Algorithms for outerplanar graph roots and graph roots of pathwidth at most 2*

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Abstract. Deciding whether a given graph has a square root is a classical problem that has been studied extensively both from graph theoretic and from algorithmic perspectives. The problem is NP-complete in general, and consequently substantial effort has been dedicated to deciding whether a given graph has a square root that belongs to a particular graph class. There are both polynomial-time solvable and NP-complete cases, depending on the graph class. We contribute with new results in this direction. Given an arbitrary input graph G, we give polynomial-time algorithms to decide whether G has an outerplanar square root, and whether G has a square root that is of pathwidth at most 2.

1 Introduction

Squares and square roots of graphs form a classical and well-studied topic in graph theory, which has also attracted significant attention from the algorithms community. A graph G is the *square* of a graph H if G and H have the same vertex set, and two vertices are adjacent in G if and only if the distance between them is at most 2 in H. This situation is denoted by $G = H^2$, and H is called a *square root* of G. A square root of a graph need not be unique; it might even not exist. There are graphs without square roots, graphs with a unique square root, and graphs with several different square roots. Characterizing and recognizing graphs with square roots has therefore been an intriguing and important problem both in graph theory and in algorithms for decades.

Already in 1967, Mukhopadhyay [24] proved that a graph G on vertex set $\{v_1, \ldots, v_n\}$ has a square root if and only if G contains complete subgraphs $\{K^1, \ldots, K^n\}$, such that each K^i contains v_i, and vertex v_j belongs to K^i if and only if v_i belongs to K^j. Unfortunately, this characterization does not yield a polynomial-time algorithm for deciding whether G has a square root. Let us

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formally call Square Root the problem of deciding whether an input graph G has a square root. In 1994 it was shown by Motwani and Sudan [23] that Square Root is NP-complete. Motivated by its computational hardness, special cases of the problem have been studied, where the input graph G belongs to a particular graph class. According to these results, Square Root is polynomial-time solvable on planar graphs [20], and more generally, on every non-trivial minor-closed graph class [25]. Polynomial-time algorithms exist also when the input graph G belongs to one of the following graph classes: block graphs [18], line graphs [21], trivially perfect graphs [22], threshold graphs [22], graphs of maximum degree 6 [3], graphs of maximum average degree smaller than $\frac{46}{11}$ [11], graphs with clique number at most 3 [12], and graphs with bounded clique number and no long induced path [12]. On the negative side, it has been shown that Square Root is NP-complete on chordal graphs [15]. A number of parameterized complexity results exist for the problem [3, 4, 11].

More interesting from our perspective, the intractability of the problem has also been attacked by restricting the properties of the square root that we are looking for. In this case, the input graph G is arbitrary, and the question is whether G has a square root that belongs to some graph class $\mathcal{H}$ specified in advance. We denote this problem by $\mathcal{H}$-Square Root, and this is exactly the problem variant that we focus on in this paper.

Significant advances have been made also in this direction. Previous results show that $\mathcal{H}$-Square Root is polynomial-time solvable for the following graph classes $\mathcal{H}$: trees [20], proper interval graphs [15], bipartite graphs [14], block graphs [18], strongly chordal split graphs [19], ptolemaic graphs [16], 3-sun-free split graphs [16], cactus graphs [10], and graphs with girth at least g for any fixed $g \geq 6$ [9]. The result for 3-sun-free split graphs has been extended to a number of other subclasses of split graphs in [17]. Observe that if $\mathcal{H}$-Square Root is polynomial-time solvable for some class $\mathcal{H}$, then this does not automatically imply that $\mathcal{H}'$-Square Root is polynomial-time solvable for a subclass $\mathcal{H}'$ of $\mathcal{H}$.

On the negative side, $\mathcal{H}$-Square Root remains NP-complete for each of the following graph classes $\mathcal{H}$: graphs of girth at least 5 [8], graphs of girth at least 4 [9], split graphs [15], and chordal graphs [15]. All known NP-hardness constructions involve dense graphs [8, 9, 15, 23], and the square roots that occur in these constructions are dense as well. This, in combination with the listed polynomial-time cases, naturally leads to the question whether $\mathcal{H}$-Square Root is polynomial-time solvable if the class $\mathcal{H}$ is "sparse" in some sense.

Motivated by the above, in this paper we study $\mathcal{H}$-Square Root when $\mathcal{H}$ is the class of outerplanar graphs, and when $\mathcal{H}$ is the class of graphs of pathwidth at most 2. In both cases, we show that $\mathcal{H}$-Square Root can be solved in polynomial time. In particular, we prove that Outerplanar (Square) Root can be solved in time $O(n^4)$ and (Square) Root of Pathwidth ≤ 2 in time $O(n^6)$. Our approach for outerplanar graphs can in fact be directly applied to every subclass of outerplanar graphs that is closed under edge deletion and that can be expressed in monadic second-order logic, including cactus graphs, for which a polynomial-time algorithm is already known.

2 Preliminaries

We consider only finite undirected graphs without loops and multiple edges. We refer to the textbook by Diestel [7] for any undefined graph terminology.

Let G be a graph. We denote the vertex set of G by V_G and the edge set by E_G. The subgraph of G induced by a subset $U \subseteq V_G$ is denoted by $G[U]$. The graph $G - U$ is the graph obtained from G after removing the vertices of U. If $U = \{u\}$, we also write $G - u$. Similarly, we denote the graph obtained from G by deleting a set of edges S, or a single edge e, by $G - S$ and $G - e$, respectively.

The *distance* $\mathrm{dist}_G(u, v)$ between a pair of vertices u and v of G is the number of edges of a shortest path between them. The *open neighborhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$, and its *closed neighborhood* is defined as $N_G[u] = N_G(u) \cup \{u\}$. For $S \subseteq V_G$, $N_G(S) = (\bigcup_{v \in S} N_G(v)) \setminus S$. Two (adjacent) vertices u, v are said to be *true twins* if $N_G[u] = N_G[v]$. A vertex v is *simplicial* if $N_G[v]$ is a clique, that is, if there is an edge between any two vertices of $N_G[v]$. The *degree* of a vertex $u \in V_G$ is defined as $d_G(u) = |N_G(u)|$. The maximum degree of G is $\Delta(G) = \max\{d_G(v) \mid v \in V_G\}$. A vertex of degree 1 is said to be a *pendant* vertex.

A *connected component* of G is a maximal connected subgraph. A vertex u is a *cut vertex* of a graph G with at least two vertices if $G - u$ has more components than G. A connected graph without cut vertices is said to be *biconnected*. An inclusion-maximal induced biconnected subgraph of G is called a *block*.

For a positive integer k, the *k-th power* of a graph H is the graph $G = H^k$ with vertex set $V_G = V_H$ such that every pair of distinct vertices u and v of G are adjacent if and only if $\mathrm{dist}_H(u, v) \leq k$. For the particular case $k = 2$, H^2 is a *square* of H, and H is a *square root* of G if $G = H^2$.

The *contraction* of an edge uv of a graph G is the operation that deletes the vertices u and v and replaces them by a vertex w adjacent to $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$. A graph G' is a contraction of a graph G if G' can be obtained from G by a series of edge contraction. A graph G' is a *minor* of G if it can be obtained from G by vertex deletions, edge deletions and edge contractions.

A graph G is *planar* if it admits an embedding on the plane such that there are no edges crossing (except in endpoints). A planar graph G is *outerplanar* if it admits a crossing-free embedding on the plane in such a way that all its vertices are on the boundary of the same (external) face. For a considered outerplanar graph, we always assume that its embedding on the plane is given. If G is a planar biconnected graph different from K_2, then for any of its embeddings, the boundary of each face is a cycle (see, e.g., [7]). If G is a biconnected outerplanar graph distinct from K_2, then the cycle C forming the boundary of the external face is unique and we call it the *boundary cycle*. By definition, all vertices of G are laying on C, and every edge is either an edge of C or a *chord* of C, that is, its endpoints are vertices of C that are non-adjacent in C. Clearly, these chords are not intersecting in the embedding. For a vertex u, we define the *canonical ordering with respect to u* as a clockwise ordering of the vertices on C starting from u. For a subset of vertices X, the canonical *canonical ordering of X with respect to u* is the ordering induced by the canonical ordering of the vertices of C. See Figure 1 a) for some examples. In our paper we use these terms for blocks of an outerplanar graphs distinct from K_2. Outerplanar graphs can also be characterized via forbidden minors as shown by Sysło [27].

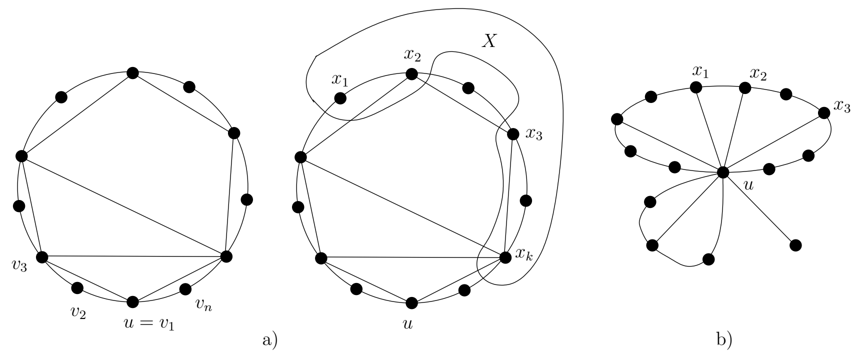

Fig. 1. a) Canonical orderings with respect to u of a biconnected outerplanar graph with vertex set $V_G = \{v_1, \ldots, v_n\}$ and a set $X = \{x_1, \ldots, x_k\}$. **b)** Example of a set $X = \{x_1, x_2, x_3\}$ that is consecutive with respect to u; notice that x_1 and x_3 are not consecutive.

Lemma 1 ([27]). *A graph G is outerplanar if and only if it does not contain K_4 and $K_{2,3}$ as minors.*

A *tree decomposition* of a graph G is a pair (T, X) where T is a tree and $X = \{X_i \mid i \in V_T\}$ is a collection of subsets (called *bags*) of V_G such that the following three conditions hold:

i) $\bigcup_{i \in V_T} X_i = V_G$,
ii) for each edge $xy \in E_G$, $x, y \in X_i$ for some $i \in V_T$, and
iii) for each $x \in V_G$ the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V_T\}, T)$ is $\max_{i \in V_T} \{|X_i| - 1\}$. The *treewidth* $\mathbf{tw}(G)$ of a graph G is the minimum width over all tree decompositions of G. If T is restricted to be a path, then we say that (X, T) is a *path decomposition* of G, The *pathwidth* $\mathbf{pw}(G)$ of G is the minimum width over all path decompositions of G. Notice that a path decomposition of G can be seen as a sequence $(X_1, \ldots, X_r)$ of bags. We always assume that the bags $(X_1, \ldots, X_r)$ are distinct and inclusion maximal, that is, there are no bags X_i and X_j such that $X_i \subset X_j$. The following fundamental results are due to Bodlaender [1], and Bodlaender and Kloks [2].

Lemma 2 ([1, 2]). *For every fixed constant c, it is possible to decide in linear time whether the treewidth or the pathwidth of a graph is at most c.*

We need the following three folklore observations about treewidth.

Observation 1 *If H is a minor (contraction) of G, then* $\mathbf{tw}(H) \leq \mathbf{tw}(G)$ *and* $\mathbf{pw}(H) \leq \mathbf{pw}(G)$.

Observation 2 *For an outerplanar graph G,* $\mathbf{tw}(G) \leq 2$.

Observation 3 *For a graph G and a positive integer k,*

$$\mathbf{tw}(G^k) \leq (\mathbf{tw}(G) + 1)\Delta(G)^{\lfloor k/2 \rfloor + 1}$$

and

$$\mathbf{pw}(G^k) \leq (\mathbf{pw}(G) + 1)\Delta(G)^{\lfloor k/2 \rfloor + 1}.$$

Let H be a square root of a graph G. We say that H is a *minimal* square root of G if $H^2 = G$, and no proper subgraph of H is a square root of G. We need the following simple observations; we provide a proof of Observation 5 in Appendix A.

Observation 4 *Let $\mathcal{H}$ be a graph class closed under edge deletion. If a graph G has a square root $H \in \mathcal{H}$, then G has a minimal square root that belongs to $\mathcal{H}$.*

Observation 5 *Let H be a minimal square root of a graph G that contains three pairwise adjacent vertices u, v, w. Then v or w has a neighbor $x \neq u$ in H such that x is not adjacent to u.*

We conclude this section by a lemma that enables us to identify some edges that are not included in any square root. The lemma is implicit in [10]; for the sake of completeness we give a full proof in Appendix A.

Lemma 3. *Let x, y be distinct neighbors of a vertex u in a graph G such that x and y are at distance at least 3 in $G - u$. Then $xu, yu \notin E_H$ for any square root H of G.*

3 Outerplanar Roots

In this section we show that it can be decided in polynomial time whether a graph has an outerplanar square root. We say that a square root H of G is an *outerplanar root* if H is outerplanar. We define the following problem:

Outerplanar Root
Input: a graph G.
Question: is there an outerplanar graph H such that $H^2 = G$?

The main result of this section is the following.

Theorem 1. Outerplanar Root *can be solved in time $O(n^4)$, where n is the number of vertices of the input graph.*

The remaining part of this section is devoted to the proof of Theorem 1. In Section 3.1 we obtain several structural results we need to construct an algorithm for Outerplanar Root. Then in Section 3.2 we construct a polynomial-time algorithm for Outerplanar Root.

3.1 Structural Lemmas

In this section we give several structural results about outerplanar square roots. Due to space restriction we omit the proofs; the details can be found in Appendix B.

Let H be an outerplanar root of a graph G, $u \in V_G$. We say that that two distinct vertices $x, y \in N_H(u)$ are *consecutive with respect to* u if x and y are in the same block F of H and there are no vertices of $N_H(u)$ between x and y in the canonical ordering of the vertices of the boundary cycle of F with respect to u. For a set of vertices $X \subseteq N_H(u)$, we say that the vertices X are *consecutive with respect to* u if the vertices of X are in the same block of H and any two vertices of X consecutive in the canonical ordering of elements of X with respect to u are consecutive with respect to u; a single-vertex set is assumed to be consecutive (see Figure 1 b for an example).

As every subgraph of an outerplanar graph is outerplanar, by Observation 4, we may restrict ourselves to minimal outerplanar roots. Let H be a minimal outerplanar root of a graph G and let $u \in V_G$. Denote by $S(G, H, u)$ a collection of all subsets X of $N_H(u)$ such that $X = N_G(x) \cap N_H(u)$ for some $x \in N_G(u) \setminus N_H(u)$. We can use $S(G, H, u)$ to find edges with both endpoints in $N_H(u)$ that are not included in a square root.

Lemma 4. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. Then for each $X \in S(G, H, u)$, X is consecutive with respect to u.*

Lemma 5. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. If for two distinct vertices $x, y \in N_H(u)$ there is no set $X \in S(G, H, u)$ such that $x, y \in X$, then $xy \notin E_H$.*

We also need the following two observations.

Lemma 6. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. If $x \in N_H(u)$ is not a pendant vertex of H, then there is $y \in N_G(u) \setminus N_H(u)$ that is adjacent to x in G.*

Lemma 7. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. Then any $X \in S(G, H, u)$ has size at most* 4.

By combining Lemmas 4 and 7 we obtain the following lemma.

Lemma 8. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. Then the following holds.*

(i) If $x, y \in N_H(u)$ do not belong to the same block of H, then for any $X \in S(G, H, u)$, $x \notin X$ or $y \notin X$.
(ii) If F is a block of H containing u and vertices $x_1, \ldots, x_k \in N_H(u)$ ordered in the canonical order with respect to u in the boundary cycle of F, then for any $X \in S(G, H, u)$, $x_i \notin X$ or $x_j \notin X$ if $i, j \in \{1, \ldots, k\}$ and $|i - j| \geq 4$.

We now state some structural results that help to decide whether an edge incident to a vertex is in an outerplanar root or not. Suppose that u and v are pendant vertices of a square root H of G and that u and v are adjacent to the same vertex of $H - \{u, v\}$. Then, in G, u and v are simplicial vertices and true twins. We use this observation in the proof of the following lemma that allows to find some pendant vertices.

Lemma 9. *Let H be a minimal outerplanar root of a graph G. If G contains at least 7 simplicial vertices that are pairwise true twins, then at least one of these vertices is a pendant vertex of H.*

We apply Lemma 3 to identify the edges incident to a vertex of sufficiently high degree in an outerplanar root using the following two lemmas.

Lemma 10. *Let G be a graph having a minimal outerplanar root H. Let also $u \in V_G$ be such that there are three distinct vertices $v_1, v_2, v_3 \in N_G(u)$ that are pairwise at distance at least 3 in $G - u$. Then for $x \in N_G(u)$, $xu \notin E_H$ if and only if there is $i \in \{1, 2, 3\}$ such that $\mathrm{dist}_{G-u}(x, v_i) \geq 3$.*

Lemma 11. *Let G be a graph having a minimal outerplanar root H such that any vertex of H has at most 7 pendant neighbors. Let also $u \in V_G$ with $d_H(u) \geq 22$. Then there are distinct $v_1, v_2, v_3 \in N_G(u)$ that are pairwise at distance at least 3 in $G - u$.*

Notice that v_1, v_2 and v_3 are in distinct components of $H - u$. We obtain that v_3 is at distance at least 3 from v_1 and v_2 in $G - u$.

The next lemma is crucial for our algorithm. To state it, we need some additional notations. Let H be a minimal outerplanar root of a graph G such that each vertex of H is adjacent to at most 7 pendant vertices. Let U be a set of vertices of H that contains all vertices of degree at least 22. For every $u \in U$ and every block F of H containing u we do the following. Consider the set $X = N_H(u) \cap V_F$ and denote the vertices of X by $x_1, \ldots, x_k$, where these vertices are numbered in the canonical order with respect to u. Then

- for $i, j \in \{1, \ldots, k\}$, delete the edge $x_i x_j$ from G if $|i - j| \geq 4$.
- for $i \in \{1, \ldots, k\}$, delete the edges $x_i y$ from G for $y \in N_H(u) \setminus V_F$.

Denote by $G'(H, U)$ the graph obtained in the end.

Lemma 12. *There is a constant c that depends neither on G nor on H such that*

$$\mathbf{tw}(G'(H, U)) \leq c.$$

3.2 The Algorithm

In this section we construct an algorithm for Outerplanar Root with running time $O(n^4)$. Let G be the input graph. Clearly, it is sufficient to solve Outerplanar Root for connected graphs. Hence, we assume that G is connected and has $n \geq 2$ vertices.

First, we preprocess G using Lemma 9 to reduce the number of pendant vertices adjacent to the same vertex in a (potential) outerplanar root of G. To do so, we exhaustively apply the following rule.

Pendants reduction. If G has a set X of simplicial true twins of size at least 8, then delete an arbitrary $u \in X$ from G.

The following claim shows that this rule is safe.

Lemma 13. *If $G' = G - u$ is obtained from G by the application of* **Pendant reduction***, then G has an outerplanar root if and only if G' has an outerplanar root.*

Proof. Suppose that H is a minimal outerplanar root of G. By Lemma 9, H has a pendant vertex $u \in X$. It is easy to verify that $H' = H - u$ is an outerplanar root of G'. Assume now that H' is a minimal outerplanar root of G'. By Lemma 9, H has a pendant vertex $w \in X \setminus \{u\}$, since the vertices of $X \setminus \{u\}$ are simplicial true twins of G' and $|X \setminus \{u\}| \geq 7$. Let v be the unique neighbor of w in H'. We construct H from H' by adding u and making it adjacent to v. It is readily seen that H is an outerplanar root of G. This completes the proof. □

For simplicity, we call the graph obtained by exhaustive application of the pendants rule G again. The following property immediately follows from the observation that any two pendant vertices of a square root H of G adjacent to the same vertex in H are true twins of G.

Lemma 14. *Every outerplanar root of G has at most 7 pendant vertices adjacent to the same vertex.*

In the next stage of our algorithm we label some edges of G *red* or *blue* in such a way that the edges labeled red are included in every minimal outerplanar root and the blue edges are not included in any minimal outerplanar root. We denote by R the set of red edges and by B the set of blue edges. We also construct a set of vertices U of G such that for every $u \in U$, the edges incident to u are labeled red or blue.

Labeling. Set $U = \emptyset$, $R = \emptyset$ and $B = \emptyset$. For each $u \in V_G$ such that there are three distinct vertices $v_1, v_2, v_3 \in N_G(u)$ that are at distance at least 3 from each other in $G - u$ do the following:

(i) set $U = U \cup \{u\}$,
(ii) set $B' = \{ux \in E_G \mid \text{there is } 1 \leq i \leq 3 \text{ s.t. } \mathrm{dist}_{G-u}(x, v_i) \geq 3\}$,
(iii) set $R' = \{ux \mid x \in N_G(u)\} \setminus B'$,
(iv) set $R = R \cup R'$ and $B = B \cup B'$,
(v) if $R \cap B \neq \emptyset$, then return a no-answer and stop.

Lemmas 10 and 11 imply the following claim.

Lemma 15. *If G has an outerplanar root, then* **Labeling** *does not stop in Step (v), and if H is a minimal outerplanar root of G, then $R \subseteq E_H$ and $B \cap E_H = \emptyset$. Moreover, every vertex $u \in V_G$ with $d_H(u) \geq 22$ is included in U.*

Next, we find the set of edges xy with $xu, yu \in R$ for some u in R that are not included in a minimal outerplanar root.

Finding irrelevant edges. Set $S = \emptyset$. For each $u \in U$ and each pair of distinct $x, y \in N_G(u)$ such that $ux, uy \in R$ do the following.

(i) If $xy \notin E_G$, then return a no-answer and stop.
(ii) If for x and y, there is no $v \in N_G(u)$ such that $vu \in B$ and $x, y \in N_G(v)$, then include xy in S.
(iii) If $R \cap S \neq \emptyset$, then return a no-answer and stop.

Combining Lemmas 15 and 5, we obtain the following claim.

Lemma 16. *If G has an outerplanar root, then* **Finding irrelevant edges** *does not stop in Steps (i) and (iii), and if H is a minimal outerplanar root of G, then $S \cap E_H = \emptyset$.*

Assume that we did not stop during the execution of **Finding irrelevant edges**. Let $G' = G - S$. We show the following.

Lemma 17. *The graph G has an outerplanar root if and only if there is a set $L \subseteq E_{G'}$ such that*

(i) $R \subseteq L$, $B \cap L = \emptyset$,
(ii) for any $xy \in E_{G'}$, $xy \in L$ or there is $z \in V_{G'}$ such that $xz, yz \in L$,
(iii) for any distinct edges $xz, yz \in L$, $xy \in E_{G'}$ or there is $u \in U$ such that $xu, yu \in R$,
(iv) the graph $H = (V_G, L)$ is outerplanar.

Proof. Let H be a minimal outerplanar root of G. By Lemma 16, $E_H \cap S = \emptyset$, i.e., $E_H \subseteq E_{G'}$. Let $L = E_H$. It is straightforward to verify that (i)–(iv) are fulfilled. Assume now that there is $L \subseteq E_{G'}$ such that (i)–(iv) hold. Then we have that $H = (V_G, L)$ is an outerplanar root of G. □

To complete the description of the algorithm, it remains to show how to check the existence of a set of edges L satisfying (i)–(iv) of Lemma 17 for given G', R and B. Notice, that if G has a minimal outerplanar root H, then G' is a subgraph of the graph $G'(H, U)$ constructed in Section 3.1 by Lemma 8. By Lemma 12, there is a constant c that depends neither on G nor on H such that $\mathbf{tw}(G'(H, U)) \leq c$. Therefore, $\mathbf{tw}(G') \leq c$ for a yes-instance. We use Lemma 2 to verify whether it holds. If we obtain that $\mathbf{tw}(G') > c$, we conclude that we have a no-instance and stop. Otherwise, we use the celebrated theorem of Courcelle [5], which states that any problem that can be expressed in monadic second-order logic can be solved in linear time on graph of bounded treewidth. It is straightforward to see that properties (i)–(iv) can be expressed in this logic. In particular, to express outerplanarity in (iv), we can use Lemma 1 and the well-known fact that the property that G contains F as minor can be expressed in monadic second-order logic if F is fixed (see, e.g., the book of Courcelle and Engelfriet [6]). It immediately implies that we can decide in linear time whether

L exists or not. Notice that we can modify these arguments such that we do not only check the existence of L but also find it. To do this, we can construct a dynamic programming algorithm for graphs of bounded treewidth that finds L.

Now we evaluate the running time of our algorithm. We can find simplicial vertices of the input n-vertex graph G in time $O(n^3)$. Therefore, **Pendant reduction** can be done in time $O(n^3)$. For every vertex u, we can compute the distances between the vertices of $N_G(u)$ in $G - u$ in time $O(n^3)$. This implies that **Labeling** can be done in time $O(n^4)$. **Finding irrelevant edges** also can be done in time $O(n^4)$ by checking $O(n^2)$ pairs of vertices x and y. Then G' can be constructed in linear time. Finally, checking whether $\mathbf{tw}(G') \leq c$ and deciding whether there is a set of edges L satisfying the required properties can be done in linear time by Lemma 2 and Courcelle's theorem [5] respectively.

Notice that we can use the same arguments to decide whether a graph G has a square root H that belongs to some subclass $\mathcal{H}$ of the class of outerplanar graphs. To be able to apply our structural lemmas, we only need the property that $\mathcal{H}$ should be closed under edge deletions. Observe also that if the properties defining $\mathcal{H}$ could be expressed in monadic second-order logic, then we can apply Courcelle's theorem [5]. It gives us the following corollary.

Corollary 1. *For every subclass $\mathcal{C}$ of the class of outerplanar graphs that is closed under edge deletions and can be expressed in monadic second-order logic, it can be decided in time $O(n^4)$ whether an n-vertex graph G has a square root $H \in \mathcal{C}$.*

4 Roots of Pathwidth at most two

Our main approach for solving Outerplanar root is general in the sense that it can be adapted to find also square roots belonging some other graph classes. In this section we show that there is an algorithm to decide in polynomial time whether a graph has a square root of pathwidth at most 2. Notice that graphs of pathwidth 1 are caterpillars, and that it can be decided in polynomial time whether a graph G has a square root that is a caterpillar by an easy adaptation of algorithms for finding square roots that are trees [20, 26].

We define the following problem:

Root of Pathwidth ≤ 2
Input: a graph G.
Question: is there a graph H such that $\mathbf{pw}(H) \leq 2$ and $H^2 = G$?

The main difference between our algorithm for Root of Pathwidth $\leq$ 2 and our algorithm for Outerplanar Root lies in the way properties of the involved graph classes are used. To show the structural results needed for this algorithm, we use the property that a potential square root has a path decomposition of width at most 2, instead of the existence of an outerplanar embedding used in the previous section.

We briefly sketch the proof of the following theorem; the details are given fully in Appendix C.

Theorem 2. Root of Pathwidth ≤ 2 *can be solved in time* $O(n^6)$*, where* n *is the number of vertices of the input graph.*

Proof. (*Sketch.*) Let G be the input graph. It is sufficient to solve Root of Pathwidth ≤ 2 for connected graphs. Hence, we assume that G is connected and has $n \geq 2$ vertices. Notice that the class of graphs of pathwidth at most 2 is closed under edge deletions. Therefore, by Observation 4, we can consider only minimal square roots.

First, we preprocess G to reduce the number of true twins that a given vertex of V_G might have. To do so, we show that there is a constant c_1 such that if W is a set of true twins of G of size at least c_1, then for any minimal square root H of G with $\mathbf{pw}(H) \leq 2$, either W has a vertex that is pendant in H or W has distinct nonadjacent vertices x, y, z with $d_H(x) = d_H(y) = d_H(y) = 2$. It allows us to show that if G has a set of true twins W of size at least $c_1 + 1$, then by the deletion of an arbitrary $u \in W$ from G we obtain an equivalent instance of Root of Pathwidth ≤ 2. From now we can assume that any set of true twins of G has size at most c_1. We need this to obtain forthcoming structural properties.

In the next stage of our algorithm we label some edges of G *red* or *blue* in such a way that the edges labeled red are included in every minimal square root of pathwidth at most 2 and the blue edges are not included in any minimal square root of pathwidth at most 2. We denote by R the set of red edges and by B the set of blue edges. We also construct a set of vertices U of G such that for every $u \in U$, the edges incident to u are labeled red or blue.

The labeling is based on the following structural property. If there is $u \in V_G$ such that there are five distinct vertices $v_1, \ldots, v_5$ in $N_G(u)$ that are at distance at least 3 from each other in $G - u$, then for any square root H with $\mathbf{pw}(H) \leq 2$, $ux \notin E_H$ for $x \in N_G(u)$ if and only if there is $i \in \{1, \ldots, 5\}$ such that $\text{dist}_{G-u}(x, v_i) \geq 3$. Respectively, if we find $u \in V_G$ with the aforementioned property that there are five distinct vertices $v_1, \ldots, v_5$ in $N_G(u)$ that are at distance at least 3 from each other in $G - u$, then we include u in U and for $x \in N_G(u)$, we label ux blue if there is $i \in \{1, \ldots, 5\}$ such that $\text{dist}_{G-u}(x, v_i) \geq 3$ and we label ux red otherwise. If we get inconsistent labelings, that is, some edge should be labeled red and blue, then we stop and report that there is no square root of pathwidth at most 2.

We show that there is a constant c_2 such that for a square root H of G with $\mathbf{pw}(H) \leq 2$, if $d_H(u) \geq c_2$, then $u \in U$ and, therefore, all the edges of G incident to u are labeled red or blue. It means that if u is a vertex of H of sufficiently high degree, then for each edge of G incident to u, we distinguish whether this edge is in a square root or not.

Next, we find the set of edges xy with $xu, yu \in R$ which for some u in U are not included in a minimal square root of pathwidth at most 2. To do it, we use Observation 5 to show that if there is no $z \in N_G(u)$ with $uz \in B$ such that $xz, yz \in E_G$, then $xy \notin E_H$ for a minimal square root H of pathwidth at most 2. Respectively, we label such edges xy blue. Again, if we get inconsistent labelings, then we stop and report that there is no square root of pathwidth at most 2.

Denote by S the set of edges labeled blue in this stage of the algorithm and let $G' = G - S$. We prove that if G has a square root of pathwidth at most 2, then there is a constant c_4 such that $\mathbf{pw}(G') \leq c_4$. The proof is based on the property that every vertex of degree at least c_2 in a (potential) square root of pathwidth at most 2 is included in U. We can verify whether $\mathbf{pw}(G') \leq c_4$ in linear time using Lemma 2. If $\mathbf{pw}(G') > c_4$, then we stop and report that there is no square root of pathwidth at most 2. Otherwise, we obtain a path decomposition of G' of width at most c_4.

Then, similarly to the proof of Theorem 1, we obtain that G has a square root of pathwidth at most 2 if and only if there is a set $L \subseteq E_{G'}$ such that

(i) $R \subseteq L$, $B \cap L = \emptyset$,
(ii) for any $xy \in E_{G'}$, $xy \in L$ or there is $z \in V_{G'}$ such that $xz, yz \in L$,
(iii) for any distinct edges $xz, yz \in L$, $xy \in E_{G'}$ or $xy \in S$,
(iv) the graph $H = (V_G, L)$ is such that $pw(H) \leq 2$.

Notice that the properties (i)–(iv) can be expressed in monadic second-oder logic. In particular, (iv) can be expressed using the property that the class of graphs of pathwidth at most 2 is defined by the set of forbidden minors given by Kinnersley and Langston in [13]. Then we use Courcelle's theorem [5] to decide in linear time whether L exists or not.

To evaluate the running time, observe that to construct U, we consider each vertex $u \in V_G$ and check whether there are 5 distinct vertices in $N_G(u)$ that are at distance at least 3 from each other in $G - u$. This can be done in time $O(n^6)$ and implies that the total running time is also $O(n^6)$. ☐

5 Conclusions

We proved that $\mathcal{H}$-Square Root is polynomial-time solvable when $\mathcal{H}$ is the class of outerplanar graphs or the class of graphs of pathwidth at most 2. The same result holds if $\mathcal{H}$ is any subclass of the class of outerplanar graphs that is closed under edge deletion and that can be expressed in monadic second-order logic (for instance, if $\mathcal{H}$ is the class of cactus graphs). We conclude by posing two questions:

– Is $\mathcal{H}$-Square Root polynomial-time solvable for every class $\mathcal{H}$ of graphs of bounded pathwidth?
– Is $\mathcal{H}$-Square Root polynomial-time solvable if $\mathcal{H}$ is the class of planar graphs?

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A Proofs omitted in Section 2

Observation 5. *Let H be a minimal square root of a graph G that contains three pairwise adjacent vertices u, v, w. Then v or w has a neighbor $x \neq u$ in H such that x is not adjacent to u.*

Proof. Since H is a minimal square root of G, $H - vw$ is not a square root of G. Because $uv, uw \in E_G$, we conclude that there is $xy \in E_G \setminus E_H$ such that H has a unique (x, y)-path P of length two and wv is an edge of this path. Then $P = xvw$ or $P = xwv$ for some $x \neq u$ that is not adjacent to u. $\square$

Lemma 3. *Let x, y be distinct neighbors of a vertex u in a graph G such that x and y are at distance at least 3 in $G - u$. Then $xu, yu \notin E_H$ for any square root H of G.*

Proof. Suppose that H is a square root of G. To obtain a contradiction, assume that $ux \in E_H$. If $uy \in E_H$, then $xy \in E_G$ contradicting the condition that $\text{dist}_{G-u}(x, y) \geq 3$. Hence, $uy \notin E_H$. Because $uy \in E_G$, there are $uz, zy \in E_H$. If $y = x$, then $xy \in E_G$; a contradiction. If $y \neq x$, then $xz \in E_G$ and xzy is a path in $G - u$ of length 2 and again we obtain a contradiction with the condition $\text{dist}_{G-u}(x, y) \geq 3$. $\square$

B Proofs omitted in Subsection 3.1

Lemma 4. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. Then for each $X \in S(G, H, u)$, X is consecutive with respect to u.*

Proof. Let $X \in S(G, H, u)$ and consider $x \in N_G(u) \setminus N_H(u)$ such that $X = N_G(x) \cap N_H(u)$. Notice that $X \neq \emptyset$, because x should be adjacent to a vertex of $N_H(u)$ since $ux \in E_G$. If $|X| = 1$, then the claims holds by definition. Assume that $|X| \geq 2$. Notice that the vertices of X are in the same block of H, because if $y, z \in N_H(u)$ are not in the same block F of H, then any vertex adjacent to y and z in G is in $N_H[u]$. Assume that $X = \{x_1, \ldots, x_k\}$ and the vertices are numbered in the canonical order with respect to u. Suppose that X is not consecutive with respect to u. Then there is y laying on C between x_{i-1} and x_i for some $i \in \{2, \ldots, k\}$ such that $uy \in E_H$ and $yx \notin E_G$. Since $x_{i-1}x, x_ix \in E_G$, H has (x_{i-1}, x) and (x_i, x)-paths P_1 and P_2 of length at most 2. Notice that these paths do not contain u, because $xu \notin E_H$, and they do not contain y, because $yx \notin E_G$. It implies that y is inside of the inner face of C in the outerplanar embedding of G; a contradiction. $\square$

Lemma 5. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. If for two distinct vertices $x, y \in N_H(u)$ there is no set $X \in S(G, H, u)$ such that $x, y \in X$, then $xy \notin E_H$.*

Proof. Suppose that for two distinct $x, y \in N_H(u)$, $xy \in E_H$. By Observation 5, there is a vertex z such that z is adjacent to x or y in H, but z is not adjacent to u in H. We obtain that x and y are adjacent to z in G and, therefore, $x, y \in N_G(z) \cap N_H(u)$. In other words, $x, y \in X = N_G(z) \cap N_H(u) \in S(G, H, u)$.

Lemma 6 *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. If $x \in N_H(u)$ is not a pendant vertex of H, then there is $y \in N_G(u) \setminus N_H(u)$ that is adjacent to x in G.*

Proof. Since x is not a pendant vertex, it has neighbors in H distinct from u. If there is $y \in N_H(x)$ such that $y \notin N_H[u]$, the claim hold. Assume that for every $y \in N_H(x)$ distinct form u, it holds that $y \in N_H(u)$. Consider such a neighbor y. By Observation 5, y has a neighbor z in H such that $z \notin N_H(u)$. Since $xy, yz \in E_H$, $xz \in E_G$. We have that $z \in N_G(u) \setminus N_H(u)$. □

Lemma 7. *Let H be a minimal outerplanar root of a graph G, and let $u \in V_G$. Then any $X \in S(G, H, u)$ has size at most 4.*

Proof. Suppose that there is $X = \{x_1, \dots, x_k\} \in S(G, H, u)$ of size at least 5. Then $X = N_G(x) \cap N_H(u)$ for $x \in N_G(u) \setminus N_H(u)$. By Lemma 4, X is consecutive with respect to u, and we assume that $x_1, \dots, x_k$ is the canonical order of the vertices of X along the boundary cycle C of the block of H with respect to u. Suppose that x lays on C. If it lays before x_3 in the canonical ordering of the vertices of C with respect to u, then x is not adjacent to x_k in G by outerplanarity; a contradiction. Similarly, if x lays after x_3, x is not adjacent to x_1 in G and we also have a contradiction. Suppose that x does not belong to C. Then x is at distance at most 2 in H from x_1 and x_k. It follows that there is a (x_1, x_k)-path P in H of length at most 4 avoiding u and at least one vertex of X but it contradicts outerplanarity. □

Lemma 9. *Let H be a minimal outerplanar root of a graph G. If G contains at least 7 simplicial vertices that are pairwise true twins, then at least one of these vertices is a pendant vertex of H.*

Proof. Let H be a minimal outerplanar root of a graph G that contains a set X of 7 simplicial vertices that are pairwise true twins.

Suppose first that X contains two vertices x and y that do not belong to the same block of H. We claim that x is a pendant vertex of H. Since x and y are adjacent in G, we obtain that $xu, yu \in E_H$ for a cut vertex u that belongs to two blocks F_x and F_y of H containing x and y respectively. To obtain a contradiction, assume that x has a neighbor $z \neq u$ in H. Clearly, z is not in F_y. It implies that $zu \in E_H$, because x and y are true twins of G and, therefore, $z \in N_G(y)$. Since H is a minimal root, $H - xz$ is not a square root of G. Because $ux, uz \in E_H$, it follows that there is a vertex $z' \neq u$ such that $xz', zz' \in E_G$ but either i) $xz' \in E_H$, $zz' \notin E_H$ and zxz' is the unique (z, z')-path of length at most 2 in H or, symmetrically, ii) $xz' \notin E_H$, $zz' \in E_H$ and xzz' is the unique (x, z')-path of length at most 2 in H. Because x and y are true twins of G and $z' \neq u$, we obtain that $yz' \in E_G$ and, therefore, $uz' \in E_H$. Hence, xuz' and zuz' are paths of H contradicting i) or ii) respectively. We conclude that x is a pendant vertex of H.

Suppose now that the vertices of X belong to the same block F of H. Let C be the boundary cycle of F and denote by $x_1, \dots, x_7$ the vertices of X numbered according to the canonical order with respect to an arbitrary vertex of C. Because the vertices of X are pairwise adjacent in G, F has a chord uv such that X has vertices in the two components of $F - \{u, v\}$. Among all such chords we choose uv and a component F' of $F - \{u, v\}$ in such a way that F' contains the minimum number of vertices of X. Assume without loss of generality that $x_1 \in V_{F'}$ and let $x_i, \dots, x_j$ for $1 < i \leq j \leq 7$ be the vertices of X in the other component F'' of $F - \{u, v\}$. Notice that F'' contains at least 3 vertices of X by the choice of uv. Assume also that v is after x_1 in the canonical ordering of the vertices of C with respect to u. Because the vertices of X are adjacent in G, they are at distance at most 2 in H. It implies that for any $x \in X \cap V_{F'}$ and any $y \in X \cap V_{F''}$, $xu, uy \in E_H$ or $xv, vy \in E_H$ by the outerplanarity of F.

Suppose that F' contains at least two vertices of X. By symmetry, we can assume that $x_1, x_2 \in V_{F'}$. It follows that $x_1u \in E_H$, $x_2v \in E_H$ but $x_1v, x_2u \notin E_H$ by the choice of uv. We obtain that $x_iu \in E_H$ and $x_jv \in E_H$, but these are intersecting chords of C; a contradiction. We conclude that x_1 is the unique vertex of X in F'. Notice that it implies that $i \leq 3$ and $j \geq 6$ but it can happen that $u = x_7$ or $v = x_2$. We also have that $x_1u \in E_H$ or $x_1v \in H$. Using symmetry we assume that $x_1u \in E_H$.

If $x_1v \notin E_H$, then $x_ku \in E_H$ for $k \in \{i, \dots, j\}$. Hence, u is adjacent to $x_1, \dots, x_6$ in this case.

Suppose that $x_1v \in E_H$. We show that either $x_ku \in E_H$ for $k \in \{i, \dots, j\}$ or $x_kv \in E_H$ for $k \in \{i, \dots, j\}$. Assume that it is not the case and there is $s \in \{i, \dots, j\}$ such that $x_su \notin E_H$ and $t \in \{i, \dots, j\}$ such that $x_tv \notin E_H$. Since $x_su \notin E_H$, $x_sv \in E_H$. Because we cannot have intersecting chords in C, $x_iu \notin E_H$. Similarly, we obtain that $x_jv \notin E_H$. Because for every $k \in \{i+1, j-1\}$, $x_ku \in E_H$ or $x_kv \in E_H$ and $j - i \geq 3$, we have that the distance between x_i and x_j in H is at least 3 by the outerplanarity of H, a contradiction. Therefore, the claim holds. By symmetry, we can assume that $x_ku \in E_H$ for $k \in \{i, \dots, j\}$. In particular, u is adjacent to $x_1, \dots, x_6$

We now show that x_3 is a pendant vertex of H. To obtain a contradiction, assume that x_3 has a neighbor y in H distinct from u. Because x_3 and x_6 are true twins of G, $yx_6 \in E_G$ and, therefore, $yu \in E_H$. Because H is a minimal outerplanar root, $H - x_3y$ is not a square root of G. Because $ux_3, uy \in E_H$, it follows that there is a vertex $z \neq u$ such that $x_3z, yz \in E_G$ but either i) $x_3z \in E_H$, $yz \notin E_H$ and yx_3z is the unique (y, z)-path in H or, symmetrically, ii) $x_3z \notin E_H$, $yz \in E_H$ and x_3yz is the unique (x_3, z)-path in H. If $x_3z \in E_H$, then by the same arguments as for y, we have that $zu \in E_H$ and we get another path yuz. Let $yz \in E_H$. If $z \in \{x_1, \dots, x_5\}$, then $zu \in E_H$ and we have the path x_3uz. Hence, $z \notin \{x_1, \dots, x_5\}$. Since z is adjacent to x_3 in G, z is a neighbor of

x_6 in G. The only possibility is that $zx_4, zx_5 \in E_H$, i.e., z lays in C between x_4 and x_5. Since z is adjacent to x_1, we obtain that $uz \in E_H$. Again, we have the path x_3uz; a contradiction. We conclude that x_3 is a pendant vertex of H. $\square$

Lemma 10. *Let G be a graph having a minimal outerplanar root H. Let also $u \in V_G$ be such that there are three distinct vertices $v_1, v_2, v_3 \in N_G(u)$ that are pairwise at distance at least 3 in $G - u$. Then for $x \in N_G(u)$, $xu \notin E_H$ if and only if there is $i \in \{1,2,3\}$ such that* $\mathrm{dist}_{G-u}(x, v_i) \geq 3$.

Proof. Let H be a minimal outerplanar root of G. If there is $i \in \{1,2,3\}$ such that $\mathrm{dist}_{G-u}(x, v_i) \geq 3$, then $xu \notin E_H$ by Lemma 3. Hence, to prove the lemma, we have to show that if $xu \notin E_H$, then there is $i \in \{1,2,3\}$ such that $\mathrm{dist}_{G-u}(x, v_i) \geq 3$. Observe that this property trivially holds if $x = v_i$ for some $i \in \{1,2,3\}$. Assume that $x \neq v_1, v_2, v_3$. Notice that $uv_i \notin E_H$ by Lemma 3 for $i \in \{1,2,3\}$.

Let $x \in N_G(u)$ such that $xu \notin E_H$.

Suppose that there is $i \in \{1,2,3\}$ such that x and v_i are in distinct components of $H - u$. Then any (x, v_i)-path in H goes through u and has length at least 4, because $ux, uv_i \notin E_H$. It follows that $\mathrm{dist}_{G-u}(x, v_i) \geq 3$.

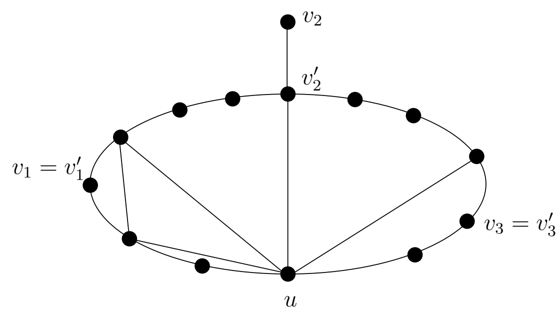

Fig. 2. The case when v_1, v_2, v_3 are in the same component of $H - u$.

Suppose now that v_1, v_2, v_3 and x are in the same component of $H - u$. Since $xu \in E_G$, there is $y \in N_H(u)$ such that $xy \in E_H$. Let F be a block of H containing u and y and denote by C its boundary cycle. Because v_1, v_2, v_3 and x are in the same component of $H - u$, each v_i is either i) a vertex of F and we let $v_i' = v_i$ in this case or ii) $v_i \notin V_F$ and there is a unique $v_i' \in V_F$ such that $v_iv_i' \in E_H$ and $v_i'u \in E_H$ (see Figure 2). Assume that v_1', v_2', v_3' and in the canonical order with respect to u in C.

Let $y \neq v_2'$. If y lays before v_2' on C in the canonical ordering with respect to u, then $\mathrm{dist}_{G-u}(x, v_3) \geq \mathrm{dist}_{G-u}(v_2, v_3) \geq 3$. Similarly, if y lays after v_2' on C in the canonical ordering with respect to u, then $\mathrm{dist}_{G-u}(x, v_1) \geq \mathrm{dist}_{G-u}(v_1, v_2) \geq 3$.

Suppose that $y = v_2'$. Notice that it means that $v_2 \neq v_2'$. If $x \notin V_F$, then y is the unique neighbor of x in H that is in F. We have that $\mathrm{dist}_{G-u}(v_1, y) =$

$\text{dist}_{G-u}(v_1, v_2) \geq 3$. If $x \in V_F$, then x is either before or after v_2' in the canonical ordering with respect to u. We have that $\text{dist}_{G-u}(x, v_3) \geq \text{dist}_{G-u}(v_2, v_3) \geq 3$ in the first case and $\text{dist}_{G-u}(x, v_1) \geq \text{dist}_{G-u}(v_1, v_2) \geq 3$ in the second. □

Lemma 11. *Let G be a graph having a minimal outerplanar root H such that any vertex of H has at most 7 pendant neighbors. Let also $u \in V_G$ with $d_H(u) \geq 22$. Then there are distinct $v_1, v_2, v_3 \in N_G(u)$ that are pairwise at distance at least 3 in $G - u$.*

Proof. Let H be a minimal outerplanar root of G such that any vertex of H has at most 7 pendant neighbors and let $u \in V_G$ be a vertex with $d_H(u) \geq 22$. Denote by P the set of pendant neighbors of u. Let $X = N_H(u) \setminus P$. Notice that $|X| \geq 15$, because $|P| \leq 7$.

Suppose that the vertices of X are in at least 3 distinct components of $H - u$. Then there are distinct blocks F_1, F_2 and F_3 of H containing u and at least one vertex of X each. By Lemma 6, there are $v_1, v_2, v_3 \in N_G(u) \setminus N_H(u)$ such that v_i is adjacent to a vertex of F_i in G for $i \in \{1, 2, 3\}$. Notice that v_1, v_2 and v_3 are in the distinct components of $H - u$. We obtain that they are pairwise at distance at least 3 in $G - u$.

Suppose now that the vertices of X are in exactly two distinct components of $H - u$. Then there are two blocks F_1 and F_2 of H containing u and the vertices of X. Since $|X| \geq 15$, we can assume that F_1 contains at least 8 vertices of X. Denote them by $x_1, \ldots, x_k$ in their order in the canonical order in the boundary cycle of F_1 with respect to u. By Lemma 6, there are $v_1, v_2 \in N_G(u) \setminus N_H(u)$ such that v_1 is adjacent to x_1 in G and v_2 is adjacent to x_k. By Lemmas 4 and 7, v_1 is not adjacent in G to $x_5, \ldots, x_k$ and v_k is not adjacent to $x_1, \ldots, x_{k-4}$. Notice that each of v_1 and v_2 is either laying on the boundary cycle of F_1 or is in another block of H containing x_1 or x_2 and x_{k-1} or x_k respectively. Then $\text{dist}_{G-u}(v_1, v_2) \geq 3$. By Lemma 6, there is $v_3 \in N_G(u) \setminus N_H(u)$ such that v_3 is adjacent to a vertex of F_2 in G. Notice that v_1, v_2 and v_3 are in the distinct components of $H - u$. We obtain that v_3 is at distance at least 3 from v_1 and v_2 in $G - u$.

Finally, assume that the vertices of X are in the same block F of H. Denote them by $x_1, \ldots, x_k$ in their order in the canonical order in the boundary cycle of F with respect to u. By Lemma 6, there are $v_1, v_2, v_3 \in N_G(u) \setminus N_H(u)$ such that v_1 is adjacent to x_1 in G, v_2 is adjacent to x_8 and v_2 is adjacent to x_k. By Lemmas 4 and 7, v_1 is not adjacent in G to $x_5, \ldots, x_k$, v_2 is not adjacent to $x_1, \ldots, x_4$ and $x_{k-3}, \ldots, x_k$, and v_k is not adjacent to $x_1, \ldots, x_{k-4}$. Notice that each of v_1, v_2 and v_2 is either laying on the boundary cycle of F or is in another block of H containing x_1 or x_2, x_7 or x_8 or x_9 and x_{k-1} or x_k respectively. Then $\text{dist}_{G-u}(v_i, v_j) \geq 3$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. □

Lemma 12. *There is a constant c that depends neither on G nor on H such that*

$$\mathbf{tw}(G'(H, U)) \leq c.$$

Proof. Let H be a minimal outerplanar root of a graph G such that each vertex of H is adjacent to at most 7 pendant vertices. Let U be a set of vertices of H containing the vertices of degree at least 22. For each vertex $u \in U$ we do the following.

– Let $F_1, \dots, F_r$ be the blocks of H containing u.
– For each $i \in \{1, \dots, r\}$, denote by $x_1^i, \dots, x_{k_i}^i$ the neighbors of u in F_i numbered according to the canonical ordering with respect to u for the boundary cycle of F_i. Assume that $x_0, \dots, x_k$ is the ordering of $N_H(u)$ obtained by the consecutive concatenation of the sequences $x_1^i, \dots, x_{k_i}^i$ for $i = 1, \dots, r$.
– Modify H as follows: delete u, replace it by a path $u_1 \dots u_k$ and make each u_i adjacent to x_{i-1} and x_i for $i \in \{1, \dots, k\}$ (see Figure 3).

Denote by $\hat{H}$ the graph obtained by the procedure. Notice that the procedure modifies degrees of vertices of H but we do not recompute U.

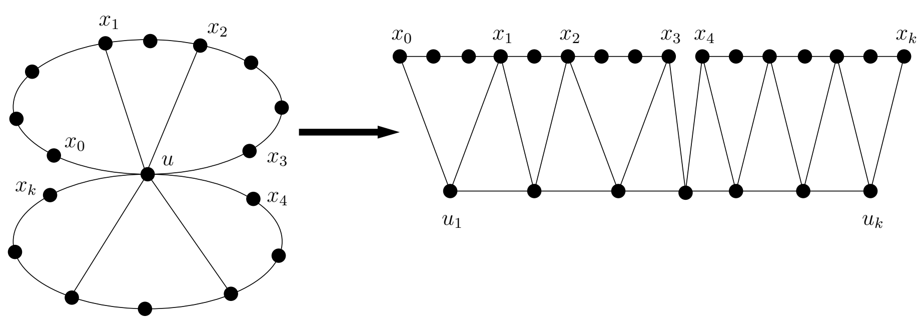

Fig. 3. Modification of H.

Notice that H is a contraction of $\hat{H}$ as H can be obtained from $\hat{H}$ by contracting the paths $u_1 \dots u_k$ constructed for $u \in U$. It is also straightforward to see that $\hat{H}$ is an outerplanar graph, because each step maintains outerplanarity (see Figure 3). By Observation 2, $\mathbf{tw}(\hat{H}) \le 2$.

We claim that $\Delta(\hat{H}) \le 42$. Let $v \in V_{\hat{H}}$. Suppose first that $v \in V_H \setminus U$. We have that $d_H(v) \le 21$ and, in particular, v has at most 21 neighbors in U in the graph H. In the construction of $\hat{H}$ each neighbor of this type is replaced by two neighbors and all other neighbors remain the same. Therefore, $d_{\hat{H}}(v) \le 42$. Suppose now that v is a vertex of one of the paths $u_1 \dots u_k$ constructed for $u \in U$. Notice, that when u is replaced by $u_1 \dots u_k$, then the degree of each vertex u_i is at most 4, and at most two neighbors of u_i could be modified in the subsequent construction steps. It implies that $d_{\hat{H}}(v) \le 6$.

We claim that $G'(H, U)$ is a minor of $\hat{H}^4$.

Let $\hat{G}$ be the graph obtained from $\hat{H}^4$ by the contraction of the paths $u_1 \dots u_k$ constructed for $u \in U$. We use the name u for the vertex obtained by contracting

$u_1 \dots u_k$ constructed for $u \in U$. Thus, we can assume that $V_{\hat{G}} = V_G$. We show that $G'(H,U)$ is a subgraph of $\hat{G}$.

We already observed that H can be obtained from $\hat{H}$ by contracting paths $u_1 \dots u_k$ constructed for $u \in U$. Hence, each edge of H is an edge of $\hat{G}$. Let xy be an edge of $G'(H,U)$ that is not an edge of H. Then, there is $u \in V_G$ such that $xu, yu \in H$. Denote by X' and Y' respectively the sets of vertices of $\hat{H}$ that are contracted to x and y respectively. If $u \notin U$, then by the construction of $\hat{H}$, there are $x' \in X'$ and $y' \in Y'$ such that $x'u, y'u \in E_{\hat{H}}$. Hence, $x'y' \in \hat{H}^4$ and $xy \in E_{\hat{G}}$. Suppose that $u \in U$. By the definition of $G'(H,U)$, the vertices x and y are in the same block F of H. Denote by $z_1, \dots, z_k$ the vertices of $N_H(u)$ in F in the canonical order with respect to u along the boundary cycle of F. We have that $x = z_i$ and $y = z_j$ for some $i, j \in \{1, \dots, k\}$. By the definition of $G'(H,U)$, $|i-j| \leq 4$. By the construction of $\hat{H}$, there are $x' \in X'$ and $y' \in Y'$ that are joined by the path $x'z_{i+1} \dots z_j y'$ in $\hat{H}$. Since this path has length at most 4, $x'y' \in \hat{H}^4$. Therefore, $xy \in \hat{G}$.

Since $G'(H,U)$ is a subgraph of $\hat{G}$ and $\hat{G}$ is a contraction of $\hat{H}^4$, we conclude that $G'(H,U)$ is a minor of $\hat{H}^4$.

Since, $G'(H,U)$ is a minor of $\hat{H}^4$, $\mathbf{tw}(G'(H,U)) \leq \mathbf{tw}(\hat{H}^4)$ by Observation 1. Because $\hat{H}$ is outerplanar, $\mathbf{tw}(\hat{H}) \leq 2$ by Observation 2, and because $\Delta(\hat{H}) \leq 42$, $\mathbf{tw}(\hat{H}^4) \leq (\mathbf{tw}(\hat{H}) + 1) \cdot 42^3$ by Observation 3. We obtain that $\mathbf{tw}(G'(H,U)) \leq 3 \cdot 42^3 = c$. $\square$

C Proof of Theorem 2

Theorem 2. Root of Pathwidth ≤ 2 *can be solved in time $O(n^6)$, where n is the number of vertices of the input graph.*

The remaining part of this section contains the proof of Theorem 2. In Section C.1, we present the structural results necessary to prove correctness of the algorithm for Root of Pathwidth ≤ 2. The algorithm itself is presented in Section C.2.

C.1 Structural Lemmas

In this section, we present structural properties of graphs with square roots of pathwidth at most 2. Recall that we assume that for any considered path decomposition, it holds that $A \nsubseteq B$ for every two distinct bags A and B of the decomposition. As the pathwidth of any subgraph of a graph of pathwidth at most 2 is upper bounded by 2, by Observation 4, we can consider only minimal roots of pathwidth at most 2.

Lemma 18. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. Suppose there exists distinct $u, v, x_1, \dots, x_k$ such that the bags $\{x_1, u, v\}$, $\{x_2, u, v\}$,..., $\{x_k, u, v\}$ appear in this order in the path decomposition of H. Then for each i, $2 \leq i \leq k-1$, $N_H(x_i) \subseteq \{u, v\}$.*

Proof. Suppose x_i has a neighbor w in H such that $w \neq u$ and $w \neq v$. There exists a bag B in the path decomposition of H that contains x_i and w. As B contains x_i, then B is between the bags $\{x_1, u, v\}$ and $\{x_k, u, v\}$ in the path decomposition and hence must contain u and v. Then $|B| \geq 4$, a contradiction with $\mathbf{pw}(H) \leq 2$. □

For two positive integers p and q, we denote by $R(p, q)$ the *Ramsey number*, that is, the smallest n such that every graph on n vertices has either a clique of size p or an independent set of size q. Those numbers are all finite by Ramsey's theorem (see, e.g., [7]).

Lemma 19. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. Then there is a constant c_1 such that for every set W of true twins in G, if $|W| \geq c_1$, then one of the following holds:*

(i) W has a vertex that is a pendant in H.
(ii) W has distinct pairwise nonadjacent vertices x, y, z of degree 2 in H such that $N_H(x) = N_H(y) = N_H(z)$.

Proof. Let $c_1 = R(4, 16)$.

We start by constructing an auxiliary graph F. Let $V_F = W$ and two vertices of F are adjacent if there exists a bag in the path decomposition of H that contains both of them. We want to show that F has an independent set of size 16. To do so, we prove that F does not contain a K_4. Suppose for the sake of contradiction that F has a K_4 and let $\{x_1, x_2, x_3, x_4\}$ be its vertex set. Let P_i be the path formed by the bags containing vertex x_i in the path decomposition of H. By the construction of F, these paths are pairwise intersecting. By the Helly property, there exists a bag containing all four vertices, a contradiction with $\mathbf{pw}(H) \leq 2$. As $|V_F| = |W| \geq R(4, 16)$ and F has no K_4, then F has an independent set of size 16.

Let $W' = \{x_1, x_2, \ldots, x_{16}\}$ be an independent set of F. By the construction of F, there are no two vertices of W' that are contained in the same bag of the path decomposition of H. Let $B_1, B_2, \ldots, B_{16}$ be bags containing the vertices of W' such that $x_i \in B_i$. As the vertices of W' are true twins in G and they are not adjacent in H, there must exist a path of length two between any two of them. Let $x_1 u x_{16}$ be such path between x_1 and x_{16}. We may consider that $u \in B_1$, otherwise we can pick bag B_1 as the one containing x_1 and u. Analogously, we may assume that $u \in B_{16}$. Since we have a path decomposition, we conclude that $u \in B_i$, $2 \leq i \leq 15$.

Suppose there are three distinct vertices x_i, x_j and x_k in W' that are not adjacent to u, with $2 \leq i < j < k \leq 15$. Let v be the vertex in a path of length two between x_i and x_k. We may assume that $B_i = \{x_i, u, v\}$ and $B_k = \{x_k, u, v\}$. Then $v \in B_j$. By Lemma 18 and by the fact that $x_j u \notin E_H$, we conclude that $d_H(x_j) = 1$ and hence (i) holds.

Suppose that at most two vertices of $W' \setminus \{x_1, x_{16}\}$ are not adjacent to u in H. Let $W'' = \{x'_1, \ldots, x'_p\}$, $p \geq 12$, be the set containing the vertices that are adjacent to u in $W' \setminus \{x_1, x_{16}\}$. If some vertex of W'' has degree one in H, then condition (i) holds. Assume this is not the case. Then all the vertices of W'' have degree at least two in H. Let $\mathcal{B}' = \{B'_1, \ldots, B'_p\}$ be bags such that $x'_i \in B'_i$. Let v be a neighbor of x'_1, $v \neq u$. We may assume $B'_1 = \{x'_1, u, v\}$. If v appears in at least five bags of $\mathcal{B}'$, then it appears in B'_1, B'_2, B'_3, B'_4 and B'_5. As x'_2, x'_3 and x'_4 do not have degree one by hypothesis, then by Lemma 18, they have degree two and hence condition (ii) holds. As x'_1 and x_{16} are true twins in G, then v must also be a neighbor of x_{16} in G. If v and x_{16} are not adjacent, let vwx_{16} be a path of length two between them. If $w \neq u$, the bag containing $\{v, w, u\}$ has to be between $\{x'_1, u, v\}$ and B_{16}. Since $p \geq 12$, either v or w appear in at least five bags of $\mathcal{B}'$. If $w = u$, then v, u and x'_1 form a triangle in H. By Observation 5, there exists a vertex z that is adjacent to either v or x'_1 that is not adjacent to u. The case when z is a neighbor of x'_1 has already been covered. Suppose $zv \in E_H$ and $zu \notin E_H$. As $d_H(z, x'_1) \leq 2$, z is adjacent to x'_1 in G, meaning that z must also be adjacent to x_{16} in G. If $zx_{16} \in E_H$, then the bag $\{v, z, u\}$ appears between bags B'_1 and the bag containing x_{16} and z. Then either v or z appear in at least five bags of $\mathcal{B}'$. If $zx_{16} \notin E_H$, let zyx_{16} be a path of length two in H. Since $p \geq 12$ and there must exist a bag containing x_{16} and y, then either v, z or y appear in at least five bags of $\mathcal{B}'$. □

If G has a set W of true twins with size greater than c_1, then we can delete an arbitrary vertex of W. The following lemma shows that the this procedure is safe.

Lemma 20. *Let W be a set of true twins in G such that $|W| \geq c_1 + 1$ and let $u \in W$. Then G has a square root H such that $\mathbf{pw}(H) \leq 2$ if and only if $G' = G - u$ has a root H' such that $\mathbf{pw}(H') \leq 2$.*

Proof. Let G be a graph that has a square root H of pathwidth at most 2. Since $|W| \geq c_1 + 1$, either there exists a vertex $v \in W$ satisfying the condition (i) of Lemma 19 or there are three vertices $v_1, v_2, v_3 \in W$ satisfying (ii). Since the vertices of W are true twins, we can assume that $u = v$ in the first case and $u = v_1$ in the second. In both cases we obtain that $H - u$ is a square root of pathwidth at most 2 for $G - u$.

On the other direction, let H' be a square root of pathwidth at most 2 of G'. Since $|W \setminus \{u\}| \geq c_1$, H' has a vertex v satisfying the condition (i) of Lemma 19 or there are three vertices $v_1, v_2, v_3 \in W$ satisfying (ii).

If v is a pendant, let w be the vertex of H' that is adjacent to v. In order to obtain a square root H for G, we attach u to w in H'. We now need to prove that $\mathbf{pw}(H) \leq 2$. We may assume v appears in only one bag in the path decomposition of H' (which also contains w). If this is not the case, we can delete all other occurrences of v and still obtain a valid path decomposition for H' of width at most two. Let A_i be the bag containing $\{w, v\}$ and A_{i+1} the following bag in the decomposition. If $w \in A_{i+1}$, we create a new bag between A_i and A_{i+1} containing $(A_i \cap A_{i+1}) \cup \{u\}$. If $w \notin A_{i+1}$, then $|A_i \cap A_{i+1}| \leq 1$, and the new bag will contain $(A_i \cap A_{i+1}) \cup \{u, w\}$. This is a path decomposition for H of width at most two.

Assume that v_1, v_2, v_3 are vertices satisfying condition (ii) of Lemma 19. Let $N_H(v_1) = N_H(v_2) = N_H(v_3) = \{w, y\}$. In order to obtain a square root H for G, we add u to $V(H')$ and make it adjacent to w and y. We now prove that $\mathbf{pw}(H) \leq 2$. Since $N_H(v_1) = N_H(v_2) = N_H(v_3) = \{w, y\}$, there is a bag in the path decomposition of H' containing $A_i = \{w, y, v_i\}$ for some $i \in \{1, 2, 3\}$. Since v_i is only adjacent to w and y, we may assume that this is the only bag in the decomposition containing v_i. Let A_{i+1} be the bag that follows A_i in the decomposition. We create a new bag between A_i and A_{i+1} containing $\{u, w, y\}$, and hence obtain a path decomposition for H of width at most two. □

For simplicity, we call the graph obtained by exhaustive application of the above procedure G again. From now on, in all our lemmas, we assume that every set of true twins of G has size bounded by the constant c_1.

Lemma 21. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. Suppose there exists distinct $u, v, x_1, \ldots, x_k$ such that the bags $\{x_1, u, v\}$, $\{x_2, u, v\}$,..., $\{x_k, u, v\}$ appear in this order in the path decomposition of H. Then $k \leq 3c_1 + 2$.*

Proof. By Lemma 18, for every $2 \leq i \leq k-1$, $N_H(x_i) \subseteq \{u, v\}$. All the vertices that are adjacent only to u are true twins in G. The same applies for the vertices that are adjacent only to v and also to those that are adjacent to both u and v. Since the size of every set of true twins in G is bounded by c_1, then $|\{x_2, \ldots, x_{k-1}\}| \leq 3c_1$ and hence $k \leq 3c_1 + 2$. □

Lemma 22. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. If $u, v \in V_H$, then the number of common neighbors of u and v in H is bounded by $c_1 + 2$.*

Proof. Suppose $N_H(u) \cap N_H(v) = \{x_1, \ldots, x_t\}$. In the path decomposition of H, for each i, we need a bag containing u and x_i and a bag containing v and x_i. As $\mathbf{pw}(H) \leq 2$, this implies the existence of one bag $B_i = \{u, v, x_i\}$ for each i. Suppose $B_1, \ldots, B_t$ appear in the path decomposition in this order. By Lemma 18 and by the fact that $\{x_2, \ldots, x_{t-1}\} \subset N_H(u) \cap N_H(v)$, the vertices $x_2, x_3, \ldots, x_{t-1}$ are true twins in G and the number of such vertices is bounded by c_1. Hence, the number of common neighbors two vertices can have is bounded by $c_1 + 2$. □

Lemma 23. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. Let $u \in V_G$ and $c_2 = 6 \cdot 21(c_1 + 2)$. If $d_H(u) \geq c_2$, then there exists five vertices $x_1, \ldots, x_5$ in $N_G(u)$ such that $d_{G-u}(x_i, x_j) \geq 3$.*

Proof. Let $u \in V_G$ be such that $d_H(u) \geq 21(c_1 + 2)$. We choose a set of bags $B_1, \ldots, B_l$ in the path decomposition of H such that $u \in B_i$ for all i and for all $x \in N_H(u)$, there exists i such that $x \in B_i$. Note that some neighbors of u might appear in more than one bag of this set. Let k_1 be the smallest integer such that $\cup_{i=1}^{k_1} B_i$ contains at least three distinct vertices of $N_H(u)$. Since u belongs to all bags, at least one of these three neighbors in $\cup_{i=1}^{k_1} B_i$ does not appear in B_{k_1}. Let

v_1 be such vertex. In general, let k_j be the smallest integer greater than k_{j-1} such that $\cup_{i=k_{j-1}}^{k_j} B_i$ contains at least five new vertices of $N_H(u)$. As u belongs to all bags, there is at least one vertex among the five that appears neither in $B_{k_{j-1}}$ nor in B_{k_j}. Let v_j be this vertex. We then obtain a set $\{v_1, \ldots, v_t\} \subset N_H(u)$ that is an independent set. Since $d_H(u) \geq 6 \cdot 21(c_1 + 2)$, we have $t \geq 21(c_1 + 2)$. For each v_i, we pick $x_i \in N_H(v_i)$ such that $x_i \neq x_j$ if $i \neq j$. Since $t \geq 21(c_1 + 2)$ and by Lemma 22, we can pick at least 21 such vertices. For simplicity, we assume that $\{v_1, \ldots, v_{21}\}$ is a set of such vertices. For each i, let A_i be a bag containing v_i and u. For $2 \leq i \leq 20$, we can assume $A_i = \{v_i, x_i, u\}$. Note that x_i and x_{i+1} might be adjacent in H, but x_i cannot be a neighbor of x_k, with $k \geq i + 2$, because of the existence of bag $\{v_{i+1}, x_{i+1}, u\}$. For the same reason, x_i cannot be adjacent to v_k for some $k \geq i + 2$. Also, if $k \geq i + 2$, all paths in H from x_i to x_k contain either x_{i+1} or v_{i+1}. The same applies for the paths from x_i to v_k for some $k \geq i + 2$. Then $\{x_1, x_6, x_{11}, x_{16}, x_{21}\}$ are vertices that are pairwise at distance at least three in $G - u$. $\square$

Lemma 24. *Let G be a graph and H be a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. If G has five vertices $x_1, \ldots, x_5 \in N_G(u)$ such that $d_{G-u}(x_i, x_j) \geq 3$, then for any $x \in N_G(u)$, $xu \notin E_H$ if and only if there exists $i \in \{1, \ldots, 5\}$ such that $d_{G-u}(x, x_i) \geq 3$.*

Proof. By Lemma 3, if there exists $i \in \{1, \ldots, 5\}$ such that $d_{G-u}(x, x_i) \geq 3$, then $xu \notin E_H$.

Let $x \in N_G(u)$ be such that $xu \notin E_H$. We need to prove that there exists i such that $d_{G-u}(x, x_i) \geq 3$. This holds if $x = x_j$ for some $j \in \{1, \ldots, 5\}$. Note also that $ux_i \notin E_H$ because $d_{G-u}(x_i, x_j) \geq 3$. Then for each i there exists v_i such that $x_i v_i \in E_H$ and $v_i u \in E_H$. If $i \neq j$, then $v_i \neq v_j$, otherwise x_i and x_j would be adjacent in G. Moreover, $v_i v_j \notin E_H$, otherwise $d_{G-u}(x_i, x_j) = 2$.

Consider that $v_1, \ldots, v_5$ appear in this order in the path decomposition of H. This implies the existence of the bags $\{v_2, x_2, u\}$, $\{v_3, x_3, u\}$ and $\{v_4, x_4, u\}$ in the path decomposition of H. Consider the bags where x appears.

If x appears before v_2, consider the distance between x and x_4. If the shortest path between x and x_4 in $G - u$ contains x_2 or x_3, then $d_{G-u}(x, x_4) \geq d_{G-u}(x, x_3) \geq 3$. If it does not contain x_2 and x_3, then it must contain either v_2, v_3 or another neighbor of u that appeared previously in the path decomposition and has no common neighbor with x_4 in H. However, $d_{G-u}(v_2, x_4) \geq 2$, otherwise $d_{G-u}(x_2, x_4) < 3$. As $d_{G-u}(v_2, x_4) \geq 2$, we have $d_{G-u}(x, x_4) \geq 3$. If x appears between $\{v_2, x_2, u\}$ and $\{v_3, x_3, u\}$, consider $d_{G-u}(x, x_5)$. By the same argument we obtain that $d_{G-u}(x, x_5) \geq 3$, because of the existence of bags $\{v_3, x_3, u\}$ and $\{v_4, x_4, u\}$. The other cases follow by symmetry. $\square$

Given a vertex $u \in V_G$ such that $d_H(u) \geq c_2$, we define $R_u = \{w \in N_G(u) \mid uw \in E_H\}$ and $B_u = \{w \in N_G(u) \mid uw \notin E_H\}$. Let $u_1, u_2 \in R_u$. If $u_1 u_2 \in E_H$, by Observation 5, there exists a vertex v that is adjacent to at least one of u_1 and u_2 in H, but it is not adjacent to u. Assume v is adjacent to u_1. This implies $v \in B_u$. Moreover, as $u_1 u_2 \in E_H$, then $vu_2 \in E_G$. In other words, if there exists an edge in H between two vertices in R_u, then there exists a vertex in B_u that is adjacent to both of them in G. We can now use this observation to identify edges that will not be in a square root of G. Let $v \in B_u$ and $X_v = \{x \in R_u \mid xv \in E_G\}$. We formalize this in the following lemma.

Lemma 25. *Let $u \in V_G$ be such that $d_H(u) \geq c_2$ and $x, y \in N_H(u)$ for a minimal square root H of G with $\mathbf{pw}(H) \leq 2$. If there is no $v \in B_u$ such that $x, y \in X_v$, then $xy \notin E_H$.*

Lemma 26. *Let G be a graph and H a minimal square root of G such that $\mathbf{pw}(H) \leq 2$. Let u be a vertex such that $d_H(u) \geq c_2$ and $v \in B_u$. Let A and A' be, respectively, the first and the last bag in the path decomposition of H containing u and a vertex of X_v. There exists a constant c_3 such that the number of bags between A and A' is bounded by $c_3 = 15c_1 + 4$.*

Proof. Let A and A' be the first and the last bag in the path decomposition of H containing u and a vertex of X_v. If v belongs to both A and A', then all the bags between A and A' (and including them) contain both u and v. Since for any two bags X, Y in the path decomposition we have $X \not\subseteq Y$, for every vertex appearing between A and A' we have exactly one new bag. The number of such vertices is bounded by $3c_1$, by Lemma 21.

If v appears before A and is not contained in A, let $x \in X_v$ be such that $x \in A'$. Since the bags containing v appear before A, x cannot be adjacent to v in H. By the definition of X_v, they are at distance two. Let $y \in V_H$ be such that $xy, yv \in E_H$. There exists a bag containing $\{v, y\}$, which is before A in the path decomposition and a bag containing $\{x, y, u\}$, which is between A and A' since $x \in X_v$. The number of bags between A and $\{x, y, u\}$ is bounded by $3c_1$, by Lemma 21, as both contain $\{y, u\}$. The same holds for the number of bags between $\{x, y, u\}$ and A', as they both contain $\{x, u\}$.

If v appears in A but not in B', let $x \in X_v$ be such that $x \in A'$. Since $xv \notin E_H$, let xyv be a path between x and v. There exists a bag containing $\{v, y, u\}$ and a bag containing $\{y, x, u\}$ that appears after $\{v, y, u\}$. The same constant $3c_1$ bounds the number of bags between A and $\{v, y, u\}$, between $\{v, y, u\}$ and $\{y, x, u\}$ and between $\{y, x, u\}$ and A'.

If v appears between A and A' but it is not contained in them, we proceed in the same way as we did in the previous cases and the worst scenario is when the vertices of X_v contained in A and A' are not adjacent to v. Let $x \in X_v \cap A$ and $y \in V_H$ be such that xyv is a path between x and v. We take x' and y' analogously with respect to A'. By Lemma 21, the constant $3c_1$ bounds the number of bags between the following pairs of bags: A and $\{x, y, u\}$, $\{x, y, u\}$ and $\{v, y, u\}$, $\{v, y, u\}$ and $\{v, y', u\}$, $\{v, y', u\}$ and $\{x', y', u\}$ and finally, $\{x', y', u\}$ and A'.

The total number of bags between A and A' is therefore bounded by $c_3 = 15c_1 + 4$. □

Let H be a minimal square root of G with $\mathbf{pw}(H) \leq 2$. Let G'_H be the subgraph of G obtained by deleting the edges between $x, y \in N_H(u)$, where $d_H(u) \geq c_2$, such that there is no $v \in B_u$ such that $x, y \in X_v$.

Lemma 27. *There exists a constant c_4 that depends neither on G nor on H such that $\mathbf{pw}(G'_H) \leq c_4$.*

Proof. Let H be a minimal square root of G. Let U be the set of vertices of H such that $d_H(u) \geq c_2$

For each $u \in U$, we do the following. Consider the bags $B_1, \ldots, B_t$ in the path decomposition of H containing u and its neighbors. Starting from B_1, we pick the first bag where a new neighbor of u appears. Let B_i be such a bag. As B_i contains at least one vertex that is not contained in B_{i-1}, we have $|B_i \cap B_{i-1}| \leq 2$ and we already know that $u \in B_i \cap B_{i-1}$. In the bags $B_1, \ldots B_{i-1}$, we replace u by u_1. We create a new bag between B_{i-1} and B_i containing u_1, u_2 and $(B_i \cap B_{i-1}) \setminus \{u\}$. In the bags $B_i, \ldots, B_t$, we replace u by u_2. In general, for every bag B_k found containing a new neighbor of u we do the following:

1. Create a new bag between B_{k-1} and B_k containing u_{j+1} and the vertices of $B_{k-1} \cap B_k$ (note that $u_j \in B_{k-1} \cap B_k$).
2. In the bags $B_k, \ldots, B_t$, replace u_j by u_{j+1}.
3. In H, add the edge between u_j and u_{j+1} and an edge between u_{j+1} and the new found neighbor of u.

Let $\hat{H}$ be the graph obtained from H by the previous procedure. Note that H is a contraction of $\hat{H}$, as H can be obtained by contracting the edges of the paths created for each vertex of U. Since we construct a path decomposition for $\hat{H}$ with the same width as the one we had for H, we have $\mathbf{pw}(\hat{H}) \leq 2$. We claim that $\Delta(\hat{H})$ is bounded. If $v \in V_H \setminus U$, then v has bounded degree in H and, in each step of the above procedure, the degree of v is maintained. The vertices u_i created for each vertex of U have degree bounded by three. Thus the graph $\hat{H}$ has indeed bounded degree.

We claim that G'_H is a minor of $\hat{H}^{c_3+1}$. Let $\hat{G}$ be obtained from $\hat{H}^{c_3+1}$ by the contraction of the paths created for each vertex of U. We can now assume that $V_{\hat{G}} = V_{G'_H}$. We now show that G'_H is a subgraph of $\hat{G}$.

Every edge of G'_H that belongs to E_H is also an edge of $\hat{H}^{c_3+1}$. Let $xy \in E_{G'_H}$ be such that $xy \notin E_H$. As H is a square root of G, there exists $u \in V_{G'_H}$ such that $xu, yu \in E_H$. Denote by X' and Y' respectively the sets of vertices of $\hat{H}$ that were contracted to x and y. If $u \notin U$, by the construction of $\hat{H}$, there are $x' \in X'$ and $y' \in Y'$ such that $x'u, y'u \in E_{\hat{H}}$ and therefore $x'y' \in \hat{H}^{c_3+1}$ and $xy \in E_{\hat{G}}$. If $u \in U$, there exists a path $u_i \ldots u_j$ in $\hat{H}$, $x' \in X$ and $y' \in Y$ such that $u_i x' \in E_{\hat{H}}$ and $u_j y' \in E_{\hat{H}}$. Since $xy \in E_{G'_H}$ and $u \in U$, we know that $x, y \in X_v$ for some v, otherwise we would have deleted this edge in the construction of G'_H. As the number of bags containing u and vertices of X_v is bounded by c_3 by Lemma 26, the length of the path $u_i \ldots u_j$ is at most c_3. This implies that $d_{\hat{H}}(x', y') \leq c_3 + 1$ and hence $x'y' \in E_{\hat{H}^{c_3+1}}$, which implies $xy \in E_{\hat{G}}$.

Since G'_H is a subgraph of $\hat{G}$ and $\hat{G}$ is a contraction of $\hat{H}^{c_3+1}$, we conclude that G'_H is a minor of $\hat{H}^{c_3+1}$. Since G'_H is a minor of $\hat{H}^{c_3+1}$, $\mathbf{pw}(G') \leq \mathbf{pw}(\hat{H}^{c_3+1})$. As $\hat{H}$ has pathwidth at most 2, $\mathbf{pw}(\hat{H}) \leq 2$. Since $\hat{H}$ has bounded degree, by Observation 3, we obtain the constant c_4 such that $\mathbf{pw}(G'_H) \leq c_4$. $\square$

C.2 The Algorithm

Let G be the input graph. It is sufficient to solve Root of Pathwidth ≤ 2 for connected graphs. Hence, we assume that G is connected and has $n \geq 2$ vertices.

First, we preprocess G using Lemma 19 to reduce the number of true twins that a given vertex of V_G might have. To do so, we exhaustively apply the following rule, which is safe by Lemma 20.

Twin reduction. If G has a set X of true twins of size at least $c_1 + 1$, then delete an arbitrary $u \in X$ from G.

For simplicity, we call the graph obtained by exhaustive application of the Twin reduction rule G again.

In the next stage of our algorithm we label some edges of G *red* or *blue* in such a way that the edges labeled red are included in every minimal square root of pathwidth at most 2 and the blue edges are not included in any minimal square root of pathwidth at most 2. We denote by R the set of red edges and by B the set of blue edges. We also construct a set of vertices U of G such that for every $u \in U$, the edges incident to u are labeled red or blue.

Labeling. Set $U = \emptyset$, $R = \emptyset$ and $B = \emptyset$. For each $u \in V_G$ such that there are five distinct vertices in $N_G(u)$ that are at distance at least 3 from each other in $G - u$ do the following:

(i) set $U = U \cup \{u\}$,
(ii) set $B' = \{ux \in E_G \mid \text{there is } 1 \leq i \leq 5 \text{ s.t. } \mathrm{dist}_{G-u}(x, v_i) \geq 3\}$,
(iii) set $R' = \{ux \mid x \in N_G(u)\} \setminus B'$,
(iv) set $R = R \cup R'$ and $B = B \cup B'$,
(v) if $R \cap B \neq \emptyset$, then return a no-answer and stop.

Lemmas 23 and 24 imply the following claim.

Lemma 28. *If G has a square root of pathwidth at most 2, then* **Labeling** *does not stop in step (v). Moreover, $R \subseteq E_H$, $B \cap E_H = \emptyset$ and every vertex $u \in V_G$ with $d_H(u) \geq c_2$ is included in U.*

Next, we find the set of edges xy with $xu, yu \in R$ for some u in U that are not included in a square root of pathwidth at most 2.

Finding irrelevant edges. Set $S = \emptyset$. For each $u \in U$ and each pair of distinct $x, y \in N_G(u)$ such that $ux, uy \in R$ and $xy \in E_G$ do the following.

(i) If there is no $v \in N_G(u)$ such that $vu \in B$ and $x, y \in N_G(v)$, then include xy in S.
(ii) If $R \cap S \neq \emptyset$, then return a no-answer and stop.

By Lemma 25, we obtain the following claim.

Lemma 29. *If G has a square root of pathwidth at most 2, then* **Finding irrelevant edges** *does not stop in step (ii), and if H is such a square root of G, then $S \cap E_H = \emptyset$.*

Assume that we did not stop during the execution of **Finding irrelevant edges**. Let $G' = G - S$. We show the following.

Lemma 30. *The graph G has a square root of pathwidth at most 2 if and only if there is a set $L \subseteq E_{G'}$ such that*

(i) $R \subseteq L$, $B \cap L = \emptyset$,
(ii) *for any* $xy \in E_{G'}$, $xy \in L$ *or there is* $z \in V_{G'}$ *such that* $xz, yz \in L$,
(iii) *for any distinct edges* $xz, yz \in L$, $xy \in E_{G'}$ *or* $xy \in S$,
(iv) *the graph* $H = (V_G, L)$ *is such that* $\mathbf{pw}(H) \leq 2$.

Proof. Let H be a square root of pathwidth at most 2 of G. By Lemma 29, $E_H \cap S = \emptyset$, i.e., $E_H \subseteq E_{G'}$. Let $L = E_H$. It is straightforward to verify that (i)–(iv) are fulfilled. Assume now that there is $L \subseteq E_{G'}$ such that (i)–(iv) hold. Then we have that $H = (V_G, L)$ is a square root of pathwidth at most 2 of G. $\square$

To complete the description of the algorithm, it remains to show how to check the existence of a set of edges L satisfying (i)–(iv) of Lemma 30 for given G', R, B and S. If G has a minimal square root H with $\mathbf{pw}(H) \leq 2$, then G' is a subgraph of G'_H by Lemmas 23 and 25. Then by Lemma 27, $\mathbf{pw}(G') \leq c_4$. Therefore, $\mathbf{pw}(G') \leq c_4$ for a yes-instance. By Lemma 2, it is possible to verify in linear time whether it holds. If $\mathbf{pw}(G') > c_4$, we conclude that we have a no-instance and stop. Otherwise, notice that properties (i) – (iv) can be expressed in monadic second order logic. In particular, (iv) can be expressed using the property that the class of graphs of pathwidth at most 2 is defined by the set of forbidden minors given by Kinnersley and Langston in [13]. Then we use Courcelle's theorem [5] to decide in linear time whether L exists or not.

Now we evaluate the running time of our algorithm. We can find classes of true twin vertices of the input n-vertex graph G in time $O(n^3)$. Therefore, **Twin reduction** can be done in time $O(n^3)$. For every vertex u, we can compute the distances between the vertices of $N_G(u)$ in $G - u$ in time $O(n^3)$. It implies that **Labeling** can be done in time $O(n^6)$. **Finding irrelevant edges** can be done in time $O(n^4)$ by checking $O(n^2)$ pairs of vertices x and y. Then G' can be constructed in linear time. Finally, checking whether $\mathbf{pw}(G') \leq c_4$ and deciding whether there is a set of edges L satisfying the required properties can be done in linear time by Lemma 2 and Courcelle's theorem [5] respectively. This implies that the total running time is also $O(n^6)$. $\square$