Strongly chordal and chordal bipartite graphs are sandwich monotone*

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Abstract

A graph class is sandwich monotone if, for every pair of its graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subset E_2$, there is an ordering $e_1, \ldots, e_k$ of the edges in $E_2 \setminus E_1$ such that $G = (V, E_1 \cup \{e_1, \ldots, e_i\})$ belongs to the class for every i between 1 and k. In this paper we show that strongly chordal graphs and chordal bipartite graphs are sandwich monotone, answering an open question by Bakonyi and Bono from 1997. So far, very few classes have been proved to be sandwich monotone, and the most famous of these are chordal graphs. Sandwich monotonicity of a graph class implies that minimal completions of arbitrary graphs into that class can be recognized and computed in polynomial time. For minimal completions into strongly chordal or chordal bipartite graphs no polynomial-time algorithm has been known. With our results such algorithms follow for both classes. In addition, from our results it follows that all strongly chordal graphs and all chordal bipartite graphs with edge constraints can be listed efficiently.

1 Introduction

A graph class is *hereditary* if it is closed under induced subgraphs, and *monotone* if it is closed under subgraphs that are not necessarily induced. Every monotone graph class is also hereditary, since removing any edge keeps the graph in the class, but the converse is not true. For example perfect graphs are hereditary but not monotone, since we can create chordless odd cycles by removing edges. Some of the most well-studied graph properties are monotone [1, 3] or hereditary [12]. Between hereditary and monotone graph classes are *sandwich monotone* graph classes. Monotonicity implies sandwich monotonicity (if we can remove any edge, we can also remove the edges in a particular order), which again implies being hereditary for graph classes that allow isolated vertices (if we can reach any subgraph in the class by removing edges, we can also reach induced subgraphs leaving the desired vertices isolated), but none of the reverse chain of implications holds. In this paper we study hereditary graph classes that are not monotone, and we resolve the *sandwich monotonicity* of two of them: strongly chordal graphs and chordal bipartite graphs.

Chordal graphs are the most famous class of graphs that are sandwich monotone [23]. Besides this, split [13], chain [15], and threshold [15] graphs are the only known sandwich monotone

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classes. On the other hand we know that cographs, interval, proper interval, comparability, permutation, and trivially perfect graphs are *not* sandwich monotone [15]. The following graph classes have been candidates of sandwich monotonicity since an open question in 1997 [2]: strongly chordal, weakly chordal, and chordal bipartite. Among the graph classes known to be sandwich monotone, chordal graphs are the only ones that are not characterized by a finite set of forbidden induced subgraphs. Similarly, strongly chordal and chordal bipartite graphs have an *infinite* set of forbidden induced subgraphs. Thus, we can say that after the chordal graphs, they are the first non-trivial graph classes for which sandwich monotonicity is proved.

Our main motivation for studying sandwich monotonicity comes from the problem of *completing* a given arbitrary graph into a graph class, meaning adding edges so that the resulting graph belongs to the desired class. For example, a *chordal completion* is a chordal supergraph on the same vertex set. A completion is *minimum* if it has the smallest possible number of added edges. The problem of computing minimum completions are applicable in several areas such as molecular biology, numerical algebra and, more generally, to areas involving graph modeling with some missing edges due to lacking data [11, 19, 22]. Unfortunately minimum completions into most interesting graph classes, including strongly chordal graphs [26, 16], are NP-hard to compute [19]. However, minimum completions are a subset of minimal completions, and hence we can search for minimum among the set of minimal. A completion is *minimal* if no subset of the added edges can be removed from it without destroying the desired property.

If a graph class is sandwich monotone then a completion into the class is minimal if and only if *no single added edge* can be removed from it [23, 13, 15], making minimality of a completion much easier to check. For a graph class that can be recognized in polynomial time, sandwich monotonicity implies that minimal completions into this class can be computed in polynomial time. More importantly, it implies that whether a given completion is minimal can be decided in polynomial time, which is a more general problem. This latter problem is so far solvable for completions into only two non sandwich monotone classes: interval graphs [14] and cographs [18]. As an example of usefulness of a solution of this problem, various characterizations of minimal chordal completions [16, 6] have made it possible to design approximation algorithms [19] and fast exact exponential time algorithms [10] for computing minimum chordal completions. A solution of this problem also allows the computation of minimal completions that are not far from minimum in practice [5]. With the results that we present in this paper, we are able to characterize minimal strongly chordal completions of arbitrary graphs and minimal chordal bipartite completions of arbitrary bipartite graphs.

In a COCOON 2008 paper, Kijima et al. give an efficient algorithm for the following problem [17]. Given two graphs on the same vertex set such that one is chordal and one is a subgraph of the other, list all chordal graphs that are sandwiched between the two graphs. In fact for the solution of this problem the only necessary property of chordal graphs is sandwich monotonicity. Hence, with our results this problem can also be solved efficiently for strongly chordal and chordal bipartite graphs.

2 Preliminaries

We consider undirected finite graphs with no loops or multiple edges. For a graph $G = (V, E)$, we denote its vertex and edge set by $V(G) = V$ and $E(G) = E$, respectively, with $n = |V|$ and $m = |E|$. For a vertex subset $S \subseteq V$, the *subgraph of G induced by S* is denoted by $G[S]$.

Moreover, we denote by $G - S$ the graph $G[V \setminus S]$ and by $G{-}v$ the graph $G[V \setminus \{v\}]$. In this paper, we distinguish between *subgraphs* and *induced subgraphs.* By a *subgraph* of G we mean a graph G' on the same vertex set containing a subset of the edges of G, and we denote it by $G' \subseteq G$. If G' contains a proper subset of the edges of G, we write $G' \subset G$. We write $G - uv$ to denote the graph $(V, E \setminus \{uv\})$.

The *neighborhood* of a vertex x of G is $N_G(x) = \{v \mid xv \in E\}$. The *closed neighborhood* of x is defined as $N_G[x] = N_G(x) \cup \{x\}$. If $S \subseteq V$, then the neighbors of S, denoted by $N_G(S)$, are given by $\bigcup_{x \in S} N_G(x) \setminus S$. We will omit the subscript when there is no ambiguity. The *length* of a path or a cycle is the number of edges on the path or the cycle. A *chord* of a cycle is an edge between two nonconsecutive vertices of the cycle. A chordless cycle on k vertices is denoted by C_k. A graph is *chordal* if it does not contain an induced C_k for $k > 3$. A *perfect elimination ordering* of a graph $G = (V, E)$ is an ordering $v_1, \ldots, v_n$ of V such that for each i, j, k, if $i < j$, $i < k$, and $v_i v_j, v_i v_k \in E$ then $v_j v_k \in E$. Rose has shown that a graph is chordal if and only if it admits a perfect elimination ordering [21].

A *clique* is a set of vertices that are pairwise adjacent, and an *independent set* is a set of vertices that are pairwise non-adjacent. A vertex is called *simplicial* if the subgraph induced by its neighborhood is a clique. Observe that a perfect elimination ordering is equivalent to removing a simplicial vertex repeatedly until the graph becomes empty. For a vertex v, the *deficiency* $D(v)$ is the set of non-edges in $N(v)$; more precisely, $D(v) = \{xy \mid vx, vy \in E, xy \notin E\}$. Thus if $D(v) = \emptyset$, v is simplicial.

A *strong elimination ordering* of a graph $G = (V, E)$ is an ordering of the vertices $v_1, \ldots, v_n$ of V such that for each i, j, k, l with $i \leq k$ and $i < l$, if $i < j$, $k < l$, $v_i v_k, v_i v_l \in E$ and $v_j v_k \in E$ then $v_j v_l \in E$. A graph is *strongly chordal* if it admits a strong elimination ordering. It is known that every induced subgraph of a strongly chordal graph is strongly chordal [8]. Moreover *every* strong elimination ordering is a perfect elimination ordering (by setting $i = k$) but the converse is not necessarily true. Thus all strongly chordal graphs are chordal.

Two vertices u and v of a graph G are called *compatible* if $N[u] \subseteq N[v]$ or $N[v] \subseteq N[u]$; otherwise they are called *incompatible.* Given two incompatible vertices u and v the *u-private neighbors* are exactly the vertices of the set $N[u] \setminus N[v]$. A vertex v of G is called *simple* if the neighbors of the vertices of $N[v]$ are linearly ordered by set inclusion, that is, the vertices of $N[v]$ are pairwise compatible. Clearly, any simple vertex is simplicial but not necessarily vice versa. An ordering $v_1, \ldots, v_n$ of a graph G is called *simple elimination ordering* if for each $1 \leq i \leq n$, v_i is simple in the graph $G_i \equiv G[\{v_i, \ldots, v_n\}]$.

Theorem 2.1 ([8])**.** *A graph is strongly chordal if and only if it has a simple elimination ordering.*

A *k-sun* (also known as *trampoline*), for $k \geq 3$, is the graph on $2k$ vertices obtained from a clique $\{c_1, \ldots, c_k\}$ on k vertices and an independent set $\{s_1, \ldots, s_k\}$ on k vertices and edges $s_i c_i, s_i c_{i+1}$, $1 \leq i < k$, and $s_k c_k, s_k c_1$.

Theorem 2.2 ([8])**.** *A chordal graph is strongly chordal if and only if it does not contain a k-sun as an induced subgraph.*

Based on the above theorem we prove the following which can also be derived by using some known results.

Lemma 2.3. *Let G be a k-sun with $K = \{c_1, \dots, c_k\}$ and $I = \{s_1, \dots, s_k\}$ as its clique and independent set, respectively. If any set of edges is removed among the vertices of K, the resulting graph is still not strongly chordal.*

Proof. Proving the statement is equivalent to proving that there is no strongly sandwich graph between the chordless cycle $C = \{c_1, s_1, c_2, s_2, \dots, c_k, s_k\}$, and the k-sun; i.e., there is no strongly chordal H such that $C \subseteq H \subseteq G$. We assume that all indices are always taken modulo k. Assume for the sake of contradiction that a strongly chordal sandwich graph H exists. Then H has at least one simple vertex. However none of the vertices s_i for $1 \leq i \leq k$ can ever be simple since they have two incompatible vertices in their neighborhood, i.e., c_i and c_{i+1}. Each of them has in fact a private neighbor with respect to the other one, namely s_{i-1} and s_{i+1}. Neither of the vertices c_i for $1 \leq i \leq k$ can be simple as well, since they can never be simplicial either in H because s_{i-1} and s_i are non-adjacent for any H. □

For our studies of strongly chordal graphs, we will need the following definitions regarding the neighborhood of a simple vertex x. We partition the sets $N(x)$ and $S(x) \equiv N(N(x)) \setminus \{x\}$. $(N_0, N_1, \dots, N_k)$ is a partition of $N(x)$ such that $N_0 = \{y \in N(x) \mid N[x] = N[y]\}$ and $N(N_0) \subset N(N_1) \cdots \subset N(N_k)$ where k is as large as possible.

These sets are also used to partition $S(x)$ into $(S_1, \dots, S_k)$ where $S_1 = N(N_1) \setminus N[x]$ and $S_i = N(N_i) \setminus (N(N_{i-1}) \cup N[x])$, for $2 \leq i \leq k$. We call the above partition a *simple partition* with respect to x.

Figure 1: The left graph is a 3-sun. It has no simple vertices. In the right graph, x and c are simple vertices, whereas a is a simplicial vertex but not simple. For the graph on the right side a simple partition with respect to x is given by the sets $N(x) = (\{y\}, \{z\})$ and $S(x) = (\{a, b\}, \{c\})$.

In the context of a minimal completion of a given graph into a graph belonging to a given class, we say that a *completion* $G' = (V, E \cup F)$ of an arbitrary graph $G = (V, E)$ is any supergraph G' of G on the same vertex set with the property that G' belongs to the given graph class. If $\mathcal{C}$ is a graph class, then we refer to G' as a $\mathcal{C}$ completion of G. For instance, a strongly chordal completion of any graph $G = (V, E)$ is the complete graph on V. The edges that are in G' but not in G are called *added edges.* A $\mathcal{C}$ completion is *minimal* if no proper subset of the added edges, when added to the input graph G, results in a graph in the class.

Although sandwich monotonicity has been a well studied property since 1976 [23], it was first given a name and a proper definition in a COCOON 2007 paper [15]:

Definition 2.4 ([15])**.** A graph class $\mathcal{C}$ is *sandwich monotone* if the following is true for any pair of graphs $G = (V, E)$ and $H = (V, E \cup F)$ in $\mathcal{C}$ with $E \cap F = \emptyset$: There is an ordering $f_1, f_2, \dots, f_{|F|}$ of the edges in F such that in the sequence of graphs $G = G_0, G_1, \dots, G_{|F|} = H$, where G_{i-1} is obtained by removing edge f_i from G_i, every graph belongs to $\mathcal{C}$.

Observation 2.5 ([23, 15])**.** *The following are equivalent on any graph class $\mathcal{C}$:*

(i) $\mathcal{C}$ *is sandwich monotone.*

(ii) *A* $\mathcal{C}$ *completion is minimal if and only if no single added edge can be removed without leaving* $\mathcal{C}$*.*

Although sandwich monotonicity implies a polynomial-time algorithm for problems related to minimal completions [15], there are other problems for which such a notion is applicable. Motivated by the work of Kijima et al. we give a connection to the problem of listing all graphs of a certain graph class under edge constraints that has been introduced in a 2008 COCOON paper [17]. In such a problem we are given two graphs G_1 and G_2 on the same vertex set such that $G_1 \subset G_2$, at least one of G_1, G_2 belongs to a graph class, and the task is to list (output) all graphs of the given graph class that contain G_1 and are contained in G_2. Such an algorithm has been explicitly given for chordal graphs that runs in polynomial time per output and in polynomial space [17]. However such an algorithm can be easily adopted for *any* sandwich monotone class by using Definition 2.4 and a simple binary partition method.

Proposition 2.6 ([17])**.** *Let* $\mathcal{C}$ *be a sandwich monotone graph class and let* G_1 *and* G_2 *be two graphs such that* $G_1 \subset G_2$ *and either* $G_1 \in \mathcal{C}$ *or* $G_2 \in \mathcal{C}$*. Given a polynomial-time algorithm for the recognition of* $\mathcal{C}$*, there is an algorithm for listing all graphs* G *in* $\mathcal{C}$ *such that* $G_1 \subseteq G \subseteq G_2$ *and the running time is polynomial in the input size per output and the memory usage is bounded by a polynomial in the input size.*

Proof. We describe such an algorithm which is actually a generalization of the algorithm given for chordal graphs [17]. Assume that we are given a pair of boundary graphs (G_1, G_2) such that $G_1 = (V, E)$, $G_2 = (V, E \cup F)$ and $G_2 \in \mathcal{C}$; the case for which $G_1 \in \mathcal{C}$ is symmetric. If there is an edge $f \in F$ such that $G_2 - f \in \mathcal{C}$ then we output $G_2 - f$ and run recursively the algorithm on the pairs of graphs $(G_1 + f, G_2)$ and $(G_1, G_2 - f)$; note that in both recursive calls the bigger graph belongs to $\mathcal{C}$. If such an edge does not exist then by the sandwich monotonicity of $\mathcal{C}$ we know that there is no other graph that belongs to $\mathcal{C}$. The correctness of the algorithm comes from Definition 2.4 and the fact that the set of the desired graphs having the edge f are output by $(G_1 + f, G_2)$ whereas the sets of the desired graphs that do not have the edge f are considered in the call $(G_1, G_2 - f)$. Furthermore the running time is polynomial per output by $|F|$ calls of the recognition algorithm for finding an appropriate edge f in G_2. □

3 Strongly chordal graphs are sandwich monotone

In this section we prove that strongly chordal graphs are sandwich monotone, and using this result we will characterize minimal strongly chordal completions of arbitrary graphs.

It is easy to see that if a single edge is added to a C_k with $k \geq 5$, then a $C_{k'}$ is created with $k' \geq 4$. First we show that a similar result holds for k-suns.

Observation 3.1. *For* $k \geq 4$*, if a single edge is added to a* k*-sun to produce a chordal graph, then a* k'*-sun with* $k > k' \geq 3$ *is created.*

Proof. Let G be a k-sun with $K = \{c_1, \ldots, c_k\}$ and $I = \{s_1, \ldots, s_k\}$ as its clique and independent set, respectively. Adding an edge between vertices of I results in a C_4, since there are always two private neighbors for any two vertices of S. Any other added edge must be between a vertex of I, say s_i, and a non-neighbor of s_i in K, say c_j, $j \neq i, i+1$. Assume first that $i < j$.

Removing the set of vertices $\{c_{i+1}, s_{i+1}, \ldots, c_{j-1}, s_{j-1}\}$ results in k'-sun with $k' = k - (j - i - 1)$. Similarly if $j < i$ then removing the set of vertices $\{s_j, c_{j+1}, \ldots, s_{i-1}, c_i\}$ results in k'-sun with $k' = k - (i - j - 1)$. $\square$

Next, we show that when a new vertex is added to a strongly chordal graph ensuring that the new vertex is simple in the larger graph, the larger graph is also strongly chordal.

Lemma 3.2. *Let x be a simple vertex in a graph $G = (V, E)$. If $G{-}x$ is strongly chordal then G is strongly chordal.*

Proof. As $G{-}x$ is a strongly chordal graph, it admits a simple elimination ordering α. Choosing x first and then processing the vertices in order α results in a simple elimination ordering for G. $\square$

The following lemma is well-known for chordal graphs, and we want to show a similar result for strongly chordal graphs.

Lemma 3.3 ([23])**.** *Let G be a chordal graph and let x be a simplicial vertex of G. Removing an edge incident to x results in a chordal graph.*

Let G be a strongly chordal graph and let x be a simplicial vertex of G. A vertex y of $N(x)$ is a *guard* if there exist at least two incompatible vertices u, v of $N(x) \setminus \{y\}$ such that y is adjacent to at least one of u-private neighbors with respect to v, and y is adjacent to at least one of v-private neighbors with respect to u. The u-private neighbors and v-private neighbors with respect to v and u, respectively, that are adjacent to y are called *guarded vertices*. If a vertex of $N(x)$ is not a guard then it is a *non-guard*.

Lemma 3.4. *Let G be a strongly chordal graph. Let x be a simplicial vertex of G and let xy be an edge of G. $G{-}\{xy\}$ is strongly chordal if and only if y is not a guard vertex for x.*

Proof. Let $G' = G{-}\{xy\}$. By Lemma 3.3 G' is chordal. By Theorem 2.2 and Observation 3.1 G' is not strongly chordal if and only if it contains a 3-sun. Vertices x and y are in the 3-sun in G' since G is a strongly chordal graph. In particular x belongs to the independent set of the 3-sun, since $N_{G'}(x)$ is a clique in G'. If y belongs to the independent set of the 3-sun then G is not chordal. Thus y belongs to the clique of the 3-sun. Therefore only if y is adjacent to at least two private neighbors of two incompatible vertices of $N_G(x)$ there is a 3-sun in G', meaning that y must be a guard in order to obtain a 3-sun in G'. $\square$

Corollary 3.5. *Let G be a strongly chordal graph and let x be a simple vertex of G. Removing an edge incident to x results in a strongly chordal graph.*

Proof. Since x is a simple vertex, none of the vertices in $N(x)$ are guards by their inclusion ordered property. Thus by Lemma 3.4 the result follows. $\square$

Let $G = (V, E)$ and $G' = (V, E \cup F)$ be two strongly chordal graphs such that $E \cap F = \emptyset$ and $F \neq \emptyset$. Let x be a simple vertex of G' and let $(N_0, N_1, \ldots, N_k)$ and $(S_1, \ldots, S_k)$ be a simple partition with respect to x. Let $u \in S_i$, $1 \leq i < k$. We denote by p_u the smallest index $i \leq p_u < k$ such that $v \in N_{p_u}$ and $vu \in E$. We define the following set of edges:

$$C(x) = \{uv \mid u \in S_i, v \in N_j, i \leq p_u < j \leq k\}.$$

Observe that $C(x)$ does not contain any edge of the form uv where $u \in S_i$ and $v \in N_i$, $1 \leq i \leq k$.

Lemma 3.6. *Let $G = (V, E)$ and $G' = (V, E \cup F)$ be two strongly chordal graphs such that $E \cap F = \emptyset$ and $F \neq \emptyset$. There exists a simple vertex x of G' such that $F \nsubseteq D(x) \cup C(x)$.*

Proof. Let x be a simple vertex in G', but not necessarily simple in G. If $F \nsubseteq D(x) \cup C(x)$ then we are done; so assume that $F \subseteq D(x) \cup C(x)$. In this case we show that any vertex $y \neq x$ that is simple both G, is also simple in G' and that $F \nsubseteq D(y) \cup C(y)$.

First we show by contradiction that if a vertex y is simple both in G and G', then $F \nsubseteq D(y) \cup C(y)$. Assume $F \subseteq D(y) \cup C(y)$, then $D(y) = \emptyset$. If not, since y is simplicial in G, we would have $N_G(y) \neq N_{G'}(y)$. Hence there would be at least one edge of F incident to y, i.e., not in $D(y) \cup C(y)$. However, if $F \subseteq C(y)$, we show that y cannot be simple in G. Let us take a simple partition $(N_0, N_1, \ldots, N_k)$ and $(S_1, \ldots, S_k)$ in G' with respect to y. We assumed that $F \neq \emptyset$, so there exists at least one edge $uv \in C(y)$ that belongs to F, such that $u \in S_i$ and $v \in N_j$ with $i < j$. By the definition of $C(y)$, there exists also a vertex $w \in N_{p_u}$ with $p_u < j$ such that $uw \in E$, and therefore $i \neq j$. This implies that $S_j \neq \emptyset$ and every edge vz with $z \in S_j$ belongs to E. Hence w and v are two incompatible neighbors of y in G, which gives the desired contradiction concluding this part of the proof.

Now we prove that y is indeed simple in G' by distinguishing two cases. For the following arguments we define $(N_0, N_1, \ldots, N_k)$ and $(S_1, \ldots, S_k)$ as a simple partition of $N(x)$ and $S(x)$ in G'. Observe also that $N_G(x) = N_{G'}(x)$.

- $y \in N(x)$: First we show that $y \in N_0$. As $F \subseteq D(x) \cup C(x)$, all edges between vertices of N_i and S_i are also in G, for every $1 \leq i \leq k$. This implies that if y is in N_i for any $i \neq 0$, it must have a neighbor $a \in S_i$ such that $ya \in E$. However this implies that also $xa \in E$ since y is simplicial in G, giving a contradiction. Now, if $y \in N_0$, then $N_{G'}(y) = N_{G'}(y)$, hence y is simple in G' because x is.

- $y \notin N(x)$: In this case let us define $N_F(y) = N_{G'}(y) \setminus N_G(y)$. Then it is easy to notice that $N_F(y) \subseteq N(x)$ and $ya \in C(x)$ for every $a \in N_F(y)$. This implies that both $N_F(y)$ and $N_G(y)$ are cliques, and if there are two incompatible vertices in $N_{G'}(y)$, then either one of them belongs to $N_F(y)$ and the other one to $N_G(y)$, or they both belong to $N_G(y)$.

  First we prove that for any two vertices $a \in N_F(y)$ and $b \in N_G(y)$, the edge ab is in $E \cup F$, i.e, that $N_{G'}(y)$ is a clique in G', and that they are compatible. If both a and b are in $N(x)$, then they are both adjacent and compatible, so we can assume that $a \in N_j$ and $b \notin N(x)$, implying that $y \in S_i$ with $i < j$. Then, by the definition of $C(x)$, there exists a vertex $c \in N_{p_y}$ such that $yc \in E$, and, by the fact that y is simplicial in G, we know that $cb \in E$ as well. This implies that a is adjacent to b because $N_G(c) \subset N_G(a)$ by the inclusion property of the neighborhood of x. Besides c and b are compatible in G because they are both in $N_G(y)$. Thus, either $N_G(c) \subseteq N_G(b)$ or $N_G(b) \subseteq N_G(c)$. In the first case x would be adjacent to b since $xc \in E$, which is a contradiction. For the second case observe that $N_{G'}(b)$ is the union of $N_G(b)$ with the endpoints of all edges in F incident to b. Since $b \in S(x)$, all such endpoints are in $N_{G'}(x)$ and we can conclude that $N_{G'}(b) \subset N_{G'}(a)$ since $N_{G'}(b) \subseteq N_G(c) \cup N_{G'}(x) \subseteq N_{G'}(c) \subset N_{G'}(a)$.

  If both a and b are in $N_G(y)$, then we only need to prove that they are compatible in G'. By using the same arguments as the previous case and replacing c with a, we can show that if either $a, b \in N(x)$ or $a \in N(x)$ and $b \in S(x)$, they are compatible. What is left is when both a and b are in $S(x)$. Assume there is a neighbor a' of a that is not adjacent to b, and a neighbor b' of b that is not adjacent to a. We show that this cannot happen, by proving the existence of the edge $a'b$. Clearly, if both aa' or bb' belonged to E, then y would not be simple in G. So let us assume that at least one of them, let us say aa', is in F, so that $a' \in N_h$ and $a \in S_g$ with $1 \leq g < h \leq k$. Then there is a vertex $c \in N_{p_a}$ so that $ca \in E$. If $bb' \in E$ then $cb \in E$ because y is simple in G, implying that $a'b \in E \cup F$ since $N_{G'}(c) \subseteq N_{G'}(a')$. If $bb' \notin E$ then $b' \in N_{G'}(x)$, in particular $b' \in N_f$, for $f < g$ since b' and a are non-adjacent. Then by $N_{G'}(b') \subseteq N_{G'}(c) \subseteq N_{G'}(a')$ we obtain $a'b \in E \cup F$.

Therefore no two vertices adjacent to y are incompatible in G' which implies that y is a simple vertex in both graphs G and G' such that $F \nsubseteq D(y) \cup C(y)$. □

The following lemma describes that in a strongly chordal graph we can turn the neighborhood of any vertex into a clique by adding edges so that the strongly chordal property is preserved. It is known that a perfect elimination ordering of a chordal graph $G = (V, E)$ is also a perfect elimination ordering for $G' = (V, E \cup D(x))$ where x is any vertex of G [23]. One might suspect that a similar result holds for strongly chordal graphs with respect to strong elimination orderings. Unfortunately this is not the case. However by considering simple elimination orderings we are able to prove the following lemma.

Lemma 3.7. *Let $G = (V, E)$ be a strongly chordal graph and let x be a vertex of G. Then $G' = (V, E \cup D(x))$ is a strongly chordal graph.*

Proof. Since G is strongly chordal it admits a simple elimination ordering. Let β be any simple elimination ordering of G. We prove that β is also a simple elimination ordering of G'. Observe first that β is a perfect elimination ordering of G' [23].

Assume for contradiction that β is not a simple elimination ordering of G'. Then there exist two adjacent vertices w_1, w_2 that are incompatible in G'_i, for some $1 \leq i \leq n$. This means that there exist at least two vertices z_1 and z_2 such that $w_1z_1, w_2z_2 \in E(G'_i)$ and $w_1z_2, w_2z_1 \notin E(G'_i)$. Since β is simple for G and we only add edges in G', at least one of the edges w_1z_1 w_2z_2 is added because of x. If both of them are added because of x then all four vertices w_1, w_2, z_1, z_2 are adjacent to x in G and w_1, w_2 are compatible in G'_i.

Without loss of generality assume that $xw_1, xz_1 \in E(G)$ so that $w_1z_1 \in E(G'_i)$. Now if $xw_2 \in E(G)$ or $xz_2 \in E(G)$ then $z_1w_2 \in E(G'_i)$ or $z_2w_1 \in E(G'_i)$, respectively, meaning that w_1, w_2 are incompatible in G'_i. Thus $xw_2, xz_2 \notin E(G)$. Remember that by assumption we have $\beta^{-1}(w_1), \beta^{-1}(w_2), \beta^{-1}(z_1), \beta^{-1}(z_2) \geq i$. We consider now the position of x in the ordering β. If $\beta^{-1}(x) < i$ then β is not a perfect elimination ordering for G since $xw_1, xz_1 \in E(G)$, $\beta^{-1}(x) < \beta^{-1}(w_1)$ and $\beta^{-1}(x) < \beta^{-1}(z_1)$. Hence we are left with the case of $\beta^{-1}(x) \geq i$. But then in such a case β is not a simple elimination ordering for G_i since w_1, w_2 are incompatible in G_i because $w_1x, w_2z_2 \in E(G)$ and $w_2x, z_2x \notin E(G)$.

Therefore in all cases we get a contradiction and thus β is a simple elimination ordering of G' which implies that G' is strongly chordal. □

In the following two statements we let $G = (V, E)$ and $G' = (V, E \cup F)$ be two strongly chordal graphs such that $E \cap F = \emptyset$ and $F \neq \emptyset$. Let x be simple vertex of G' such that $F \nsubseteq D(x) \cup C(x)$. By Lemma 3.6 such a vertex exists. We denote by $(N_0, N_1, \ldots, N_k)$ and $(S_1, \ldots, S_k)$ a simple partition with respect to x in G'.

Observation 3.8. *Let $H = (V, E \cup D(x) \cup C(x))$.*

1. *For any two vertices $u \in N_i$ and $v \in N_j$, $1 \leq i < j \leq k$, $N_H[u] \subseteq N_H[v]$.*

2. *Let $u \in N_i$ and $v \in N_j$ be two incompatible vertices in H, $1 \leq i, j \leq k$. Then $i = j$ and their private neighbors in H are exactly their private neighbors in G.*

3. *For every edge $yu \in C(x)$ such that $u \in S_i$ and $y \in N_j$, $i < j$, there is a vertex $w \in N_{p_u}$ such that $i \leq p_u < j$, $wu \in E$ and $N_H[w] \subseteq N_H[y]$.*

Proof. First observe that because of $D(x)$, the set $N_H(x)$ is a clique and contains exactly the vertices of the sets $N_0, N_1, \ldots, N_k$. Assume that the first statement does not hold. Let u' be a vertex adjacent to u and non-adjacent to v in H. If $u'u \in C(x)$, then $p_{u'} < i$ and u' is adjacent to every vertex of $N_i, \ldots, N_j, \ldots, N_k$. If $u'u \notin C(x)$, then it must be $p_{u'} = i$, and u' is adjacent to every vertex of $N_{i+1}, \ldots, N_j, \ldots, N_k$, proving the first statement and that the only case for which $N_H[u] \nsubseteq N_H[v]$, can be when $i = j$. Assume then that $u, v \in N_j$ are incompatible in H, and let u' and v' be two private neighbors in H with respect to v and u, respectively; that is, uu', vv' are edges of H and $u'v, v'u$ are non-edges of H. We show that both uu' and vv' are in E. Assume for the sake of contradiction that uu' in F (the case for $vv' \in F$ is purely symmetric). Then $uu' \in C(x)$, and $u' \in S_i$ such that $i \leq p_{u'} < j$. This implies that u' must be adjacent to all vertices of N_j in H, including v, giving the desired contradiction and proving the second statement. For showing the third, notice that by the definition of $C(x)$ we have that $i \leq p_u < j$, and that there is always a vertex $w \in N_{p_u}$ such that $wu \in E$. Hence by applying the first statement it follows that $N_H[w] \subseteq N_H[y]$, concluding the proof. □

For the next statement observe that any simple (or strong) elimination ordering of G is not necessarily a simple (or strong) elimination ordering for H.

Lemma 3.9. *The graph $H = (V, E \cup D(x) \cup C(x))$ is a strongly chordal graph.*

Proof. First notice that by Lemma 3.7 the graph $H' = (V, E \cup D(x))$ is strongly chordal. Because the vertices adjacent to x are the same in all considered graphs, we denote them by $N(x)$. To what follows we denote by G the graph H'. Also notice that every added edge in H is incident to vertices of $N(x)$. Finally we need to point out that the set $S(x)$ is defined in G', but we will use it in H, so that the vertices are the same, but the edges between $N(x)$ and $S(x)$ are those defined in H.

We start by proving that H is chordal. Assume for a contradiction that H has an induced chordless cycle C of length greater than 3, then we show that there exists a chordless cycle also in G. Since both G' and $H[V \setminus N(x)] = G[V \setminus N(x)]$ are chordal, and x is simplicial, a chordless cycle in H must have at least one vertex in $N(x)$, but not more than two since $N(x)$ is a clique. The rest of the cycle consists in a chordless path P completely contained outside $N(x)$, so that $H[P] = G[P]$.

1. $C \cap N(x) = \{u, v\}$: As part of the cycle, u and v must be incompatible in H. By the second statement of Observation 3.8 we know that $u, v \in N_i$ for some $1 \leq i \leq k$ and their private neighbors are the same in H and in G. Then a chordless cycle that contains u and v in H results in a chordless cycle in G, which is a contradiction since G is chordal.

2. $C \cap N(x) = \{y\}$: In this case, let us define $a \in S(x)$ and $b \in S(x)$ as the neighbors of y in C. If both ya and yb are in E, then we get a contradiction as the cycle exists in G as well. So we need to distinguish two more cases, where at least one of the two edges is in F.

(a) $ya \in F$ and $yb \in E$: This implies that $ya \in C(x)$, and, by the third statement of Observation 3.8, there exists a vertex $w \in N_{p_a}$ such that $wa \in E$. As $wy \in E$, if a and b are private neighbors of w and y respectively, we get the same case as when C intersects $N(x)$ in two vertices. Therefore w must be adjacent to b both in H and G. Furthermore w is non-adjacent to any other vertex of P, since $N_H[w] \subseteq N_H[y]$ and by assumption y is non-adjacent to any other vertex of P. However this implies the existence of the cycle $P \cup \{w\}$ in H as well as in G, since both wa and wb are in E.

(b) $ya \in F$ and $yb \in F$: In this case, by the third statement of Observation 3.8, there are two vertices $w_1 \in N_{p_a}$ and $w_2 \in N_{p_b}$, such that w_1a and w_1b are in E. If $w_1 = w_2$ we know that $P \cup \{w_1\}$ is a chordless cycle both in G and in H because $N_H[w_1] \subset N_H[y]$ and therefore w_1 is not adjacent to any vertex of $P \setminus \{a, b\}$. If $w_1 \neq w_2$ and at least one of w_1b or w_2a is in E, then P forms a chordless cycle with either w_1 or w_2, respectively. Finally, if $w_1 \neq w_2$ and neither of w_1b and w_2a are in H, then $P \cup \{w_1, w_1\}$ is a chordless cycle in G. Notice that all these cycles exist in G as all their edges belong to E, and neither w_1 nor w_2 are incident to any vertex of $P \setminus \{a, b\}$ as we observed for the case $w_1 = w_2$.

Next we show that H is strongly chordal. This part of the proof makes use of Lemma 2.3 and Observation 3.8. The underlying idea is to assume for contradiction that there is a l-sun in H, and then, by using Observation 3.8, find a partial sun where all edges between the vertices of the independent set and the vertices of the clique of the sun are also in G. At this point we apply Lemma 2.3 and show that G is not strongly chordal, getting the desired contradiction. As we proved that H is chordal, we can assume that, if H is not strongly chordal, it must contain an l-sun, with $l \geq 3$. Let $K = \{c_1, \ldots, c_l\}$ and $I = \{s_1, \ldots, s_l\}$ be the vertices of the clique and the independent set, respectively, of the l-sun. Recall that all vertices of an l-sun are incompatible and thus no vertex is simple.

Let us start by showing that x cannot be part of the l-sun. Assume that x belongs to the l-sun. Then it must belong to I because its neighborhood is a clique, and in particular $N(x) \cap K = \{c_i, c_j\}$ for some $1 \leq i < j \leq l$. Hence, it must also be that $N(x) \cap I = \emptyset$. Since c_i and c_j must be incompatible, by the second statement of Observation 3.8 they both belong to the same N_h for some $1 \leq h \leq k$ and their private neighbors are the same in G and H. At this point it is important to notice that the neighbors of c_i and c_j in $I \setminus \{x\}$, are, in fact, among such private neighbors. Since the rest of the l-sun must be completely contained in $V \setminus N[x]$, we conclude that the only edges of the l-sun that might not be in G, are those from c_i and c_j to the rest of K. Therefore applying Lemma 2.3 we obtain that G is not strongly chordal.

Now we continue by assuming that x is not part of the l-sun and observing that at least one vertex of $K \cup I$ belongs to $N(x)$. In particular, if $I \cap N(x) \neq \emptyset$, then $|I \cap N(x)| = 1$ and thus $|K \cap N(x)| \leq 2$. Otherwise, if $I \cap N(x) = \emptyset$, then $1 \leq |K \cap N(x)| \leq l$. Based on this observation we will distinguish the following cases.

1. $|K \cap N(x)| \geq 2$: Let us start by considering $|K \cap N(x)| > 2$ or $\{c_i, c_j\} \in K \cap N(x)$ such that c_i and c_j are not consecutive in K, i.e., they do not have a common neighbor in I and therefore $I \cap N(x) = \emptyset$. Let $C_N = K \cap N(x)$. Observe that every vertex in $N(C_N) \cap I$ is a private neighbor in H. By the second statement of Observation 3.8, all edges between C_N and I are also in G. Since the rest of the l-sun is completely contained in $V \setminus N[x]$, we apply Lemma 2.3 and reach a contradiction that G is not strongly chordal. If $\{c_i, c_j\} \in K \cap N(x)$ such that c_i and c_j are consecutive and have a common neighbor s in I, we might have a

problem because we cannot guarantee that sc_i and sc_j are in E. However, in this case, we can create another l-sun by replacing s with x and get a contradiction because x cannot be part of a sun.

2. $I \cap N(x) = \{s\}$ and $K \cap N(x) = \emptyset$: Let us define c_1 and c_2 as the two neighbors of s in K. If both sc_1 and sc_2 are in E, then the whole l-sun is also in G, so at least one of them must belong to F and therefore to $C(x)$.
   (a) $sc_1 \in F$ and $sc_2 \in E$: Using the third statement of Observation 3.8 there exists $w \in N_{p_{c_1}}$ such that $wc_1 \in E$ and $N_H[w] \subseteq N_H[s]$. Notice that $c_1c_2 \in E$, so that $wc_2 \in E$ or we have a chordless cycle $\{w, s, c_2, c_1\}$ in G. Then replacing s with w in the l-sun, we get a new l-sun, where all edges belong to E, reaching a contradiction to the strongly chordal graph G. Notice that w is not adjacent to any other vertex of the l-sun than c_1 and c_2 because s is not and $N_H[w] \subseteq N_H[s]$.
   (b) $sc_1 \in F$ and $sc_2 \in F$: By the third statement of Observation 3.8 there exist w_1 and w_2 such that $w_1c_1 \in E$, $w_2c_2 \in E$, $N_H[w_1] \subseteq N_H[s]$ and $N_H[w_2] \subseteq N_H[s]$. If $w_1 = w_2$, then we get the same situation as the previous case. Otherwise let us point out that $\{w_1, w_2, c_2, c_1\}$ forms a chordless cycle in G unless either w_1c_2 or w_2c_1 or both are in E. In each case we can choose either w_1 or w_2, accordingly, to replace s in the l-sun and create a new one that is completely contained also in G.

3. $I \cap N(x) = \{s_1\}$ and $K \cap N(x) = \{c_2\}$: We define c_1 as the other neighbor of s_1 in K, s_2 as the other neighbor of c_2 in I and c_3 as the second neighbor of s_2 in K. In this case if both $s_1c_1 \in E$ and $c_2s_2 \in E$ we apply Lemma 2.3 and reach a contradiction. Otherwise we distinguish three cases.
   (a) $s_1c_1 \in F$ and $c_2s_2 \in E$: By the third statement of Observation 3.8, there exists $w \in N(x)$ such that $wc_1 \in E$ and $N_H[w] \subseteq N_H[s_1]$. Then we replace s_1 with w in the l-sun and notice that all edges between K and $I \setminus \{s\} \cup \{w\}$ are in E. Thus by Lemma 2.3 we reach a contradiction.
   (b) $s_1c_1 \in E$ and $c_2s_2 \in F$: By the third statement of Observation 3.8, there exists $w \in N(x)$ such that $ws_2 \in E$ and $N_H[w] \subseteq N_H[c_2]$. Since $w \in N(x)$, we know that $ws_1 \in E$. Then we replace c_2 with w in the l-sun, so that all edges between $K \setminus \{c_2\} \cup \{w\}$ and I are in E and use Lemma 2.3 to get the desired contradiction.
   (c) $s_1c_1 \in F$ and $c_2s_2 \in F$: By the third statement of Observation 3.8, there exist $w_1 \in N(x)$ such that $w_1c_1 \in E$ and $N_H[w_1] \subseteq N_H[s_1]$ and $w_2 \in N(x)$ such that $w_2s_2 \in E$ and $N_H[w_2] \subseteq N_H[c_2]$. Observe that $w_1w_2 \in E$. If $w_1 \neq w_2$ then we can replace s_1 with w_1 and c_2 with w_2 in the l-sun, so that we create a new (possibly partial) l-sun where all the edges between the vertices of the independent set $I \setminus \{s\} \cup \{w_1\}$ and the vertices of the clique $K \setminus \{c_2\} \cup \{w_2\}$ are in E. Hence we apply Lemma 2.3 and obtain that G is not strongly chordal. If $w_1 = w_2$, then the independent set $I \setminus \{s_1, s_2\} \cup \{w_1\}$ and the clique $K \setminus \{c_2\}$ induce an $(l-1)$-sun that is also contained in G since all its edges are in E, concluding the proof of this case.

4. $I \cap N(x) = \emptyset$ and $K \cap N(x) = \{c_2\}$: We define s_1 and s_2 the two neighbors of c_2 in I, and c_1 and c_3 the other neighbor of s_1 and s_2, respectively, in K. Notice that the whole l-sun except c_2 is in $V \setminus N[x]$. If the two edges c_2s_1 and c_2s2 are in E, we apply Lemma 2.3 and

obtain a contradiction. We then distinguish the two cases in which exactly one of them is in F, does not matter which, or both of them are.

(a) $c_2s_1 \in F$ and $c_2s_2 \in E$: By the third statement of Observation 3.8, there exists $w \in N(x)$ such that $ws_1 \in E$ and $N_H[w] \subseteq N_H[c_2]$. Observe that $wc_2 \in E$. Now, according to whether c_1c_2 and c_3c_2 belong to F or not, we have different chordless cycles in G involving w,c_2 and s_1, and some or all of c_1, c_3 and s_2. However, in each case, such cycle cannot exist in G, so at least the edge wc_1 must be in E.

If w is not adjacent to neither c_3 nor s_2, then both c_2c_3 and c_2c_1 must be in G not to have a chordless cycle. However now there is a 3-sun in G involving x with $\{w, c_2, c_1\}$ as its clique and $\{x, s_1, c_3\}$ as its independent set, which is a contradiction. If w is adjacent to c_1 and c_3 in G, but not to s_2, then $c_2c_3 \in E$ or $\{c_2, w, c_3, s_2\}$ is a C_4 in G. In this case, if $c_2c_1 \notin E$, then we get a 3-sun in G with $\{w, c_2, c_3\}$ its clique and $\{c_1, x, s_2\}$ its independent set. Thus $c_2c_1 \in E$. Now if $c_2c_i \notin E$ for any $4 \leq i \leq l$ or $l = 3$ then we have a 4-sun with clique $\{w, c_2, c_1, c_3\}$ and independent set $\{x, s_1, s_2, c_i\}$ or $\{x, s_1, s_2, s_3\}$. We conclude that $c_2c_i \in E$ for every $4 \leq i \leq l$, that is, K is a clique also in G. Observe that at this point we also have $wc_i \in E$ for every $1 \leq i \leq l$. In fact, if there is an i for which $wc_i \notin E$ then we have a 3-sun with clique $\{w, c_1, c_2\}$ and independent set $\{x, s_1, c_i\}$. Then since $ws_2 \notin E$, we have that $\{w, c_1, c_2, \ldots, c_l\}$ and $\{x, s_1, s_2, \ldots, s_l\}$ form an $(l + 1)$-sun in G reaching a contradiction. Hence as a final contradiction in this case we consider that w being adjacent to s_1 and s_2 in G. Observe that w is not adjacent to any other vertex in I than s_1 and s_2 since $N_H[w] \subseteq N_H[c_2]$. In this case we replace c_2 with w in the l-sun and apply Lemma 2.3 to obtain that G is not strongly chordal because now all edges between I and $K \setminus \{c_2\} \cup \{w\}$ are in E.

(b) $c_2s_1 \in F$ and $c_2s_2 \in F$: By the third statement of Observation 3.8, there exist $w_1 \in N(x)$ such that $w_1s_1 \in E$ and $N_H[w_1] \subseteq N_H[c_2]$, and $w_2 \in N(x)$ such that $w_2s_1 \in E$ and $N_H[w_2] \subseteq N_H[c_2]$. If $w = w_1 = w_2$, then wc_1 and wc_2 are in E or w, s_1, c_1, c_2, s_2 is a chordless cycle in G. Hence we replace c_2 with w in the l-sun and apply Lemma 2.3 to reach a contradiction. If $w_1 \neq w_2$ then we know that w_1c_1, w_2c_3 and at least one of w_1c_3 or w_2c_1 must be in E. If not, because $w_1w_2 \in E$, $\{w_1, w_2, s_2, c_3, c_2, s_1\}$ induces a chordless cycle in G. Also notice that if $w_1c_3 \in E$ then $w_1s_2 \notin E$, and if $w_2c_1 \in E$ then $w_2s_1 \notin E$, for otherwise w_1 or w_2, respectively, can be used as w in the case when $w_1 = w_2 = w$ in order to yield a contradiction. Given this, if $w_1c_3 \in E$ then $w_2s_1 \notin E$ and if $w_2c_1 \in E$ then $w_1s_2 \notin E$, for otherwise $\{w2, s_1, c_1, c_3\}$ and $\{w_1, c_1, c_3, s_2\}$, respectively, would induce chordless cycles in G. We can therefore reduce ourselves to three cases, two of which are equivalent. (i) $w_1c_3 \in E$ and $w_2c_1 \notin E$ (symmetric to $w_1c_3 \notin E$ and $w_2c_1 \in E$): In this case we have a 3-sun in G, with $\{w_1, w_2, c_3\}$ its clique and $\{x, c_1, s_2\}$ its independent set. (ii) $w_1c_3 \in E$ and $w_2c_1 \in E$: If $l = 3$, then let $I = \{s_1, s_2, s_3\}$. We know that neither w_1 nor w_2 is adjacent to s_3 because c_2 is not and $N_H[w_1] \subseteq N_H[c_2]$ and $N_H[w_2] \subseteq N_H[c_2]$. Hence there is a 4-sun in G with $\{w_1, w_2, c_2, c_3\}$ its clique and $\{x, s_1, s_2, s_3\}$ its independent set. Now we consider the case for $l \geq 4$. If there exists a vertex $c_i \in K$ such that $c_iw_1 \notin E$ and $c_iw_2 \notin E$, then we can use c_i as we used s_3 in the previous case. If every vertex $c_i \in K$ is adjacent to both w_1 and w_2 in G, then we get an $(l + 1)$-sun in G, with $K \setminus \{c_2\} \cup \{w_1, w_2\}$ its clique and $I \cup \{x\}$ its independent set. Finally, if there is at least one vertex $c_i \in K$ such that $c_i w_1 \notin E$ and $c_i w_2 \in E$ (or vice-versa), then we get a 3-sun with $\{w_1, w_2, c_1\}$ its clique and $\{x, s_1, c_i\}$ its independent set.

In all cases we reach a contradiction to the existence of an l-sun in H implying that H is strongly chordal. $\square$

Now we are occupied with the necessary tools for proving the next important property of strongly chordal graphs.

Lemma 3.10. *Let $G = (V, E)$ and $G' = (V, E \cup F)$ be two strongly chordal graphs such that $E \cap F = \emptyset$ and $F \neq \emptyset$. Then there exists an edge $f \in F$ such that $G' - f$ is a strongly chordal graph.*

Proof. We prove the statement by induction on the number of vertices $|V|$. If $|V| \leq 3$ all graphs are strongly chordal and the statement holds. Assume that the statement is true for $|V| - 1$ vertices. Let x be a simple vertex of G' such that $F \nsubseteq D(x) \cup C(x)$. By Lemma 3.6 such a vertex always exist in G'. If there is an edge f of F incident to x then by Corollary 3.5 $G' - f$ is strongly chordal. Otherwise, no edge of F is incident to x and by Lemma 3.6 there is always an edge of F incident to a vertex of $V \setminus N_{G'}[x]$.

Now let $G_x = (V \setminus \{x\}, E \cup D(x) \cup C(x))$. G_x is strongly chordal by Lemma 3.9. Also the graph $G'_x = G'[V \setminus \{x\}]$ is strongly chordal as an induced subgraph of a strongly chordal graph. Notice that the added edges of G'_x are given by the set $F \setminus (D(x) \cup C(x))$ which by Lemma 3.6 is non-empty. Furthermore it is important to notice that G_x is a subgraph of G'_x since the set of edges $D(x)$, $C(x)$, and $P(x)$ are edges of G'. Both graphs G_x and G'_x are on $|V| - 1$ vertices and by the induction hypothesis there is always an edge $f \in F \setminus (D(x) \cup C(x))$ such that $G'_x - f$ is strongly chordal.

Let $(N_0, N_1, \ldots, N_k)$ and $(S_1, \ldots, S_k)$ be a simple partition with respect to x in G'. Let us now show if the edge that is picked at the induction step is between vertices of $N(x)$ and $S(x)$ then there is an edge $f = uv \in F \setminus (D(x) \cup C(x))$ such that $u \in N_i$ and $v \in S_i$ and $G'_x - f$ is strongly chordal. If at the induction step uv is picked in such a way then we are done. Thus assume that $u \in N_j$ and $v \in S_i$ for $i < j$. We show first that there is an edge of the form $v'v \in F \setminus (D(x) \cup C(x))$ where $v' \in N_i$. By definition of $C(x)$, we know that $p_v \geq j > i$ because $uv \in F \setminus (D(x) \cup C(x))$. Then by the smallest choice of p_v, every edge incident to v and the vertices of $N_i, \ldots, N_{p_v - 1}$ belongs to $F \setminus (D(x) \cup C(x))$. Hence there is an edge of the form $v'v \in F \setminus (D(x) \cup C(x))$ where $v' \in N_i$. Now we show that the graph $G'_x - \{v'v\}$ is strongly chordal by using the fact that $G'_x - \{uv\}$ is strongly chordal. Assume for the sake of contradiction that $G'_x - \{v'v\}$ is not strongly chordal. Since we remove only a single edge from a strongly chordal graph G'_x, there is a chordless cycle on four vertices or by Observation 3.1 there is a 3-sun in $G'_x - \{v'v\}$. If there is a chordless cycle in $G'_x - \{v'v\}$ then let v_a and v_b be the two non-adjacent vertices that are both adjacent to v' and v in G'_x. By the fact that $N_{G'_x}[v'] \subseteq N_{G'_x}[u]$ ($v' \in N_i$ and $u \in N_j$), u is adjacent to both vertices in G'_x and then we reach a contradiction to the chordal graph $G'_x - \{uv\}$. If there is a 3-sun in $G'_x - \{v'v\}$ then we consider two cases: (i) if v' belongs to the clique of the 3-sun then by $N_{G'_x}[v'] \subseteq N_{G'_x}[u]$ we know that u is also adjacent to the vertices of the clique of the 3-sun; thus we reach a contradiction to the strongly chordal graph $G'_x - \{uv\}$. (ii) If v' belongs to the independent set of the 3-sun then by $N_{G'_x}[v'] \subseteq N_{G'_x}[u]$, u is adjacent to the two vertices of the clique. If u is adjacent to at least one further vertex of the 3-sun then $G'_x - \{uv\}$ is not chordal. Otherwise the graph $G'_x - \{uv\}$ has a 3-sun and thus in both cases we reach a contradiction. Therefore if the edge f of the graph $G'_x - f$ is between vertices of $N(x)$ and $S(x)$ then there is an edge $f' = uv$ such that $f' \in F \setminus (D(x) \cup C(x))$, $u \in N_i$, $v \in S_i$ and the graph $G'_x - \{uv\}$ is strongly chordal.

Next we show that $G' - f$ is strongly chordal by constructing the graph according to $G'_x - f$. We prove that $G' - f$ is strongly chordal by using Lemma 3.2 and showing that x is a simple vertex in $G' - f$. If f is not between $N(x)$ and $S(x)$ then x is simple in $G' - f$ since x is simple in G'. Otherwise because of the previous result there is an edge $f \in F \setminus (D(x) \cup C(x) \cup P(x))$ such that the endpoints of f belong to the sets N_i and S_i for some $1 \leq i \leq k$. Let u and v be the endpoints of f such that $u \in N_i$ and $v \in S_i$. Remember that in G' we have $N_{G'}(N_{i-1}) \subseteq N_{G'}(N_i)$. In the graph $G' - f$ we have that $N_{G'-f}(N_{i-1}) \subseteq N_{G'-f}(u) \subset N_{G'-f}(N_i \setminus \{u\})$ meaning that there is an inclusion set property for $N(x)$. Therefore x is a simple vertex of $G' - f$ and thus $G' - f$ is strongly chordal which completes the proof. □

Theorem 3.11. *Strongly chordal graphs are sandwich monotone.*

Proof. Given G and G' as in the premise of Lemma 3.10 we know that $G' - f$ is strongly chordal graph. The same argument can be applied to G and $G'' = G' - f$ repeatedly, until we get the smaller graph G. □

By the previous theorem and Observation 2.5 we have the following property of minimal strongly chordal completions.

Corollary 3.12. *Let $G = (V, E)$ be an arbitrary graph and let $G' = (V, E \cup F)$ be a strongly chordal graph such that $E \cap F = \emptyset$ and $F \neq \emptyset$. G' is a minimal strongly chordal completion if and only if no edge of F can be removed from G' without destroying the strongly chordal property.*

It is known that an edge f can be removed from a chordal graph if and only if f is not the unique chord of a C_4 [23]. We prove a similar characterization for strongly chordal graphs. We call a *chord of a 3-sun* an edge between a vertex of the independent set and a vertex of the clique.

Lemma 3.13. *Let G be a strongly chordal graph and let f be an edge of G. $G - f$ is a strongly chordal graph if and only if f is not the unique chord of a C_4 or the unique chord of a 3-sun.*

Proof. Let $G - f$ be a non-strongly chordal graph. Assume first that $G - f$ has a chordless cycle of length greater than 4. Then adding back f is not sufficient to kill the cycle. If $G - f$ has a k-sun, for $k > 3$, as an induced subgraph then adding back the edge f would create another k'-sun, $k' \geq 3$, by Observation 3.1 and thus it wouldn't be strongly chordal. Thus we conclude that G is not a strongly chordal graph reaching a contradiction. □

The previous lemma leads to the following characterization of minimal strongly chordal completions.

Theorem 3.14. *Let $G = (V, E)$ be a graph and $G' = (V, E \cup F)$ be a strongly chordal completion of G. G' is a minimal strongly chordal completion if and only if every $f \in F$ is the unique chord of a C_4 or a 3-sun in G'.*

Proof. If G' is a minimal strongly chordal completion of G then $G' - f$ is not strongly chordal for any edge $f \in F$ by definition of minimality. Thus $G' - f$ contains either a chordless cycle or a chordless 3-sun as an induced subgraph by Lemma 3.13. If every edge $f \in F$ is the unique chord of a C_4 or a 3-sun in G' then $G' - f$ is not strongly chordal and by Corollary 3.12 G' is minimal. $\square$

4 Chordal bipartite graphs are sandwich monotone

A bipartite graph $B = (X, Y, E)$ is *chordal bipartite* if it does not contain an induced C_k for $k \geq 6$. In this section we show that chordal bipartite graphs are sandwich monotone, answering an open question of Bakony and Bono [2]. Our approach is to make use of a well known relationship between the classes of strongly chordal graphs and chordal bipartite graphs and Lemma 3.10. Using our result, we are able to characterize minimal chordal bipartite completions of arbitrary bipartite graphs.

Theorem 4.1 ([7]). *Given bipartite graph $B = (X, Y, E)$, let G be the graph obtained from B by adding edges between pairs of vertices in X so that X becomes a clique. Then, B is chordal bipartite if and only if G is strongly chordal.*

Lemma 4.2. *Let $B = (X, Y, E)$ and $B' = (X, Y, E \cup F)$ be chordal bipartite graphs with $E \cap F = \emptyset$ and $F \neq \emptyset$. Then, there exists an edge $f \in F$ such that $B' - f$ is chordal bipartite.*

Proof. Let $C = \{vw \mid v \in X, w \in X, v \neq w\}$. First construct the following graphs: $G = ((X \cup Y), (E \cup C))$ and $G' = ((X \cup Y), (E \cup C \cup F))$. By Theorem 4.1, G and G' are strongly chordal. By, Lemma 3.10, there exists $f \in F$ such that $G' - f$ is strongly chordal. The desired chordal bipartite graph $B' - f$ is obtained from $G' - f$, via Theorem 4.1, by simply deleting all the edges in C. $\square$

Hence the next theorem follows.

Theorem 4.3. *Chordal bipartite graphs are sandwich monotone.*

From the theorem above and Observation 2.5 we have the following corollary.

Corollary 4.4. *Let $B = (X, Y, E)$ be an arbitrary bipartite graph and let $B' = (X, Y, E \cup F)$ be a chordal bipartite graph such that $E \cap F = \emptyset$. B' is a minimal chordal bipartite completion of B if and only if for any $f \in F$, $B' - f$ is not chordal bipartite.*

Lemma 4.5. *Let B be a chordal bipartite graph and f be an edge of B. $B - f$ is a chordal bipartite graph if and only if f is not the unique chord of a C_6 in B.*

Proof. If f is the unique chord of a C_6 in B, then $B - f$ contains a C_6 and hence is not chordal bipartite. For the other direction, observe that if the deletion of a single edge from a chordal bipartite graph creates an induced cycle on six or more vertices, then the created induced cycle must have exactly six vertices. Thus, if $B - f$ is not chordal bipartite, then f must be the unique chord of a C_6 in B. $\square$

Finally, we have the following characterization of minimal chordal bipartite completions:

Theorem 4.6. *Let $B = (X, Y, E)$ be a bipartite graph and let $B' = (X, Y, E \cup F)$ be a chordal bipartite completion of B. B' is a minimal chordal bipartite completion if and only if every $f \in F$ is the unique chord of a C_6 in B'.*

Proof. Follows from Corollary 4.4 and Lemma 4.5. □

5 Concluding remarks

We have proved that strongly chordal graph and chordal bipartite graphs are sandwich monotone. The best running time for recognizing those graphs is $\mathcal{O}(\min\{m \log n, n^2\})$ [20, 25]. Hence by applying a simple algorithm proposed in [15] we obtain algorithms for computing minimal completions into both graph classes of arbitrary graphs with running time $\mathcal{O}(n^4(\min\{m \log n, n^2\}))$. We strongly believe that such a running time can be improved. Furthermore problems that involve listing all strongly chordal graphs (or chordal bipartite graphs) between a given pair of graphs where at least one of the input pair is strongly chordal (or chordal bipartite) can be efficiently solved by generalizing the results in [17] and applying Proposition 2.6.

A graph is weakly chordal if neither the graph nor its complement contains a chordless cycle longer than 4. Despite their somewhat misleading name, chordal bipartite graphs are exactly the graphs that are both weakly chordal and bipartite. Minimum weakly chordal completions are NP-hard to compute [4], and we do not yet know whether minimal weakly chordal completions can be computed or recognized in polynomial time. We would like to know whether weakly chordal graphs are sandwich monotone. The resolution of this question in the affirmative would answer the above questions about minimal weakly chordal completions. Another interesting question to resolve is whether minimum chordal bipartite completions are NP-hard to compute. This is widely believed, but no proof of it exists to our knowledge. Note that the related sandwich problem was solved only quite recently [9, 24].

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