applying Lemma 7. We will, therefore, strengthen the lemma to at least cover all even cycles.

For this, let us extend our notation a bit. For two vertices u, v let us denote by $\mathcal{A}_{uv}$ the set of $S \in \mathcal{A}$ containing both of u and v, by $\mathcal{A}_{u\overline{v}}$ the set of $S \in \mathcal{A}$ containing u and but not v, and by $\mathcal{A}_{\overline{uv}}$ the set of $S \in \mathcal{A}$ containing neither of u and v.

Lemma 10. *Let G be a bipartite graph. Let y and z be two neighbours of a vertex x so that $N^2(x) \subseteq N(y) \cup N(z)$. Then one of y and z is unstable.*

Proof. We may assume that $|\mathcal{A}_{y\overline{z}}| \leq |\mathcal{A}_{\overline{y}z}|$. Now, from $N^2(x) \subseteq N(y) \cup N(z)$ we deduce that $\mathcal{A}_{yz} = \mathcal{A}_{N(x)}$. Thus, by Lemma 6, we obtain $|\mathcal{A}_{yz}| \leq |\mathcal{A}_x|$. Since $\mathcal{A}_x \subseteq \mathcal{A}_{\overline{yz}}$ it follows that $|\mathcal{A}_y| = |\mathcal{A}_{y\overline{z}}| + |\mathcal{A}_{yz}| \leq |\mathcal{A}_{\overline{y}z}| + |\mathcal{A}_{\overline{yz}}| = |\mathcal{A}_{\overline{y}}|$. As $|\mathcal{A}| = |\mathcal{A}_y| + |\mathcal{A}_{\overline{y}}|$, we see that y is unstable. □

Again, the lemma generalises a fact that is well known for the set formulation of the union-closed sets conjecture: If one of the sets in the union-closed set $\mathcal{X}$ contains exactly two elements then one of the two elements will lie in at least half of the members of $\mathcal{X}$; see Sarvate and Renaud [19].

We give an application of Lemma 10 to a class of graphs derived from *circular interval graphs*. The class of circular interval graphs plays a fundamental role in the structure theorem of claw-free graphs of Chudnovsky and Seymour [3]. Circular interval graphs are defined as follows: Let a finite subset of a circle be the vertex set, and for a given set of subintervals of the circle consider two vertices to be adjacent if there is an interval containing them both. This class is equivalent to what is known as the proper circular arc graphs.

Circular interval graphs are not normally bipartite. The only exceptions are even cycles and disjoint unions of paths. Nevertheless, we may obtain a rich class of bipartite graphs from circular interval graphs: For any circular interval graph, partition its vertex set and delete every edge with both its endvertices in the same class. We call any graph arising in this manner a *bipartitioned circular interval graph*.

Theorem 11. *Every bipartitioned circular interval graph with at least one edge satisfies the union-closed sets conjecture.*

Proof. Consider a bipartitioned circular interval graph defined by intervals $\mathcal{I}$, and let x be a non-isolated vertex of the graph.

For every neighbour u of x we choose an interval $I_u \in \mathcal{I}$ containing both x and u. If $\bigcup_{v \in N(x)} I_v$ covers the whole circle, then there are already two such intervals I_y and I_z that cover the circle. Clearly, every vertex not in the same

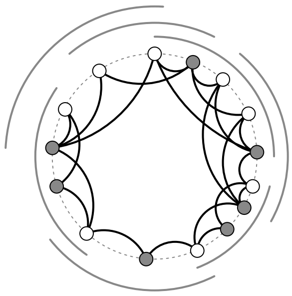

Figure 1: A bipartitioned circular interval graph

bipartition class as y and z is adjacent to at least one of them. In particular, $N^2(x) \subseteq N(y) \cup N(z)$.

So, let us assume that there is a point p on the circle that is not covered by any I_v, $v \in N(x)$. We choose y as the first neighbour of x from p in clockwise direction, and z as the first neighbour of x from p in counterclockwise direction. Then y, v, z appear in clockwise order for every $v \in N(x)$ and $v' \in I_y \cup I_z$ for every vertex v' so that y, v', z appear in clockwise order.

Let us show that again $N^2(x) \subseteq N(y) \cup N(z)$. For this consider a $u \in N^2(x)$, and a neighbour w of x that is adjacent to u. Thus, there is a $J \in \mathcal{I}$ containing both u and w. If y, u, z appear in clockwise order, then $u \in I_y \cup I_z$, which implies $u \in N(y) \cup N(z)$. If not, then J meets one of y or z as y, w, z appear in clockwise order. Thus, by virtue of J, the vertex u is adjacent to at least one of y and z.

In both cases, we apply Lemma 10 in order to see that one of y and z is unstable. As the choice of x was arbitrary, we find unstable vertices in both bipartition classes. □

4 Discussion

Lemmas 7 and 10 generalise the cases when there is a vertex x of degree 1 or 2. Then, one of the neighbours of x is unstable. Unfortunately, none of the neighbours of a degree 3 vertex have to be unstable. This is, for instance, the case for the vertex v in Figure 2 on the left.

Again, this is not new, in the sense that it corresponds directly to an observation of Sarvate and Renaud [18] in the set formulation: A set of size three need not contain any element appearing in at least half of the member sets of the union-closed set.

As chordal bipartite graphs are exactly the $(C_6, C_8, C_{10}, \ldots)$-free graphs one may be tempted to generalise Theorem 9 by allowing one more even cycle, the 6-cycle, as induced subgraph. While Lemma 7 is no longer strong enough even for the C_6, Lemma 10 easily takes care of any graph with a degree 2 vertex in each bipartition class. In general, however, Lemma 10 turns out to be too weak as well to prove the conjecture for $(C_8, C_{10}, C_{12}, \ldots)$-free graphs: The graph on the right in Figure 2 is of that form but has no vertices covered by Lemma 10.

We close with two open problems.

1. Does every cubic bipartite graph satisfy the union-closed sets conjecture?

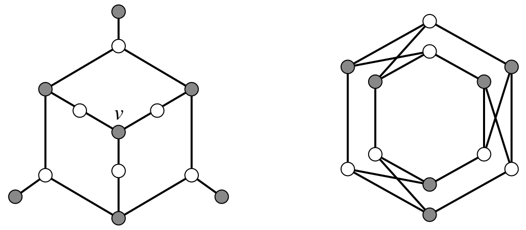

Figure 2: Left: No neighbour of v is unstable. Right: Lemmas 7 or 10 not applicable

2. What is the complexity to decide whether a given vertex is unstable?

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