Computing role assignments of proper interval graphs in polynomial time*

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Abstract. A homomorphism from a graph G to a graph R is locally surjective if its restriction to the neighborhood of each vertex of G is surjective. Such a homomorphism is also called an R-role assignment of G. Role assignments have applications in distributed computing, social network theory, and topological graph theory. The Role Assignment problem has as input a pair of graphs (G, R) and asks whether G has an R-role assignment. This problem is NP-complete already on input pairs (G, R) where R is a path on three vertices. So far, the only known non-trivial tractable case consists of input pairs (G, R) where G is a tree. We present a polynomial time algorithm that solves Role Assignment on all input pairs (G, R) where G is a proper interval graph. Thus we identify the first graph class other than trees on which the problem is tractable. As a complementary result, we show that the problem is Graph Isomorphism-hard on chordal graphs, a superclass of proper interval graphs and trees.

1 Introduction

Graph homomorphism is a natural way to generalize graph coloring: there is a homomorphism from a graph G to the complete graph on k vertices if and only if G is k-colorable. A *homomorphism* from a graph $G = (V_G, E_G)$ to a graph $R = (V_R, E_R)$ is a mapping $r : V_G \to V_R$ such that $r(u)r(v) \in E_R$ whenever $uv \in E_G$. A homomorphism r from G to R is *locally surjective* if the following is true for every vertex u of G: for every neighbor y of $r(u)$ in R, there is a neighbor v of u in G with $r(v) = y$. We also call such an r an *R-role assignment*. See Figure 1 for an example.

Role assignments originate in the theory of social behavior [9, 19]. A role graph R models roles and their relationships, and for a given society we can ask if its individuals can be assigned roles such that relationships are preserved: each person playing a particular role has exactly the roles prescribed by the model

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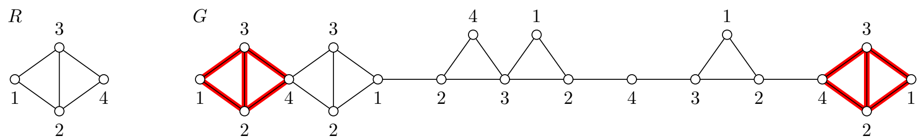

Fig. 1. A graph R and a proper interval graph G with an R-role assignment.

among its neighbors. Role assignments are also useful in the area of distributed computing, in which one of the fundamental problems is to arrive at a final configuration where all processors have been assigned unique identities. Chalopin et al. [6] show that, under a particular communication model, this problem can be solved on a graph G representing the distributed system if and only if G has no R-role assignment for a graph R with fewer vertices than G. Role assignments are useful in topological graph theory as well, where a main question is which graphs G allow role assignments to planar graphs R [21].

The Role Assignment problem has as input a pair of graphs (G, R) and asks whether G has an R-role assignment. It is NP-complete on arbitrary graphs G even when R is any fixed connected bipartite graph on at least three vertices [12]. Hence, for polynomial time solvability, our only hope is to put restrictions on G. So far, the only known non-trivial graph class that gives tractability is the class of trees: Role Assignment is polynomial time solvable on input pairs (G, R) where G is a tree and R is arbitrary [13]. Are there other graph classes on which Role Assignment can be solved in polynomial time?

We show that Role Assignment can be solved in polynomial time on input pairs (G, R) where G is a proper interval graph and R is arbitrary. Our work is motivated by the above question and continues the research direction of Sheng [23], who characterizes proper interval graphs that have an R-role assignment for some fixed role graphs R with a small number of vertices. Proper interval graphs, also known as unit interval graphs or indifference graphs, are widely known due to their many theoretical and practical applications [5, 15, 22]. By our result, they form the first graph class other than trees on which Role Assignment is shown to be polynomial time solvable. To obtain our algorithm we prove structural properties of clique paths of proper interval graphs related to role assignments. This enables us to give an additional result, namely a polynomial time algorithm for the problem of deciding whether there exists a graph R with fewer vertices than a given proper interval graph G such that G has an R-role assignment. Recall that this problem stems from the area of distributed computing [6]. It is co-NP-complete in general [7]. Finally, to indicate that Role Assignment might remain hard on larger graph classes, we show that it is Graph Isomorphism-hard for input pairs (G, R) where G belongs to the class of chordal graphs, a superclass of both proper interval graphs and trees.

2 Preliminaries

All graphs considered in this paper are undirected, finite and simple, i.e., without loops or multiple edges. A graph is denoted $G = (V_G, E_G)$, where V_G is the set of vertices and E_G is the set of edges. We will use the convention that $n = |V_G|$ and $m = |E_G|$. For a vertex u of G, $N_G(u) = \{v \mid uv \in E_G\}$ denotes the set of *neighbors* of u, also called the *neighborhood* of u. The *degree* of a vertex u is $\deg_G(u) = |N_G(u)|$. A graph $H = (V_H, E_H)$ is a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. If G is isomorphic to H we write $G \simeq H$. For $U \subseteq V_G$, the graph $G[U] = (U, \ \{uv \in E_G \mid u, v \in U\})$ is called the subgraph of G *induced* by U. A graph is *complete* if it has an edge between every pair of vertices. A set of vertices $A \subseteq V_G$ is a *clique* if $G[A]$ is complete. A clique is *maximal* if it is not a proper subset of any other clique.

Let u and v be two vertices of a graph G. Then a *path* between u and v is a sequence of distinct vertices $P = u_1 u_2 \cdots u_p$ starting at $u_1 = u$ and ending at $u_p = v$, where each pair of consecutive vertices u_i, u_{i+1} forms an edge of G. If uv is an edge as well we obtain a *cycle*. Sometimes we fix an orientation of P. In that case we write $u_i \overrightarrow{P} u_j = u_i u_{i+1} \cdots u_j$ and $u_j \overleftarrow{P} u_i = u_j u_{j-1} \cdots u_i$ to denote the subpath from u_i to u_j, or from u_j to u_i, respectively. The *length* of a path or cycle is its number of edges. The set of vertices of a path or cycle P is denoted by V_P. A graph is *connected* if there is a path between every pair of vertices. A *connected component* of G is a maximal connected subgraph of G.

Let $A_1, \ldots, A_p$ be a sequence of sets. We use the shorthand notation $A_{\leq i} = A_1 \cup \cdots \cup A_i$ and $A_{\geq i} = A_i \cup \cdots \cup A_p$ for $i = 1, \ldots, p$.

2.1 Chordal, interval, and proper interval graphs

A graph isomorphic to the graph $K_{1,3} = (\{a, b_1, b_2, b_3\}, \ \{ab_1, ab_2, ab_3\})$ is called a *claw* with *center* a and *leaves* b_1, b_2, b_3. A graph is called *claw-free* if it does not have a claw as an induced subgraph. An *asteroidal triple (AT)* in a graph G is a set of three mutually nonadjacent vertices u_1, u_2, u_3 such that G contains a path P_{ij} from u_i to u_j with $P_{ij} \cap N_G(u_k) = \emptyset$ for all distinct $i, j, k \in \{1, 2, 3\}$. A graph is called *AT-free* if it does not have an AT.

A graph is *chordal* if it contains no induced cycle of length at least 4. A graph is an *interval graph* if intervals of the real line can be associated with its vertices such that two vertices are adjacent if and only if their corresponding intervals overlap. Interval graphs are a subclass of chordal graphs: a chordal graph is an interval graph if and only if it is AT-free [18].

The following characterization of interval graphs is also well known. Let G be a connected graph with maximal cliques $K_1, \ldots, K_p$ and let $\mathcal{K}_v$ denote the set of maximal cliques in G containing vertex $v \in V_G$. Then G is an interval graph if and only if G has a *clique path* [14], i.e., a path $P = K_1 \cdots K_p$ such that for each $v \in V_G$ the set $\mathcal{K}_v$ induces a connected subpath in P. We say that the maximal cliques of G are the *bags* of P. A bag K_i *introduces* a vertex u of G if $u \in K_i$ for $i = 1$ or $u \in K_i \setminus K_{i-1}$ for some $i \geq 2$. In that case, by the definition of a clique path, u is not in a bag K_h with $h \leq i - 1$. If $u \in K_i$ for $i = p$ or

$u \in K_i \setminus K_{i+1}$ for some $i \leq p-1$, then we say that K_i *forgets* u. Note that every bag introduces at least one vertex, and forgets at least one vertex. Because G is connected, we also observe that each bag, except K_1, contains at least one vertex from a previous bag. We denote the index of the bag in P that introduces a vertex u by $f_P(u)$ and the index of the bag that forgets u by $l_P(u)$. We say that u *transcends* a vertex v in P if $f_P(u) < f_P(v)$ and $l_P(v) < l_P(u)$. A clique path has at most n bags, and can be constructed in linear time (see e.g. [14]).

An interval graph is *proper interval* if it has an interval representation in which no interval is properly contained in any other interval. An interval graph is a proper interval graph if and only if it is claw-free [22]. Equivalently, a chordal graph is a proper interval graph if and only if it is AT-free and claw-free. Chordal graphs, interval graphs, and proper interval graphs can all be recognized in linear time, and have at most n maximal cliques (see e.g. [5, 15]). The following theorem will be heavily used in our proofs.

Theorem 1 ([16]). *A connected chordal graph is a proper interval graph if and only if it has a unique clique path in which no vertex transcends any other vertex.*

Two adjacent vertices u and v of a graph G are *twins* if $N_G(u) \cup \{u\} = N_G(v) \cup \{v\}$. Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. Note that two vertices u and v of G are twins if and only if $f_P(u) = f_P(v)$ and $l_P(u) = l_P(v)$. We partition V_G into sets of twins. A vertex that has no twin appears in its twin set alone. We order the twin sets with respect to P, and label them $T_1, \ldots, T_S$, in such a way that $i < j$ if and only if for all $u \in T_i, v \in T_j$, either $f_P(u) < f_P(v)$, or $f_P(u) = f_P(v)$ and $l_P(u) < l_P(v)$. We call $T_1, \ldots, T_s$ the *ordered twin sets* of G. The following observation immediately follows from this definition, and is even valid for interval graphs that are not proper.

Observation 1 *Let G be a connected proper interval graph with clique path $P = K_1 \ldots K_p$ and ordered twin sets $T_1, \ldots, T_s$. Then for $h = 1, \ldots, s-1$, there exists a bag that contains twin sets T_h and T_{h+1}. Furthermore, if a bag contains twin sets T_b and T_c with $b < c$ then it contains twin sets $T_{b+1}, \ldots, T_{c-1}$ as well.*

2.2 Role assignments

If r is a homomorphism from G to R and $U \subseteq V_G$, then we write $r(U) = \bigcup_{u \in U} r(u)$. Recall that r is an R-role assignment of G if $r(N_G(u)) = N_R(r(u))$ for every vertex u of G. Graph R is called a *role graph* and its vertices are called *roles*. We use n and m to refer to the number of vertices and edges of G.

Observation 2 ([12]) *Let G be a graph and let R be a connected graph such that G has an R-role assignment. Then each role $x \in V_R$ appears as a role of some vertex $u \in V_G$. Furthermore, if $|V_G| = |V_R|$ then $G \simeq R$.*

Lemma 1 ([12]). *Let G and R be two graphs such that G has an R-role assignment r, and let x, $y \in V_R$ be roles connected by a path $z_1 \cdots z_\ell$ in R, with $x = z_1$ and $y = z_\ell$. Then for each $u \in V_G$ with $r(u) = x$ there exists a vertex*

$v \in V_G$ and a path $t_1 \cdots t_\ell$ in G, with $u = t_1$ and $v = t_\ell$, such that $r(t_i) = z_i$ for $i = 1, \ldots, \ell$.

Our first result, given in Theorem 2, shows that chordal graphs, interval graphs, and proper interval graphs are closed under role assignments, and it is needed in Section 3. Its proof is given in the appendix. Note that, for each of the three statements in Theorem 2, the reverse implication is not valid. In order to see this let G be the 6-cycle and R be the 3-cycle.

Theorem 2. *Let G be a graph and let R be a connected graph such that G has an R-role assignment.*

(i) If G is a chordal graph then R is a chordal graph.
(ii) If G is an interval graph then R is an interval graph.
(iii) If G is a proper interval graph then R is a proper interval graph.

3 Role assignments on proper interval graphs

We start with the following key result. Note that this result is easy to verify for paths.

Theorem 3. *Let G and R be two connected proper interval graphs such that G has an R-role assignment r. Let P and P' be the clique paths of G and R, respectively. Then the bags of P and P' can be ordered such that $P = K_1 \cdots K_p$ and $P' = L_1 \cdots L_q$, with $q \leq p$, and $r(K_i) = L_i$, for $i = 1, \ldots, q$.*

Proof. By the definition of a role assignment, $|V_G| \geq |V_R|$ holds. Assume first that $|V_G| = |V_R|$. Then, as a result of Observation 2, G and R are isomorphic. By Theorem 1 the clique paths of G and R are unique. Hence the ordering of the bags in each path is unique up to reversal. We can try each direction for one of the paths, and the statement of the theorem holds.

For the rest of the proof, assume that $|V_G| > |V_R|$. Then at least one vertex of R is the role of more than one vertex of G. Let x be such a role. Then there exist vertices u and u' in G with $r(u) = r(u') = x$. Assume $l_P(u) = h$ and $f_P(u') = i$, where we may assume that $h < i$ because K_h and K_i are cliques, and vertices with the same role can not be adjacent. Let x be chosen in such a way that every vertex in $K_{\leq i-1}$ has a unique role, i.e., $|r(K_{\leq i-1})| = |K_{\leq i-1}|$.

Claim 1. Every vertex of R occurs as a (unique) role of a vertex of $K_{\leq i-1}$.

We prove this claim by contradiction. Suppose there is a role y that does not occur as a role of a vertex in $K_{\leq i-1}$. As a result of Observation 2, there exists a vertex v in G with $r(v) = y$. Let $f_P(v) = j$. Since y does not appear as a role on $K_{\leq i-1}$, we find that $j \geq i$. We may even chose v such that there is no vertex in $K_{\leq j-1}$ with role y. Because K_j is a clique, we find that v is the only vertex of K_j with role y.

Let $Q' = z_1 \cdots z_\ell$, with $x = z_1$ and $y = z_\ell$, be a shortest path between x and y in R. By Lemma 1 we find that G contains a path $Q_1 = t_1 \cdots t_\ell$, with $u = t_1$

and $v' = t_\ell$, such that $r(t_i) = z_i$ for $i = 1, \dots, \ell$. Since Q' is a shortest path from x to y in R, and there is no other vertex in K_j with role y, our choice of v implies that we may assume that $v = v'$.

By the same reasoning we find a path $Q_2 = t'_1 \cdots t'_\ell$, with $u' = t'_1$ and $v = t'_\ell$, such that $r(t'_i) = z_i$ for $i = 1, \dots, \ell$. Hence Q_1 and Q_2 are two paths with $r(V_{Q_1}) = r(V_{Q_2}) = V_{Q'}$ and $|V_{Q_1}| = |V_{Q_2}| = |V_{Q'}|$. Consequently, u is not on Q_2 and u' is not on Q_1. However, since $l_P(u) = h < f_P(u') = i \leq f_P(v) = j$ and K_i, K_j are cliques, we find that Q_1 contains a neighbor w of u'.

Suppose $i = j$. Then u' and v are neighbors in G, and consequently, xy is an edge of R. This means that u and v are neighbors in G. Hence, there is a bag in P containing both of them. This means that $h = l_P(u) \geq f_P(v) = j$. However, this is not possible since $h < i \leq j$.

Suppose $i < j$. Then $w = t_2$ as otherwise r maps the path $u'w\overrightarrow{Q_1}t_\ell$ to a path from x to y in R that is shorter than Q'. By the same reasoning, we find that w is the only neighbor of u' on Q_1. Since Q_1 is a shortest path and $uu' \notin E_G$, this means that G contains an induced claw with center t_2 and leaves u, u', t_3, which contradicts the assumption that G is a proper interval graph. This completes the proof of Claim 1.

By Claim 1 we find that $r(K_{\leq i-1}) = V_R$, and consequently, as $|r(K_{\leq i-1})| = |K_{\leq i-1}|$, we obtain $|K_{\leq i-1}| = |V_R|$. Let r' be the restriction of r to $K_{\leq i-1}$.

Claim 2. r' is an R-role assignment of $G[K_{\leq i-1}]$.

We prove Claim 2 as follows. Suppose r' is not an R-role assignment of $G[K_{\leq i-1}]$. Because r is a homomorphism from G to R, we find that r' is an homomorphism from $G[K_{\leq i-1}]$ to R. Hence, there must exist a vertex $t \in K_{\leq i-1}$ and vertices $z, z' \in V_R$ with $r'(t) = r(t) = z$, $zz' \in E_R$ and $z' \notin r'(N_G(t))$. Since r is an R-role assignment of G, we find that $z' \in r(N_G(t))$. Hence $l_P(t) \geq i + 1$. Consequently, as $t \in K_{\leq i-1}$, we find that t belongs to K_i. We proceed as follows. Since $r(K_{\leq i-1}) = V_R$, there exists a vertex $t' \in K_{\leq i-1}$ with $r'(t') = r(t') = z'$. By definition of r, we find that t' has a neighbor s in G with $r(s) = z$. Because t has no neighbor with role z', we find that t and t' are not adjacent in G. Hence $s \neq t$ holds. Since every vertex of $K_{\leq i-1}$ has a unique role and vertex $t \in K_{\leq i-1}$ already has role z, we find that $s \notin K_{\leq i-1}$. This means that K_i does not only contain t but also contains t'. However, since K_i is a clique, t and t' must be adjacent. With this contradiction we have completed the proof of Claim 2.

Due to Claim 2 and the aforementioned observation that $|K_{\leq i-1}| = |V_R|$, we may apply Observation 2 and obtain that $G[K_{\leq i-1}]$ is isomorphic to R. By Theorem 1, the clique paths of $G[K_{\leq i-1}]$ and R are unique. Hence, $i = q + 1$, and the statement of the theorem follows. □

Note that Theorem 3 is not valid for interval graphs, which can be seen with the following example. Let G be the path $u_1u_2u_3u_4$ to which we add a vertex u_5 with edge u_2u_5 and a vertex u_6 with edge u_3u_6. Let $P = K_1 \cdots K_5$ be a clique path of G with $K_1 = \{u_1, u_2\}$, $K_2 = \{u_2, u_5\}$, $K_3 = \{u_2, u_3\}$, $K_4 = \{u_3, u_6\}$ and $K_5 = \{u_3, u_4\}$. Let R be the 4-vertex path 1234. The unique clique path of R is

$P' = L_1L_2L_3$ with $L_1 = \{1, 2\}$, $L_2 = \{2, 3\}$ and $L_3 = \{3, 4\}$. However, we find that G has an R-role assignment r with $r(u_1) = r(u_5) = 1$, $r(u_2) = 2$, $r(u_3) = 3$, and $r(u_4) = r(u_6) = 4$.

Also note that we can apply Theorem 3 twice depending on the way the bags in the clique path of the proper interval graph G are ordered. This leads to a rather surprising corollary that might be of independent interest.

Corollary 1. *Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. If G has an R-role assignment and R is connected, then $R \simeq G[K_{\leq i}]$ and $R \simeq G[K_{\geq p-i+1}]$, for some $1 \leq i \leq p$.*

As an illustration of Corollary 1 we have indicated the two copies of R in G with bold edges in Figure 1. Due to Theorem 2 we do not need to restrict R to be a proper interval graph in the statement of the above corollary. Hence for any two connected graphs G and R, where G is proper interval with $|V_G| > |V_R|$, if G has an R-role assignment then G contains two (not necessarily vertex-disjoint) induced subgraphs isomorphic to R.

Theorem 3 only shows what an R-role assignment r of a proper interval graph G looks like at the beginning and end of the clique path of G. To derive our algorithm, we need to know the behavior of r in the middle bags as well. We therefore give the following result, which is valid when R has at least three maximal cliques and the number of maximal cliques in G is not too small. Its proof is given in the appendix. The special cases when R has just one or two maximal cliques or G has few maximal cliques will be dealt with separately in the proof of Theorem 4.

Lemma 2. *Let G be a connected proper interval graph with clique path $P = K_1 \cdots K_p$. Let R be a connected proper interval graph with clique path $P' = L_1 \cdots L_q$ and ordered twin sets $X_1, \ldots, X_t$, such that r is an R-role assignment of G with $r(K_q) = L_q$. Let T be the subset of K_q that consists of all vertices with roles in X_t. Then the following holds if $q \geq 3$ and $p \geq 2q + 1$.*

(i) If there is a vertex in T not in K_{q+1}, then there exists an index $i \geq q+1$ such that $K_{\geq q+1} \setminus K_{\leq q} \subseteq K_{\geq i}$ and the restriction of r to $K_{\geq i}$ is an R-role assignment of $G[K_{\geq i}]$ with $r(K_i) = L_q$. Furthermore, if $i > q+1$ then $r(K_h) \subseteq X_t$ for $h = q+1, \ldots, i-1$.

(ii) If all vertices in T are in K_{q+1}, then there exists an index $i \geq q+1$ such that $T = K_{\leq i-1} \cap K_i$ and $T \cap K_{i+1} = \emptyset$, and the restriction of r to $K_{\geq i}$ is an R-role assignment of $G[K_{\geq i}]$ with $r(K_i) = L_q$.

Let G and R be two connected proper interval graphs with clique paths $P = K_1 \cdots K_p$ and $P' = L_1 \cdots L_q$, respectively. Then a *starting R-role assignment* of $G[K_{\leq i}]$ is a mapping $r : K_{\leq i} \to V_R$ for some $1 \leq i \leq p$ such that for all $u \in K_{\leq i} \setminus K_{i+1}$ we have that $r(N_G(u)) = N_R(r(u))$. Note that a starting R-role assignment of $G[K_{\leq p}] = G$ is an R-role assignment of G. We say that $v \in K_{\leq i} \cap K_{i+1}$ is *missing* role x if x is a neighbor of $r(v)$ that is not in $r(N_G(v))$. Let $X_1, \ldots, X_t$ be the ordered twin sets of R. We denote the set of missing roles of v that are in X_c by $M_c(v)$. We say that r can be *finished by* r^* if r^* is an R-role assignment of G with $r^*(u) = r(u)$ for all $u \in K_{\leq i}$.

The following lemma is important for our algorithm.

Lemma 3. *Let G and R be two connected proper interval graphs. Let G have clique path $P = K_1 \cdots K_p$, and let R have ordered twin sets $X_1, \ldots, X_t$. Let $r : K_{\leq i} \to V_R$ be a starting R-role assignment of $G[K_{\leq i}]$ for some $1 \leq i \leq p$. Then $K_{\leq i} \cap K_{i+1}$ does not contain two vertices u, v such that $M_c(u) \setminus M_c(v) \neq \emptyset$ and $M_c(v) \setminus M_c(u) \neq \emptyset$ for some $1 \leq c \leq t$.*

Proof. In order to derive a contradiction, assume that such vertices u and v exist. Let $x \in M_c(u) \setminus M_c(v)$ and $y \in M_c(v) \setminus M_c(u)$. Because u misses x and $x \in X_c$, we find that $r(u)$ is adjacent to all roles in $X_c \setminus \{r(u)\}$. Hence $r(u)$ is adjacent to $y \in X_c$, unless $r(u) = y$. However, the latter case is not possible, because in that case v, being adjacent to u, would not miss y. So, indeed $r(u)$ and y are adjacent. From $y \in M_c(v) \setminus M_c(u)$ we then deduce that u already has a neighbor $w \in K_{\leq i}$ with role $r(w) = y$. Since v misses y and R contains no self-loop, we find that $r(v) \neq y$, and consequently $w \neq v$. Since v misses y, the edge uw must be in a bag before v got introduced. Hence, we obtain $f_P(u) < f_P(v)$. Analogously, we get $f_P(v) < f_P(u)$. This is not possible due to Theorem 1. □

We are now ready to present our main result.

Theorem 4. Role Assignment *can be solved in polynomial time on input pairs (G, R) where G is a proper interval graph and R is an arbitrary graph.*

Proof. First we give an algorithm with running time $\mathcal{O}(n^3)$ that takes as input a *connected* proper interval graph G and a *connected* graph R, and decides whether G has an R-role assignment.

If $|V_R| > n$ or R is not a proper interval graph, then we know by respectively Observation 2 and Theorem 2 that the answer is `NO`. These conditions can be checked in linear time, as explained in the preliminaries. Thus we assume that $|V_R| \leq n$ and R is a proper interval graph.

Let G have clique path $P = K_1 \cdots K_p$. Recall that P can be constructed in linear time. Let R have clique path $P' = L_1 \cdots L_q$ and ordered twin sets $X_1, \ldots, X_t$. Because $|V_R| \leq n$, we find that $q \leq p$ and that we can compute P' and the ordered twin sets in $\mathcal{O}(|V_R| + |E_R|) = \mathcal{O}(n^2)$ time. Since Lemma 2 applies only when $q \geq 3$, we distinguish between the cases where $q = 1$, $q = 2$, and $q \geq 3$.

Case 1. $q = 1$**.** Then R is a complete graph. By Theorem 3, we find that $|K_1| = |L_1|$ and we give each vertex in K_1 a different role. This yields a starting R-role assignment r of $G[K_1]$.

Suppose $i \geq 1$ and that we have extended r to a starting R-role assignment of $G[K_{\leq i}]$. By Lemma 3 we can order the vertices in $K_i \cap K_{i+1}$ as $u_1, \ldots, u_b$ such that $M_1(u_a) \subseteq M_1(u_{a+1})$ for $a = 1, \ldots, b-1$. We assign different roles to the vertices of $K_{i+1} \setminus K_i$, where we first use the roles of $M_1(u_a)$ before using any roles of $M_1(u_{a+1})$ for $a = 1, \ldots, b-1$. If we have used all the roles and there are still vertices in K_{i+1} with no role yet, we output NO. Otherwise we must check if the resulting mapping is a starting R-role assignment of $G[K_{\leq i+1}]$. If this is not the case, we output NO, because any R-role assignment is a starting role assignment of $G[K_{\leq i+1}]$. If this is the case, we stop if $i+1=p$, because a starting R-role assignment of $G[K_{\leq p}]=G$ is an R-role assignment of G; otherwise we repeat the above procedure with $i:=i+1$.

It is clear that this algorithm is correct. It runs in $\mathcal{O}(n^3)$ time, because ordering the vertices in $K_i \cap K_{i+1}$ takes $\mathcal{O}(n^2)$ time and there are $\mathcal{O}(n)$ bags.

Case 2. $q=2$**.** The algorithm for this case goes along the same lines as the previous case. Due to space restrictions we therefore moved it to the appendix.

Case 3. $q \geq 3$**.** First suppose $p \leq 2q$. By Theorem 3, both $G[K_{\leq q}]$ and $G[K_{\geq p-q+1}]$ must be isomorphic to R and have an R-role assignment, in case G has an R-role assignment. Because $p \leq 2q$, every vertex of G is in $K_{\leq q} \cup K_{\geq p-q+1}$. Hence, there are just four possibilities of assigning roles to vertices of G, namely two possibilities for $K_{\leq q}$ combined with two possibilities for $K_{\geq p-q+1}$. We check if one of them leads to an R-role assignment of G. Verifying whether a mapping $V_G \rightarrow V_R$ is an R-role assignment of G can be done in $\mathcal{O}(n^3)$ time by considering each vertex and checking if it has the desired roles occurring in its neighborhood.

Suppose $p \geq 2q+1$. We first check if $G[K_{\leq q}]$ is isomorphic to R. This can be done in linear time [4]. If $G[K_{\leq q}]$ is not isomorphic to R then we output NO due to Theorem 3. Suppose $G[K_{\leq q}] \simeq R$ and that without loss of generality we have a starting R-role assignment r of $G[K_{\leq q}]$ with $r(K_i)=L_i$ for $i=1,\ldots,q$. We now check whether we are in situation (i) or (ii) of Lemma 2. Then in both situations we can determine in $\mathcal{O}(n)$ time the desired index i and afterwards we continue with the graph $G[K_{\geq i}]$ unless we found no starting R-role assignment of $G[K_{\leq i}]$; in that case we output NO. The total running time of this procedure is $\mathcal{O}(n^3)$.

We have thus presented and proved the correctness of an algorithm with running time $\mathcal{O}(n^3)$ for testing whether a connected proper interval graph G has an R-role assignment for a connected graph R. If G is disconnected then we run the algorithm on each connected component separately. The total running time is still $\mathcal{O}(n^3)$. It remains to study the case when R is disconnected. In this case we cannot assume that $|V_R| \leq |V_G|$. Let c_R be the number of connected components of R. By the definition of a role assignment, G has an R-role assignment if and only if each connected component of G has an R'-role assignment for some connected component R' of R. Hence we can run our algorithm on every pair of connected components of G and R. This gives a total running time $\mathcal{O}(n^3 \cdot c_R)$, which is clearly polynomial. □

Recall that the problem of testing if a graph G has an R-role assignment for some smaller graph R is co-NP-complete in general [7]. Theorem 4 together with Corollary 1 has the following consequence.

Corollary 2. *There exists a polynomial time algorithm that has as input a proper interval graph G and that tests whether there exists a graph R with $|V_R|<|V_G|$ such that G has an R-role assignment.*

Proof. Let G be a proper interval graph on n vertices. First assume that G is connected. Let $P = K_1 \dots K_p$ be the clique path of G. Recall that $p \leq n$. By Corollary 1 we find that G only has an R-role assignment if $R \simeq G[K_{\leq i}]$ for some $1 \leq i \leq p$. This means that we need to apply the $\mathcal{O}(n^3)$ time algorithm for connected proper interval graphs of Theorem 4 at most $p \leq n$ times. Hence we find that testing whether G has an R-role assignment for some graph R with $|V_R| < |V_G|$ takes $\mathcal{O}(n^4)$ time.

Now assume that G is disconnected. Let $G_1, \dots, G_a$ with $a \geq 2$ be the connected components of G. For $j = 1, \dots, a$ we define $n_j = |V_{G_j}|$. As long as $j \leq a - 1$ we do as follows. We consider G_j and check if G_j has an R_j-role assignment for some role graph R_j with $|V_{R_j}| \leq n_j$. If so, then we replace connected component G_j by connected component R_j in G, i.e., we output $R = G_1 \oplus \dots G_{j-1} \oplus R_j \oplus G_j \oplus \dots \oplus G_a$, where $\oplus$ denotes the disjoint union operation on graphs. Suppose not. Then we consider G_{j+1}. If $j = a$ and we did not find a suitable role graph R in this way, then we output `NO`. Because we need $\mathcal{O}(n_j^4)$ time for each G_j and $n = n_1 + \dots + n_a$, the total running time of this algorithm is $\mathcal{O}(n^4)$, which is polynomial, as desired. □

As a consequence, we have in fact a stronger result: given a proper interval graph G, we can list in polynomial time all graphs R (up to isomorphism) with $|V_R| < n$ such that G has an R-role assignment.

4 Complementary results and an open question

A homomorphism r from a graph G to a graph R is *locally injective* if $|r(N_G(u))| = |N_G(u))|$ for every $u \in V_G$, and r is *locally bijective* if $r(N_G(u)) = N_R(r(u))$ and $|r(N_G(u))| = |N_G(u))|$ for every $u \in V_G$. Locally injective homomorphisms, also called *partial coverings*, have applications in frequency assignment [10] and telecommunication [11]. Locally bijective homomorphisms are also called *coverings* and have applications in topological graph theory [20] and distributed computing [1, 2]. The corresponding decision problems, called PARTIAL COVER and COVER respectively, are NP-complete for arbitrary G even when R is fixed to be the complete graph on four vertices [11, 17].

In this section, to give a complete picture, we study the computational complexity of all three locally constrained homomorphisms on chordal, interval, and proper interval graphs. Our findings can be summarized in the table below, where the three problems have input (G, R) and the left column indicates the graph class that G belongs to.

	PARTIAL COVER	COVER	ROLE ASSIGNMENT
Chordal	NP-complete	GI-complete	GI-hard
Interval	NP-complete	Polynomial	?
Proper Interval	NP-complete	Polynomial	Polynomial

We start with the following result, which allows us to conclude several of the entries in the above table, and which is interesting on its own. Its proof is given in the appendix.

Theorem 5. *Let G be a chordal graph and let R be a connected graph. Then there exists a locally bijective homomorphism from G to R if and only if every connected component of G is isomorphic to R.*

It is known that Graph Isomorphism is Graph Isomorphism-complete even for pairs (G, R) where G and R are chordal graphs [3]. Hence, Theorem 5 and Theorem 2 immediately imply that Cover is Graph Isomorphism-complete for pairs (G, R) where G is a chordal graph. On the other hand, Cover is polynomial time solvable on interval graphs, and hence also on proper interval graphs, since isomorphism between two interval graphs can be checked in polynomial time [4].

Unfortunately, the problem Partial Cover remains NP-complete even on proper interval graphs, and therefore also on interval graphs and chordal graphs. To see this, observe that a complete graph G allows a locally injective homomorphism to an arbitrary graph R if and only if R contains G as a subgraph. This gives a reduction from the well-known NP-complete problem Clique.

Finally, we present one more complexity result on the Role Assignment problem.

Theorem 6. Role Assignment *is* Graph Isomorphism*-hard on input pairs (G, R) where G is a chordal graph.*

Proof. As we argued above, Cover is Graph Isomorphism-complete on input pairs (G, R) where both G and R are chordal graphs. We give a polynomial time reduction from Cover to Role Assignment on chordal graphs. Suppose we are given two connected chordal graphs G and R, and ask whether G allows a locally bijective homomorphism to R. Then we may without loss of generality assume that G and R have the same number of vertices. The reason is that by Theorem 5 the answer can be `YES` only if $|V_G| = |V_R|$. In that case, by Observation 2, G allows a locally bijective homomorphism to R if and only if G allows a locally surjective homomorphism to R. This completes the reduction and the proof. □

Hence, unless Graph Isomorphism is polynomial time solvable, we do not have hope of solving Role Assignment in polynomial time on chordal graphs. Note, however, that if R is fixed, i.e., not a part of the input, then Role Assignment, Cover, and Partial Cover can all be solved in linear time on chordal input graphs G. This is because, in order to get a `YES` answer, G cannot have a larger clique than $|V_R|$ and thus has treewidth bounded by $|V_R|$, which is a constant. Since these problems are expressible in monadic second order logic, and the treewidth of a chordal graph can be computed in linear time, linear time solvability follows from the well-known result of Courcelle [8].

We conclude with the following the open question, corresponding to the question mark in the table.

What is the computational complexity of Role Assignment *on input pairs (G, R) when G is an interval graph?*

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Appendix

The following definition will be used in some of the proofs given here. For a subset X of V_R, we write $r^{-1}(X) = \{u \in V_G \mid r(u) \in X\}$. Let R' be a subgraph of R. Then $G[r^{-1}(V_{R'})]$ is the *preimage* of R' in G.

Proof of Theorem 2

Proof. Let G and R be two graphs such that G has an R-role assignment r. We may without loss of generality assume that G is connected; otherwise we consider each connected component of G separately. Before we proceed with the proof of the theorem, observe first that the preimage of an induced cycle C in R contains an induced cycle of length at least $|V_C|$ and the preimage of an (induced) tree T contains an (induced) tree isomorphic to T.

Proof of (i). Suppose G is chordal. If R is not chordal then R contains an induced cycle C of length at least four. By the observation above, the preimage of C in G contains an induced cycle of length at least four. This contradicts the assumption that G is chordal, hence R must be chordal.

Proof of (ii). Suppose G is an interval graph. From (i) we know that R is chordal. Suppose R is not an interval graph. Lekkerkerker and Boland give a list of minimal forbidden induced subgraphs of interval graphs [18]. According to this list, if R is chordal but not interval, there are 4 types of graphs one of which R must contain as an induced subgraph:

- the graph F_1 obtained from a claw by subdividing its edges exactly once, i.e., the graph with vertices $x, y_1, y_2, y_3, z_1, z_2, z_3$, and edges $xy_1, y_1z_1, xy_2, y_2z_2, xy_3, y_3z_3$.
- the graph F_2 obtained from a cycle $x_1 \cdots x_6x_1$ after adding edges x_1x_3, x_3x_5, x_3x_6 and a new vertex y with edge x_6y.
- for $k \geq 4$, the graph F_3^k that is obtained from a path $x_1 \cdots x_k$ after adding two new vertices y, z with edges x_iy for $i = 2, \ldots, k-1$ and yz.
- for $k \geq 3$, the graph F_4^k that is obtained from a cycle $C_{k+2} = x_1 \cdots x_kzyx_1$ after adding edges x_iy and x_iz for $i = 2, \ldots k-1$ and adding a new vertex z' with edges yz' and zz'.

Suppose R contains F_1 as an induced subgraph. Consider in G a vertex with role x and apply Lemma 1 three times starting from x. Then we find that G contains an induced F_1. Since G is an interval graph, this is not possible. Hence R does not contain F_1 as an induced subgraph.

Suppose R contains an induced F_2. Let u be a vertex of G with role x_6. Then u is on a cycle D of G of length $6d$ for some $d \geq 1$ such that the vertices of D have roles in repeated order $x_1, \ldots, x_6$. Note that D does not have to be an induced cycle of G. Also, u has a neighbor v in G with role y. Let s, s', t, t' be four vertices on D such that $ss'ut't$ forms a subpath of D and $r(s) = x_2$, $r(s') = x_1$, $r(t) = x_4$ and $r(t') = x_5$. Since D is a cycle, we can take a shortest path P_{st} in $G[V_D]$ from s to t not passing through a vertex from $\{s', t', u\}$. Suppose P_{st}

contains a neighbor u' of v. If v has more than one neighbor on P_{st}, then we choose u' to be the one closest to s on P_{st}. Vertex u' must have role x_6. Now we have a cycle $D' = u'vus's\overrightarrow{P_{st}}u'$, which is not necessarily chordless. However, since we took P_{st} to be shortest and we took u' to be closest to s, the only chords possible on D' are incident to u or s'.

Suppose u has a neighbor w on D' neither equal to s' nor to v. Assume that w is chosen closest to u'. Because u and u' have the same role, namely role x_6, we find that u and u' are not adjacent. This means that $w \neq u'$. Then the cycle $u'vuw\overrightarrow{P_{st}}u'$ is a chordless cycle on at least four vertices. This is not possible, because G is chordal. Hence, the only neighbors of u on D' are s' and v.

Suppose s' has a neighbor w' on D' neither equal to s nor to u. Assume that w' is chosen closest to u' (with possibly $w' = u'$). Then the cycle $u'vus'w'\overrightarrow{P_{st}}u'$ is a chordless cycle on at least four vertices. This is not possible, because G is chordal. Hence, the only neighbors of s' on D' are s and u.

From the above we find that D' is chordless. Since D' contains at least five vertices and G is chordal, this is not possible. We conclude that P_{st} avoids any neighbor of v. Then, due to the paths P_{st}, $ss'uv$ and $tt'uv$, we find that $\{s, t, v\}$ forms an AT. This is not possible, because G is AT-free. Hence R does not contain an induced F_2.

Suppose R contains an induced F_3^k for some $k \geq 4$. Let $t_2, t_3 \in V_G$ have roles x_2, x_3, respectively. Because G is chordal, we may without loss of generality assume that t_2 and t_3 have a common neighbor u with role y. Let v be a neighbor with role z. Let t_1 be a neighbor of t_2 with role x_1. We apply Lemma 1 to find a path $t_3 \cdots t_k$ in G such that $r(t_i) = x_i$ for $i = 3, \ldots, k$. We consider the paths $t_1 \cdots t_k$, t_1t_2uv and $t_k \cdots t_3uv$ in order to obtain that $\{t_1, t_k, v\}$ is an AT. This is not possible, because G is AT-free. Hence R does not contain an induced F_3^k.

Suppose R contains an induced F_4^k for some $k \geq 3$. Let $u, v \in V_G$ have roles y, z, respectively. Because G is chordal, we may without loss of generality assume that u and v have a common neighbor w with role z'. We also deduce that uv is an edge of a cycle D of G of length $d(k+2)$ for some $d \geq 1$ such that the vertices of D have roles in repeated order $z, y, x_1, \ldots, x_k$. Suppose that we chose u, v, w and D such that d is minimal over all triangles $uvwu$ with $r(u) = y$, $r(v) = z$ and $r(w) = z'$. Note that D does not have to be an induced cycle of G.

Let s, t be the two vertices such that $suvt$ forms a subpath of D; note that $r(s) = x_1$ and $r(t) = x_k$. Since D is a cycle, we can take a shortest path P_{st} in $G[V_D]$ from s to t not using edge uv. Suppose P_{st} contains a neighbor u' of w that is neither u nor v. Note that w is not adjacent to s because $r(s) = x_1$ and $r(w) = z'$ are not adjacent in R. Hence, u' can be chosen in such a way that u' is the only vertex on $s\overrightarrow{P_{st}}u'$ that is adjacent to w. Because u' is adjacent to w and w has role z', we find that u' must have role y or z. We consider each case.

Suppose $r(u') = y$. Then u and u' have the same role. Consequently, u and u' are not adjacent. Let v' be the neighbor of u on $s\overrightarrow{P_{st}}u'$ that is closest to u'. Because u' and u are not adjacent, we find that $v' \neq u'$. Recall that u' is the only vertex on $s\overrightarrow{P_{st}}u'$ that is adjacent to w. We then find that the cycle $uv'\overrightarrow{P_{st}}u'wu$

is a chordless cycle on at least four vertices. This is not possible, because G is chordal. Hence $r(u') \neq y$.

Suppose $r(u') = z$. If uu' is an edge then we could take a shorter cycle D' of length $d'(k+2)$ with $d' < d$, and d would not be minimal. Hence uu' is not an edge. This means we can apply the same arguments as in the previous case, which leads us to conclude that $r(u') \neq z$. Because $r(u') \in \{y, z\}$, we get a contradiction. We conclude that P_{st} avoids any neighbor of w.

Now we do as follows. Besides P_{st} we also consider the paths suw and tvw. We then find that $\{s, t, w\}$ forms an AT in G. This is not possible, because G is an interval graph. Consequently, R does not contain an induced F_4^k. This means that R must be an interval graph as well.

Proof of (iii). Suppose G is a proper interval graph. From the above, R is an interval graph. If R has a claw then, by the observation in the beginning of the proof, its preimage in G contains a claw, contradicting the assumption that G is proper interval. Hence, R must claw-free and consequently a proper interval graph. □

Proof of Lemma 2

Proof. Note that $t \geq 6$ because $q \geq 3$. We also observe that the restriction of r to $K_{\leq q}$ is an R-role assignment by Theorem 3.

Proof of (i). Suppose $u \in T$ is not in K_{q+1}. Let $i \geq q+1$ be chosen to be the smallest index such that $|K_i| \geq |L_q|$. Hence, if $i > q+1$ then $|K_h| < |L_q|$ for $q+1 \leq h \leq i-1$. Note that such an index i exists, because $p - q + 1 \geq 2q + 1 - q + 1 \geq q + 2$ and either $r(K_{p-q+1}) = L_q$ or $r(K_p) = L_q$, due to Theorem 3.

We first show that if $i > q+1$ then $r(K_h) \subseteq X_t$ for $q+1 \leq h \leq i-1$. In order to derive a contradiction, suppose there exists an index h' with $q+1 \leq h' \leq i-1$ such that $K_{h'}$ contains a vertex with role in $X_{\leq t-1}$. Choose h' in such a way that $r(K_h) \subseteq X_t$ for $q+1 \leq h \leq h'-1$.

Claim 1. $f_P(v) = h'$ for all $v \in K_{h'}$ with $r(v) \in X_{\leq t-1}$.

We prove Claim 1 as follows. Let $v \in K_{h'}$ with $r(v) \in X_{\leq t-1}$. First suppose $v \in K_{\leq q}$. Since $r(v) \in X_{\leq t-1}$, we find that $r(v)$ is adjacent to some vertex in $X_{\leq t-1}$ to which $r(u)$ is not adjacent, i.e., $f_{P'}(r(v)) < f_{P'}(r(u)) = q$. By Theorem 3 and our assumption that $r(K_q) = L_q$, we have $r(K_a) = L_a$ for $a = 1, \ldots, q$. Hence, $f_P(v) < f_P(u)$. Since $l_P(u) = q$ and $l_P(v) \geq q+1$, this means that v transcends u. This is not possible due to Theorem 1. We conclude that v does not appear in $K_{\leq q}$. By our choice of h', we then find that v is in $K_{h'} \setminus K_{\leq h'-1}$, so $f_P(v) = h'$ indeed. This completes the proof of Claim 1.

We claim that $K_{h'}$ contains a vertex with role in X_t. If $h' \geq q+2$ then this claim follows from the definition of a clique path, which implies that $K_{h'-1} \cap K_{h'} \neq \emptyset$, and our choice of h', which implies $r(K_{h'-1}) \subseteq X_t$. Suppose $h' = q+1$. If all vertices of T are not in K_{q+1}, then there must be a vertex $v^* \in K_{q+1} \cap K_q$ with role in $X_{\leq t-1}$. Then $f_P(v^*) \leq q \leq h'-1$ and this is not possible due to Claim 1. Hence, indeed, $K_{h'}$ contains a vertex with role in X_t. Consequently, we find that $r(K_{h'}) \subset L_q$.

By definition of a clique path, $K_{h'} \setminus K_{h'+1} \neq \emptyset$. Let $u' \in K_{h'} \setminus K_{h'+1}$, so $l_P(u') = h'$. We claim that $r(u') \in X_t$. If not then $r(u') \in X_{\leq t-1}$, and consequently, $f_P(u') = h'$ by Claim 1. However, we have $r(K_{h'}) \subset L_q$ and $|K_{h'}| < |L_q|$. This, together with $f_P(u') = l_P(u') = h'$, implies that u' misses at least one role of L_q in its neighborhood. This is not possible. Hence, $r(u') \in X_t$ indeed. We need this vertex u' in the proof of the following claim, and also in the rest of the proof.

Claim 2. There exists a vertex in $K_{h'}$ with role in X_{t-1}.

We prove Claim 2 as follows. In order to derive a contradiction, suppose there is no vertex in $K_{h'}$ with role in X_{t-1}. Let v^* be a vertex in $K_{h'}$ with $r(v^*) \in X_{\leq t-1}$. Then we find that $r(v^*) \in X_{\leq t-2}$. Let s be a neighbor of v^* with role in X_{t-1}. Because $f_P(v^*) = h'$ by Claim 1 and $s \notin K_{h'}$, we find that $l_P(v^*) \geq h'+1$ and $f_P(s) \geq h'+1$. Since $r(v^*)$ belongs to $X_{\leq t-2}$, and $r(s)$ and $r(u')$ are both in $X_{\geq t-1}$, there exists a neighbor v' of v^* with $r(v')$ adjacent to neither $r(u')$ nor $r(s)$ in R. Hence v' is adjacent to neither u' nor s in G. Since $l_P(u') = h'$ and $f_P(s) \geq h'+1$, we find that u' and s are not adjacent. However, then G has an induced claw with center v^* and leaves s, u', v', which contradicts the assumption that G is a proper interval graph. Hence we have proven Claim 2.

By Claim 2, there exists a vertex $v \in K_{h'}$ with $r(v) \in X_{t-1}$. Because $|K_{h'}| < |L_q|$, there exists a role $x \in L_q$ that is not in $r(K_{h'})$. This means that v is in $K_{h'+1}$; otherwise v will not get its required neighbor with role x. Let w be this neighbor, so $r(w) = x$.

Claim 3. There is no neighbor s of v that has $f_P(s) \geq h'+1$ and $r(s) \in X_t$.

We prove Claim 3 as follows. Suppose v is adjacent to such a vertex s. Then, since $r(v)$ belongs to $X_{\leq t-1}$, and $r(s)$ and $r(u')$ are both in X_t, there exists a neighbor v' of v with $r(v')$ adjacent to neither $r(u')$ nor $r(s)$. Hence v' is adjacent to neither u' nor s. Since $l_P(u') = h'$ and $f_P(s) \geq h'+1$, we find that u' and s are not adjacent. However, then G has an induced claw with center v and leaves s, u', v', which contradicts the assumption that G is a proper interval graph. Hence we have proven Claim 3.

Claim 3 implies that $x \in X_{\leq t-1}$, because v is adjacent to w with $f_P(w) \geq h'+1$ and $r(w) = x$. Let $z_\ell \in X_1$ and let $Q' = z_1 \cdots z_\ell$, with $x = z_1$, be a shortest path in R from x to a role $z_\ell \in X_1$. By Lemma 1 we find that G contains a path $Q = t_1 \cdots t_\ell$ with $t_1 = w$ such that $r(t_i) = z_i$ for $i = 1, \ldots, \ell$. Because Q' is shortest, we find that $t_2 \overrightarrow{Q} t_\ell$ contains no vertex with role in L_q. Because $K_{h'}$ only contains vertices with role in L_q and $w \notin K_{h'}$, this implies that $f_P(t_i) \geq h'+1$ for $i = 1, \ldots, \ell$.

Because w has role x and $x \in L_q$, we find that all roles of X_t appear as roles of neighbors of w. Because $r(u') \in X_t$, this means that w has a neighbor w' with $r(w') = r(u')$. Note that $f_P(w') \geq h'+1$, because $f_P(w) \geq h'+1$. Because

$l_P(u') = h'$, this implies that u' and w' are two different vertices that are not adjacent.

We claim that $\ell \geq 3$. In order to see this, we first observe that v is not adjacent to a vertex with role in X_1. This is because $r(v) \in X_{t-1}$ is already adjacent to a role in X_t, namely $r(u')$, and then $q = 2$, whereas we assume that $q \geq 3$. Suppose $\ell = 1$. Then $x = z_1 \in X_1$, and v is adjacent to a vertex, namely w, with role $r(w) = x \in X_1$. This is not possible, as we just observed. Suppose $\ell = 2$. Then $r(t_2) = z_2 \in X_1$, and we find that v is not adjacent to t_2, again due to the above observation. Since $r(w') \in X_t$, we also find that t_2 and w' are not adjacent. By Claim 3, v and w' are not adjacent. Then G has an induced claw with center w and leaves v, w', t_2, which contradicts the assumption that G is a proper interval graph. So, $\ell \geq 3$ indeed.

We claim that t_ℓ, u', w' form an AT in order to get a contradiction (recall that a proper interval graph is AT-free). To show this we first prove that t_ℓ, u', w' are three different vertices that are not adjacent to one another. We already deduced that u' and w' are two different non-adjacent vertices. Because $l_P(u') = h'$ and $f_P(t_\ell) \geq h'+1$, we also find that u' and t_ℓ are two different non-adjacent vertices. As $t > 1$, vertices t_ℓ and w' with roles in X_1 and X_t, respectively, are different and non-adjacent.

Since Q' is a shortest path in R from x to a vertex in X_1, we find that Q neither contains v nor w', because these vertices have a role in L_q. Recall that w' has a role in X_t and that w, the first vertex of Q, has role in $X_{\leq t-1}$. Then we can also use the fact that Q' is a shortest path to deduce that w' has no neighbor on $t_2\overrightarrow{Q}t_\ell$.

In order to have a path from u' to t_ℓ, we claim that v is adjacent to t_2. Suppose not. By Claim 3, we find that v and w' are not adjacent. Since w' has no neighbor on $t_2\overrightarrow{Q}t_\ell$, vertices t_2 and w' are not adjacent. However, then G has an induced claw with center w and leaves t_2, v, w', which contradicts the assumption that G is a proper interval graph. Hence, v and t_2 are adjacent. This implies that G indeed contains such a path, namely the path $u'vt_2\overrightarrow{Q}t_\ell$.

We consider the three paths $u'vww'$, $u'vt_2\overrightarrow{Q}t_\ell$ and $w'w\overrightarrow{Q}t_\ell$. In order to finish our claim that $\{t_\ell, u', w'\}$ is an AT, we show that t_ℓ has no neighbor on vw, that u' has no neighbor on $w\overrightarrow{Q}t_{\ell-1}$, and that w' has no neighbor on $vt_2\overrightarrow{Q}t_{\ell-1}$.

Consider t_ℓ. Recall that v is not adjacent to a vertex with role in X_1. This is because $r(v) \in X_{t-1}$ is already adjacent to a role in X_t, namely $r(u')$, and then $q = 2$, whereas we assume that $q \geq 3$. Hence t_ℓ with role $z_\ell \in X_1$ is not adjacent to v. Since Q is a shortest path in R, we find that Q' is an induced path in G. We already showed that $\ell \geq 3$, i.e., Q' contains at least three vertices. Hence we find that t_ℓ is not adjacent to $w = t_1$ Consider u'. Because $l_P(u') = h$ and each vertex in $w\overrightarrow{Q}t_{\ell-1}$ is introduced in bag $K_{h'+1}$ or later, we find that u' has no neighbor on $w\overrightarrow{Q}t_{\ell-1}$. Consider w'. We already deduced that w' and v are not adjacent, and that w' has no neighbor on $t_2\overrightarrow{Q}t_{\ell-1}$.

The above indeed implies that the vertices t_ℓ, u', w' form an AT, contradicting the assumption that G is proper interval. We conclude that $r(K_h) \subseteq X_t$ for $q+1 \leq h \leq i-1$.

Because $r(K_h) \subseteq X_t$ for $q+1 \leq h \leq i-1$, every vertex v with $q+1 \leq f_P(v) \leq i-1$ has a role in X_t and still needs a neighbor with role in $X_{\leq t-1}$. Hence $K_{\geq q+1} \setminus K_{\leq q} \subseteq K_{\geq i}$.

We claim that K_i contains a vertex with role in X_t. If $i \geq q+2$, then any vertex in $K_{i-1} \cap K_i$ has role in X_t. Hence, for $i \geq q+2$, this claim is true. Suppose $i = q+1$. Let $s^* \in K_{\leq q} \cap K_{q+1}$. If $s^* \notin T$, then every vertex of T is in K_{q+1}; otherwise s^* will transcend such a vertex (we can prove this using the same arguments as in the proof of Claim 1). So, also for $i = q+1$, the claim is true.

Because $r(K_i) \cap X_t \neq \emptyset$ and $|K_i| \geq |L_q|$, we find that $r(K_i) = L_q$ (and hence $|K_i| = |L_q|$). Since all vertices in K_i with role in $X_{\leq t-1}$ are not in $K_{\leq i-1}$, we find that the restriction of r to $K_{\geq i}$ is an R-role assignment. This proves (i).

Proof of (ii). Suppose all vertices in T are in K_{q+1}. Because $t > 1$, we find that $L_1 \cap T = \emptyset$. Let $i \geq q+1$ be such that K_i contains a vertex $u \in T$, whereas K_{i+1} does not contain u, so $l_P(u) = i$. Since $L_1 \cap T = \emptyset$ and either $r(K_{p-q+1}) = L_1$ or $r(K_p) = L_1$ due to Theorem 3, such an index i exists. We choose $i \geq q+1$ to be the smallest index with this property, i.e., $T \subseteq K_h$ for $q+1 \leq h \leq i$. We observe that $r(K_i) \subseteq L_q$, because $T \subseteq K_i$.

Let C be the set of vertices in $K_{\leq q} \cap K_i$ with role in $X_{\leq t-1}$. We claim that C is empty. In order to prove this, suppose there exists a vertex $u^* \in C$. Using the same arguments as in Claim 1 of the proof of (i), we obtain $l_P(u^*) = l_P(u) = i$ and without loss of generality that $r(u^*) \in X_{t-1}$.

Let $L_q = X_b \cup \cdots \cup X_t$ for some $b \leq t-1$. Let $v^* \in K_i \setminus K_{\leq i-1}$, so $f_P(v^*) = i$. Because $T \subset K_i$, we find that $r(v^*) \in X_{\geq b} \cap X_{\leq t-1}$. Then $v^* \in K_{i+1}$ is adjacent to a neighbor s with $r(s) \in X_{b-1}$. Since $f_P(v) = i$ and a vertex with role in X_{b-1} is not adjacent to a vertex in T (which has a role in X_t), we find that $l_P(v) \geq i+1$ and $l_P(s) \geq i+1$. Since $r(u^*) \in X_{\leq t-1}$, roles $r(u^*)$ and $r(s)$ are adjacent. Hence, s is adjacent to a neighbor s' with $r(s') = r(u^*)$. Because $l_P(u^*) = i$ and $f_P(s) \geq i+1$, we find that u^* and s' are two different and non-adjacent vertices. Let s'' be a neighbor of s' with $r(s'') \in X_t$. Then $f_P(s'') \geq i+1$, and we find that s'' and u^* are non-adjacent either.

Let $z_\ell \in X_1$ and let $Q' = z_1 \cdots z_\ell$ with $z_1 = x$ be a shortest path from x to z_ℓ in R. By Lemma 1 we find that G contains a path $Q = t_1 \cdots t_\ell$ with $t_1 = s$ such that $r(t_i) = z_i$ for $i = 1, \ldots, \ell$.

We claim that $\ell \geq 2$. Suppose $\ell = 1$. Then vertex s' with $r(s') = r(u^*) \in X_{t-1}$ is adjacent to a vertex with role in X_1, namely s. This means that $q = 2$, whereas we assume that $q \geq 3$ holds. Hence, $\ell \geq 2$ indeed.

We claim that s' and v^* are adjacent. Suppose not. Recall that $r(s) \in X_{b-1}$, and that s' and v^* each have a role in $X_{\geq b}$. Then, because Q' is a shortest path, we find that t_2 is neither adjacent to s' nor to v^*. This means that G contains an induced claw with center s and leaves s', t_2, v^*, contradicting the assumption that G is proper interval. We conclude that indeed $s'v^*$ is an edge.

We claim that s'' and v^* are not adjacent. Suppose they are. Recall that $r(v^*) \in X_{\leq t-1}$ and u^*, s'' each have a role in X_t. This means that v^* is adjacent to a vertex v' with role in $X_{\leq b-1}$, whereas u^*, s'' are both not adjacent to such v'. Since u^* and s'' are not adjacent, we find that G contains an induced claw with center v^* and leaves s'', u^*, v', contradicting the assumption that G is proper interval. Hence s'' and v^* are not adjacent.

We claim that s and s'' are not adjacent. Suppose they are. Then G contains an induced claw with center s and leaves s'', t_2, v^*. This is not possible, as we saw before. Hence, indeed s and s'' are not adjacent.

We now consider the three paths $u^*v^*s's''$, $u^*v^*st_2$, and $t_2ss's''$ and deduce from the above claims that t_2, u^*, s'' form an AT, contradicting the assumption that G is proper interval. We conclude that $C = \emptyset$.

Because $C = \emptyset$, we have $K_{\leq q} \cap K_i = T$. This means that every $v \in K_i \setminus T$ has $f_P(v) \geq q+1$. Because $T \subset K_h$ for $q+1 \leq h \leq i$, every v with $q+1 \leq f_P(v) \leq i-1$ belongs to K_i and still need all its neighbors with roles in $X_{\leq b-1}$. Hence the restriction of r to $K_{\leq i}$ is an R-role assignment. Because $T \subset K_i$, we deduce that $r(K_i) = L_q$ and $r(K_{i+1}) = L_{q-1}$ after applying Theorem 3. Hence $T \cap K_{i+1} = \emptyset$. This completes the proof of (ii). □

Proof of "Case 2. $q = 2$" of Theorem 4

We check in linear time whether $G[K_{\leq 2}] \simeq R$. If not, then we output `NO` due to Theorem 3. Otherwise, we assume without loss of generality that $G[K_{\leq 2}]$ has an R-role assignment r with $r(K_i) = L_i$ for $i = 1, 2$. We note that r is a starting R-role assignment of $G[K_{\leq 2}]$.

From the above, we may suppose that $i \geq 2$ and that we have extended r to a starting R-role assignment of $G[K_{\leq i}]$. We must extend r to $K_{\leq i+1}$ by assigning roles to $K_{i+1} \setminus K_i$. We say that a choice of r on $K_{i+1} \setminus K_i$ is *right* if our extension of r to $K_{\leq i+1}$ eventually leads to an R-role assignment of G.

Note that $q = 2$ implies that R consists of exactly three twin sets such that $X_1 = L_1 \setminus L_2$, $X_2 = L_1 \cap L_2$, and $X_3 = L_2 \setminus L_1$. For $j = 1, 2, 3$, let $X'_j = r(K_i \cap K_{i+1}) \cap X_j$. Let $Y = r(K_{i+1} \setminus K_i)$ be the set of roles assigned to the vertices of $K_{i+1} \setminus K_i$ after we have extended r to $K_{\leq i+1}$. Before we explain how to do this, we first prove the following two claims.

Claim 1. If $X'_1 \cup X'_3 = \emptyset$ then all vertices in $K_i \cap K_{i+1}$ either all miss all roles in X_1 and no role in X_3, or else they all miss all roles in X_3 and no role in X_1.

We prove Claim 1 as follows. Assume that r is a right choice and that $X'_1 \cup X'_3 = \emptyset$. Then all vertices in $K_i \cap K_{i+1}$ have role in X'_2. First suppose $K_i \cap K_{i+1}$ contains a vertex u, such that u has a neighbor $v \in K_{\leq i}$ with $r(v) \in X_1$ and a neighbor $w \in K_{\leq i}$ with $r(w) \in X_3$. Because a vertex with role in X_1 is not adjacent to a vertex with role in X_3, there is no bag containing both v and w. Then we may without loss of generality assume that $l_P(v) < f_P(w)$. Because $X'_3 = \emptyset$, we find that $l_P(u) \geq i + 1 > l_P(w)$. Because uv is an edge, there is a bag containing u and v. Hence $f_P(u) \leq l_P(v)$. This means that $f_P(u) \leq l_P(v) < f_P(w) \leq l_P(w) < l_P(u)$. Consequently, u transcends w. This is not possible due to Theorem 1. We conclude that every vertex in $K_i \cap K_{i+1}$ either misses all roles in X_1 and no role in X_3, or else misses all roles in X_3 and no role in X_1.

Now suppose $K_i \cap K_{i+1}$ contains two vertices u, u' such that u misses all roles in X_1 and no role in X_3, whereas u' misses all roles in X_3 and no role in X_1. Then there exists a neighbor v of u with $l_P(v) \leq i-1$ and role in X_3, and there exists a neighbor v' of u' with $l_P(v') \leq i-1$ and role in X_1. Note that vu' is not an edge in G, because u' misses all roles in X_3 and $r(v) \in X_3$; similarly, $v'u$ is not an edge. Because roles in X_1 are not adjacent to roles in X_3, we may without loss of generality assume that $l_P(v) < f_P(v')$. Because u and v are neighbors, there is a bag of P that contains both of them. This means that $f_P(u) \leq l_P(v)$, and consequently, $f_P(u) < f_P(v')$. This, together with $l_P(v') < i < l_P(u)$, implies that u and v' are adjacent. This is not possible, since we concluded earlier that $v'u$ is not an edge. Consequently, we have proven Claim 1.

Claim 2. Let $u \in K_i \cap K_{i+1}$. If u misses a role in X_1, then Y contains no role in X_3, and if u misses a role in X_3, then Y contains no role in X_1; otherwise r is not a right choice.

We prove Claim 2 as follows. Suppose $u \in K_i \cap K_{i+1}$ misses role $x \in X_3$, and Y contains a role in X_1. Then, by definition of Y, there is a vertex $u' \in K_{i+1} \setminus K_i$ with $r(u') \in X_1$. Note that $f_P(u') = i+1$. Because u misses role x, it needs a neighbor v with $f_P(v) \geq i+1$ and role x. Because a vertex with role in X_1 is not adjacent to a vertex with role in X_3 and $f_P(u') = i+1$, we find that $l_P(u') < f_P(v)$. Because u and v are neighbors, we get $f_P(v) \leq l_P(u)$. Then $f_P(u) < i+1 = f_P(u')$ and $l_P(u') < f_P(v) \leq l_P(u)$. Hence u transcends u'. This is not possible due to Theorem 1, and we have proven Claim 2.

We will now do as follows. Because a role in X_1 is not adjacent to a role in X_3, we know that Y will either contain no role from X_1 or else no role from X_3. If $X'_1 \neq \emptyset$ then it is immediately clear that Y contains no role in X_3. If $X'_3 \neq \emptyset$ then it is immediately clear that Y contains no role in X_1. Below we show how to decide in $\mathcal{O}(n)$ time whether Y contains no role from X_1 or no role from X_3 in the case when $X'_1 = X'_3 = \emptyset$.

If $X'_3 = \emptyset$, then we pick a vertex $u \in K_i \cap K_{i+1}$ and check in $\mathcal{O}(n)$ time if u misses a role in X_1. We apply Claim 1 and either find that all vertices in $K_i \cap K_{i+1}$ miss all roles in X_1 and no roles in X_3, or they all miss all roles in X_3 and no role in X_1. We apply Claim 2 in order to find that, in the first case, Y contains no role in X_3, and in the second case, Y contains no role in X_1. From the above we deduce that there are three cases:

(i) $X'_1 \neq \emptyset$, and consequently, $X'_3 = \emptyset$ and Y contains no role from X_3;
(ii) $X'_3 \neq \emptyset$, and consequently, $X'_1 = \emptyset$ and Y contains no role from X_1;
(iii) $X'_1 = X'_3 = \emptyset$ and we found that Y contains no role from X_3;
(iv) $X'_1 = X'_3 = \emptyset$ and we found that Y contains no role from X_1;

We assume without loss of generality that we are in case (ii) or (iv). This means that Y only contains roles from $(X_2 \backslash X'_2) \cup (X_3 \backslash X'_3)$. Before we continue we need two new claims.

Claim 3. If Y contains a role from X_2 then Y contains all roles from $X_3 \setminus X'_3$; otherwise r is not a right choice.

We prove Claim 3 as follows. Let $v \in K_{i+1} \setminus K_i$ have role $r(v) \in X_2$, whereas role $x \in X_3 \setminus X'_3$ does not belong to Y. Note that $f_P(v) = i+1$. Because Y contains no role from X_1, and $X'_1 = \emptyset$, we find that v has no neighbor in K_{i+1} that has its role in X_1. Then, as $f_P(v) = i+1$ and $r(v) \in X_2$, we find that v will need a neighbor w with role $r(w) \in X_1$ in a later bag, so $f_P(w) \geq i+2$. Because $f_P(v) = i+1$ and $x \notin Y$, we find that v also needs a neighbor w' with role $r(w') = x$ in a later bag, so $f_P(w') \geq i+2$. Because $r(w) \in X_1$ and $r(w') \in X_3$, we find that w and w' are not adjacent in G. Let u be a vertex in K_{i+1} with $l_P(u) = i+1$. Because $l_P(u) = i+1$ and $f_P(w) \geq i+2$, we find that u and w are not adjacent. For the same reason, u and w' are not adjacent. Then we find that G contains an induced claw with center v and leaves u, w, w', which contradicts the assumption that G is proper interval. Hence we have proven Claim 3.

Claim 4. If $K_{i+1} \setminus K_i$ contains two vertices v and v' such that $r(v) \in X_2$ and $r(v') \in X_3$ then $l_P(v) > l_P(v')$; otherwise r is not a right choice.

We prove Claim 4 as follows. Suppose v and v' are two vertices in $K_{i+1} \setminus K_i$ such that $r(v) \in X_2$ and $r(v') \in X_3$, whereas $l_P(v) \leq l_P(v')$. Because $f_P(v) = f_P(v')$ and $l_P(v) \leq l_P(v')$, all neighbors of v are neighbors of v'. This means that also a neighbor of v with role in X_1 is a neighbor of v'. Because v' has a role in X_3, this is not possible. Hence, we have proven Claim 4.

Due to Claims 3-4 we do as follows. By Lemma 3 we can order the vertices in $K_i \cap K_{i+1}$ as $u_1, \ldots, u_b$ such that $M_3(u_a) \subseteq M_3(u_{a+1})$ for $a = 1, \ldots, b-1$. Note that $M_3(u_b) \subseteq X_3 \setminus X'_3$. This enables us to order the roles of X_3 by first ordering the roles of $M_3(u_1)$ in an arbitrary way, followed by the vertices of $M_3(u_2)$ in arbitrary order and so on, until we finish off with the remaining vertices of $X_3 \setminus X'_3$ placed in some arbitrary order. We call such an ordering a *safe* ordering. We also safely order the vertices of $X_2 \setminus X'_2$. We now order the vertices of $K_{i+1} \setminus K_i$ as $v_1, \ldots, v_d$ such that $l_P(v_i) \leq l_P(v_{i+1})$ for $i = 1, \ldots, d-1$. We consider the vertices of $K_{i+1} \setminus K_i$ in order $v_1, \ldots, v_d$ and try to assign different roles to them by first using the roles of $X_3 \setminus X'_3$ ordered safely and then using the roles of $X_2 \setminus X'_2$ ordered safely. If this is not possible (i.e., there are too many vertices in $K_{i+1} \setminus K_i$) then we output `NO`. Otherwise we check if r is a starting R-role assignment of $G[K_{\leq i+1}]$. If this is not the case, we output `NO` because any R-role assignment is a starting role assignment of $G[K_{\leq i+1}]$. If this is the case, we stop if $i+1 = p$, because a starting R-role assignment of $G[K_{\leq p}]$ is an R-role assignment of G. If $i+1 < p$, we repeat the above procedure with $i := i+1$.

Note that this algorithm runs in $\mathcal{O}(n^2)$ time per bag. Because there are at most n bags, we find an $\mathcal{O}(n^3)$ total running time. □

Proof of Theorem 5

Proof. If G is disconnected then we consider each connected component of G separately. Assume that G is connected. If G is isomorphic to R, then the identity mapping from G to R is our desired locally bijective homomorphism.

For the reverse implication, suppose that there exists a locally bijective homomorphism r from G to R. Because any locally bijective homomorphism is also locally surjective, we can apply Theorem 2 in order to find that R is chordal. For the same reason we can apply Observation 2 in order to find that each vertex in R appears as a role of at least one vertex in G. We claim that each vertex in R appears as a role of exactly one vertex in G. In order to derive a contradiction, suppose there exists a vertex $x \in V_R$ such that $r^{-1}(x)$ has size at least two.

Let v and v' be two different vertices of G belonging to $r^{-1}(x)$. Let P be a shortest path from v to v' in G. Because P is shortest, P is an induced path. From the definition of a locally bijective homomorphism we deduce the following two statements. Firstly, because two vertices with the same role cannot be adjacent, we find that $|V_P| \neq 2$. Secondly, because a vertex has no two neighbors with the same role, we find that $|V_P| \neq 3$. Hence, P is an induced path with $|V_P| \geq 4$. This, together with $r(v) = r(v') = x$, means that $r(P)$ forms an induced cycle D in R with $|V_D| = |V_P| - 1$. Because R is chordal, D must consist of three vertices, say $D = xyzx$. Consequently, $|V_P| = 4$ holds.

Let C be the connected component of $G[r^{-1}(x) \cup r^{-1}(y) \cup r^{-1}(z)]$ that contains v and v'. By definition of a locally bijective homomorphism, every vertex is of degree two in D. This means that D is an induced cycle in G. Because every vertex of P belongs to D, and $|V_P| = 4$, we find that $|V_D| \geq 4$. This contradicts our assumption that G is chordal. We conclude that indeed each vertex in R appears as a role of exactly one vertex in G. This means that r is an isomorphism between G and R, and we find that $G \simeq R$, as desired. $\square$