Maximum number of minimal feedback vertex sets in chordal graphs and cographs*

Jean-François Couturier[1], Pinar Heggernes[2], Pim van 't Hof[2], and Yngve Villanger[2]

[1] LITA, Université de Lorraine, 57045 Metz Cedex 01, France.
couturier@univ-metz.fr
[2] Department of Informatics, University of Bergen, N-5020 Bergen, Norway.
{pinar.heggernes,pim.vanthof,yngve.villanger}@ii.uib.no

Abstract. A feedback vertex set in a graph is a set of vertices whose removal leaves the remaining graph acyclic. Given the vast number of published results concerning feedback vertex sets, it is surprising that the related combinatorics appears to be so poorly understood. The maximum number of minimal feedback vertex sets in a graph on n vertices is known to be at most 1.864^n. However, no examples of graphs having 1.593^n or more minimal feedback vertex sets are known, which leaves a considerable gap between these upper and lower bounds on general graphs. In this paper, we close the gap completely for chordal graphs and cographs, two famous perfect graph classes that are not related to each other. We prove that for both of these graph classes, the maximum number of minimal feedback vertex sets is $10^{\frac{n}{5}} \approx 1.585^n$, and there is a matching lower bound. Our results also imply that the maximum number of feedback vertex sets in circular arc graphs is at most $O(1.585^n)$.

1 Introduction

The study of maximum number of vertex subsets satisfying a given property in a graph has always attracted interest and found applications in combinatorics and computer science. Especially during the last decades there has been a tremendous increase of interest in exponential time algorithms, whose running times often rely on the maximum number of certain objects in graphs [13]. A classical example is the highly cited and widely used result of Moon and Moser [22], who showed that the maximum number of maximal cliques and maximal independent sets, respectively, in any graph on n vertices is $3^{\frac{n}{3}} \approx 1.442^n$. More recently, maximum numbers of minimal dominating sets, minimal feedback vertex sets, minimal subset feedback vertex sets, minimal separators, and potential maximal cliques in general graphs, and minimal feedback vertex sets in tournaments have been studied [10–12, 14, 15].

A *feedback vertex set* in a graph is a set of vertices whose removal from the graph results in an acyclic graph. Computing a feedback vertex set of minimum cardinality

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or minimum weight is by now one of the most well-studied NP-hard problems in graph algorithms. In 1985 Wang, Lloyd, and Soffa [27] classified this to be the least understood of the classical NP-complete problems. Since then a large number of papers have been published on the topic, especially during recent years. In 2002, Schwikowski and Speckenmeyer [25] showed that all minimal feedback vertex sets of a given graph can be enumerated with polynomial delay. In 2008, Fomin, Gaspers, Pyatkin, and Razgon [10] showed that the maximum number of minimal feedback vertex sets in a graph on n vertices is at most 1.864^n. This immediately implies an algorithm with running time $O(1.864^n)$ for listing all minimal feedback vertex sets of a graph, and for computing a feedback vertex set of minimum weight. It also shows the impact of good upper bounds on the maximum number of important objects in graphs. Although for some objects, e.g., maximal independent sets [22], the known upper bound on their maximum number matches the known lower bound, this is unfortunately not the case for all objects, and in particular for minimal feedback vertex sets. In fact, we do not have examples of graphs that have 1.593^n or more minimal feedback vertex sets [10], and hence there is a considerable gap between the upper and lower bounds on the maximum number of minimal feedback vertex sets in graphs.

Motivated by the results of Fomin et al. [10] and the mentioned gap, we turn our attention to graph classes with the objective of narrowing the gap on graphs with particular structure. In this paper we close the gap completely on chordal graphs and cographs. In particular, we show that the maximum number of minimal feedback vertex sets in these graph classes is at most $10^{\frac{n}{5}} \approx 1.585^n$, and that this also matches the lower bound. Our results are obtained by purely combinatorial arguments, whereas the upper bound in [10] and most of the other cited upper bounds are obtained by algorithmic tools. One of the computational implications of our results is that all minimal feedback vertex sets in chordal graphs and cographs can be listed in time $O(1.585^n)$, using the algorithm of Schwikowski and Speckenmeyer [25]. Furthermore, our result on chordal graphs implies that circular arc graphs have at most $O(n^2)\, 10^{\frac{n}{5}}$ minimal feedback vertex sets, and thus all minimal feedback vertex sets in circular arc graphs can also be listed in time $O(1.585^n)$. Another implication is that the search for better lower bounds on the maximum number of minimal feedback vertex sets in general graphs can discard all chordal graphs and cographs.

Although a lot of attention has been given to graph classes when it comes to tractability of optimization problems, they have been left largely unexplored when it comes to counting, enumerating, and determining the maximum number of objects, apart from a few recent results [7, 17, 23, 24]. Chordal graphs and cographs are famous and well-studied subclasses of perfect graphs with many applications in real-life problems, like sparse matrix computations, perfect phylogeny, VLSI, and computer vision [2, 16, 26]. Many problems that are NP-hard in general can be solved in polynomial time on these graph classes, and this is also the case for the problem of computing a feedback vertex set of minimum weight [6, 26]. Hence our motivation and results are not related to efficient computation of a feedback vertex set of minimum weight or cardinality in these graph classes (the same applies to several of the results in [7, 17, 23, 24]). However, these results might have computational implications for other, seemingly unrelated, optimization problems on the studied graph classes. For example, using Moon and Moser's [22] upper bound on the maximum number of maximal independent sets, Lawler [20] gave an algorithm for graph coloring, which was the fastest for over two decades. A faster algorithm for graph coloring was obtained by Eppstein [9] by improving the upper bound for maximal independent sets of small size.

Our paper is organized as follows. The next section gives definitions and background on graph classes and minimal feedback vertex sets, and it provides an example that constitutes our lower bound on the maximum number of minimal feedback vertex sets in chordal graphs, cographs, and circular arc graphs. In Section 3, we prove the matching upper bound for chordal graphs, and in section 4, we prove the matching upper bound for cographs. We conclude with the mentioned implications for circular arc graphs and some open questions in Section 5.

2 Preliminaries

We work with simple undirected graphs. We denote such a graph by $G = (V, E)$, where V is the set of vertices and E is the set of edges of G. We adhere to the convention that $n = |V|$. The set of *neighbors* of a set of vertices $S \subset V$ is the set $N_G(S) = \{u \notin S \mid v \in S, uv \in E\}$. A vertex is *universal* if it is adjacent to every other vertex. The subgraph of G induced by a set $S \subset V$ is denoted by $G[S]$. A graph is *connected* if there is a path between every pair of its vertices. A maximal connected subgraph is called a *connected component*. A set $S \subseteq V$ is called an *independent set* if $uv \notin E$ for every pair of vertices $u, v \in S$, and S is called a *clique* if $uv \in E$ for every pair of vertices $u, v \in S$. An independent set or a clique is *maximal* if no proper superset of it is an independent set or a clique, respectively. A complete graph on n vertices is denoted by K_n.

2.1 Graph classes

A *chord* of a cycle (or path) is an edge between two non-consecutive vertices of the cycle (or path). A graph is *chordal* if every cycle of length at least 4 has a chord. A chordal graph has at most n maximal cliques. A *clique tree* of a graph G is a tree T, whose set of nodes is the set of maximal cliques of G, that satisfies the following: for every vertex v of G, the nodes of T that correspond to maximal cliques of G

containing v induce a connected subtree of T. A graph has a clique tree if and only if it is chordal [4]. Chordal graphs can be recognized, and a clique tree can be constructed, in linear time [1].

The *disjoint union* operation on graphs takes as input a collection of graphs and outputs the collection as one graph, without adding any edges. The *complete join* operation on graphs takes as input a collection of graphs and adds edges between every pair of vertices that belong to two different graphs in the collection. A graph is a *cograph* if it can be generated from single-vertex graphs with the use of disjoint union and complete join operations. Cographs are exactly the graphs that do not contain chordless paths of length at least 4 as induced subgraphs, and they can be recognized in linear time [5].

A *circular arc graph* is the intersection graph of a set of arcs on a circle (equivalently, paths on a cycle), i.e., every vertex can be assigned an arc on the circle so that two vertices are adjacent if and only if their arcs intersect. These graphs form an interesting graph class as they can have an exponential number of maximal cliques [26], but have $O(n^2)$ minimal separators [19]. A graph is a *proper circular arc graph* if it has a circular arc representation where no arc properly contains another arc. Circular arc and proper circular arc graphs can be recognized, and their circular arc representations can be constructed, in linear time [18, 21, 26].

Chordal graphs and cographs are subclasses of perfect graphs, but they are not related to each other. They form two of the most well-studied graph classes. Circular arc graphs do not form a subclass of perfect graphs or a superclass of chordal graphs or cographs. All three mentioned graph classes are closed under taking induced subgraphs. More details about these graph classes and perfect graphs can be found in the books by Brandstädt et al. [2] and Golumbic [16].

2.2 Minimal feedback vertex sets and our lower bound

A set $S \subseteq V$ is called a *feedback vertex set* if $G[V \setminus S]$ is a forest. A feedback vertex set S is *minimal* if no proper subset of S is a feedback vertex set. In that case, $G[V \setminus S]$ is a *maximal induced forest* of G. Hence there is a bijection between the minimal feedback vertex sets and the maximal induced forests of a graph, and the number of minimal feedback vertex sets is equal to the number of maximal induced forests. For a disconnected graph, the total number of minimal feedback vertex sets is the product of the numbers of minimal feedback vertex sets of its connected components. We will use these facts extensively in our arguments.

Lemma 1 gives a lower bound on the number of minimal feedback vertex sets of chordal graphs, cographs, and circular arc graphs. In the next two sections, we will give matching upper bounds for chordal graphs and cographs.

Lemma 1. *There are infinite families of chordal graphs, cographs, and circular arc graphs on n vertices that have $10^{\frac{n}{5}}$ minimal feedback vertex sets.*

Proof. For any positive integer k, let G_k be a graph on $n = 5k$ vertices that is the disjoint union of k copies of K_5, as shown in Fig. 1. G_k is a chordal graph, a cograph, and a circular arc graph. Observe that each edge in K_5 is a maximal induced forest of K_5, and K_5 has no other maximal induced forests. Hence K_5 has 10 maximal induced forests, or equivalently, minimal feedback vertex sets. Consequently, for every k, G_k has $10^k = 10^{\frac{n}{5}} \approx 1.585^n$ minimal feedback vertex sets. This provides an infinite family of graphs $\mathcal{G} = \{G_k \mid k \geq 1\}$ that constitute our lower bound example for all three mentioned graph classes. $\square$

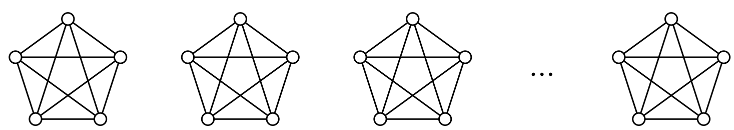

Fig. 1. The infinite family $\mathcal{G}$ of graphs that have $10^{\frac{n}{5}}$ minimal feedback vertex sets. This family belongs to all three classes of chordal graphs, cographs, and circular arc graphs.

Finally, we conclude this section by stating the aforementioned result of Moon and Moser [22] on the number of maximal independent sets of a graph, which will be used in Sections 4 and 5.

Theorem 1 ([22]). *Every graph on n vertices has at most $3^{\frac{n}{3}}$ maximal independent sets.*

3 A tight bound for chordal graphs

In this section, we provide the upper bound on the number of minimal feedback vertex sets in chordal graphs that matches the lower bound given in Section 2. For our arguments, it is more convenient to work with maximal induced forests instead of minimal feedback vertex sets.

We will in fact prove a slightly stronger statement than previously announced, answering the following question: Given a chordal graph $G = (V, E)$ and a set of vertices $F \subseteq V$, what is the maximum number of maximal induced forests in G that contain all vertices of F? Clearly, if $G[F]$ is not a forest the answer is 0, and if F is empty the answer is exactly the maximum number of minimal feedback vertex sets of G.

Theorem 2. *Let $G = (V, E)$ be a chordal graph and let $F \subseteq V$ be such that $G[F]$ is a forest. Then G has at most $10^{\frac{n-|F|}{5}}$ maximal induced forests containing F.*

Proof. For any graph $G = (V, E)$ and any subset $F \subseteq V$, we define $\mu(G, F)$ to be the number of maximal induced forests of G containing F. Let G and F be as in the premises of the theorem; we prove that $\mu(G, F) \leq 10^{\frac{n-|F|}{5}}$. If we prove the bound on connected chordal graphs, then it trivially applies to disconnected chordal graphs, as explained in Section 2. Consequently, we assume that G is connected. Let T be a clique tree of G and let k be the number of nodes in T. We prove the theorem by induction on k. Recall that $n = |V|$.

The base case is when $k = 1$, thus G is a complete graph on $n \geq 1$ vertices. If $n = 1$, then there is a unique maximal induced forest, and since $|F| \leq 1$ and $1 \leq 10^0 \leq 10^{\frac{1-|F|}{5}}$, the theorem holds in this case. If $n \geq 2$, then since G is complete, every maximal induced forest contains exactly 2 vertices. In particular, this implies that $|F| \in \{0, 1, 2\}$. The vertices of F have to appear in every maximal induced forest, which limits the number of choices of vertices from $V \setminus F$ to be in the forest. Clearly, if $|F| = 0$ then $\mu(G, F) = \binom{n}{2}$; if $|F| = 1$ then $\mu(G, F) = n-1$; and if $|F| = 2$ then $\mu(G, F) = 1$. It is easy to verify that $\binom{n}{2} \leq 10^{\frac{n}{5}}$, $n-1 \leq 10^{\frac{n-1}{5}}$, and $1 \leq 10^{\frac{n-2}{5}}$, when $n \geq 2$. Thus we can conclude that there are at most $10^{\frac{n-|F|}{5}}$ maximal induced forests in G that contain F. This completes the base case.

Let $k \geq 2$, which means in particular that G is not a complete graph. According to the base case, our induction hypothesis is that $\mu(G, F) \leq 10^{\frac{n-|F|}{5}}$ for all chordal graphs whose clique trees have at most $k - 1$ nodes.

Consider a clique tree T of our graph G, which has k nodes. Let X_ℓ be a maximal clique corresponding to a leaf of T, and let X_p be the maximal clique corresponding to the parent of X_ℓ in T. Let $L = X_\ell \setminus X_p$ and $C = X_\ell \cap X_p$. An illustration is given in Fig. 2. Observe that $C = N_G(L)$, and $X_\ell = L \cup C$. Hence, the vertices of L appear only in X_ℓ and in no other clique of G. In particular, removing L or some superset of L from G results in a chordal graph that has at most $k - 1$ maximal cliques, and therefore has a clique tree with at most $k - 1$ nodes. Also, by construction, $|L| \geq 1$ and $|C| \geq 1$; hence $|X_\ell| \geq 2$.

If $|X_\ell| = 2$, then L consists of a single vertex x, and C consists of a single vertex y. In particular, x has no other neighbor than y in G, which means that x is a part of every maximal induced forest, regardless of whether or not x belongs to F. Consequently $\mu(G, F) = \mu(G[V \setminus \{x\}], F \setminus \{x\})$ in this case. Since $G[V \setminus \{x\}]$ has a clique tree with $k - 1$ nodes, we can apply the induction hypothesis and obtain that $\mu(G, F) \leq 10^{\frac{n-1-|F|+1}{5}} = 10^{\frac{n-|F|}{5}}$ if $x \in F$, and $\mu(G, F) \leq 10^{\frac{n-1-|F|}{5}} < 10^{\frac{n-|F|}{5}}$ if $x \notin F$.

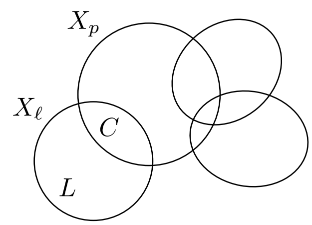

Fig. 2. The maximal cliques of G that are used in the inductive step of the proof.

For the rest of the proof we can thus assume that $|X_\ell| \geq 3$. Since C is a clique, every set $A \subseteq V$ such that $G[A]$ is a maximal induced forest in G satisfies exactly one of the following three cases.

Case 0. $|A \cap C| = 0$. In this case, since L is also a clique, A cannot contain more than 2 vertices of L. However, since $G[A]$ is a maximal induced forest, if $|L| \geq 2$ then $|A \cap L| = 2$, and if $|L| = 1$ then $|A \cap L| = 1$. Furthermore, $G[A \setminus L] = G[A \setminus X_\ell]$ is a maximal induced forest of $G[V \setminus X_\ell]$, since there are no edges in G between vertices in $A \cap L$ and vertices in $A \setminus L$.

Case 1. $|A \cap C| = 1$. In this case, by the same arguments as in the previous case, $|A \cap L| = 1$. Let u be the single vertex in $A \cap C$. Then $G[(A \setminus X_\ell) \cup \{u\}]$ is a maximal induced forest of $G[(V \setminus X_\ell) \cup \{u\}]$, since the vertex in $A \cap L$ does not impose any restrictions for this subgraph.

Case 2. $|A \cap C| = 2$. In this case, $A \cap L = \emptyset$ and $A \cap (X_p \setminus X_\ell) = \emptyset$, since selecting any vertex of L or any vertex of $X_p \setminus X_\ell$ creates a cycle of length 3. Let u and v be the two vertices in $A \cap C$. Then $G[(A \setminus (X_\ell \cup X_p)) \cup \{u, v\}]$ is a maximal induced forest of $G[(V \setminus (X_\ell \cup X_p)) \cup \{u, v\}]$, since u and v are both in A.

Based on the analyses of the cases above, let us define the following corresponding values that will help us in the further analysis.

- $\mu_0 = \mu(\, G[V \setminus X_\ell],\, F \setminus X_\ell\,)$
- $\mu_1 = \max\,\{\mu\,(\,G[(V \setminus X_\ell) \cup \{u\}],\, (F \setminus X_\ell) \cup \{u\}\,) \mid u \in C\}$
- $\mu_2 = \max\,\{\mu\,(\,G[(V \setminus (X_\ell \cup X_p)) \cup \{u, v\}],\, (F \setminus (X_\ell \cup X_p)) \cup \{u, v\}\,) \mid u, v \in C\}$

Before we give upper bounds for μ_0, μ_1, μ_2, we first show that $\mu(G, F)$ is maximized when $F \cap (X_\ell \cup X_p) = \emptyset$. To do this, we have to analyze how F intersects with C. Since $G[F]$ is a forest, it is clear that F contains $0, 1$, or 2 vertices of C.

If $|F \cap C| = 2$, then let u and v be the two vertices of $F \cap C$. Notice that $F \cap L = \emptyset$ in this case. This places us directly in Case 2 above, as every maximal forest of G containing F must contain u and v, and therefore cannot contain any other vertex of $X_\ell \cup X_p$. For the same reasons, $F \cap (X_\ell \cup X_p) = \{u, v\}$. Consequently, $\mu(G, F) = \mu(G[(V \setminus (X_\ell \cup X_p)) \cup \{u, v\}], F) \leq \mu_2$. Since $|C| \geq 2$ in this case, we also obtain that $\mu(G, F) \leq \mu_2 \leq (|C| - 1)\mu_2$, which will be helpful later.

If $|F \cap C| = 1$, then let $F \cap C = \{u\}$. In this case, either $|F \cap L| = 1$ or $|F \cap L| = 0$. Assume first that $|F \cap C| = 1$ and $|F \cap L| = 1$, and let $F \cap L = \{x\}$. Every maximal forest of G containing F must contain u and x, and cannot contain any other vertex of X_ℓ, which means that we are in Case 1 above. In particular $F \cap X_\ell = \{u, x\}$, and hence $\mu(G, F) = \mu(G[(V \setminus X_\ell) \cup \{u\}],\ F \setminus \{x\}) \leq \mu_1$. Assume next that $|F \cap C| = 1$ and $F \cap L = \emptyset$. Then a maximal induced forest of G containing F can contain either one or two vertices of C, which means that we are either in Case 1 or in Case 2 above. Case 1 implies that every forest of this type contains a vertex of L in addition to u. Case 2 implies that $|C| \geq 2$ and every forest of this type contains a vertex of $C \setminus \{u\}$ in addition to u. Consequently, $\mu(G, F) \leq |L|\mu_1 + (|C| - 1)\mu_2$. We can now conclude that, whenever $|F \cap C| = 1$, we have that $\mu(G, F) \leq |L|\mu_1 + (|C| - 1)\mu_2$, since we always have $|L| \geq 1$.

If $|F \cap C| = 0$, then $|F \cap L| \in \{0, 1, 2\}$. If $|F \cap C| = 0$ and $|F \cap L| = 2$, then we are exactly in Case 0 above, since no induced forest containing F can contain a vertex of C. Clearly $\mu(G, F) \leq \mu_0$ in this case. If $|F \cap C| = 0$ and $|F \cap L| = 1$, then let $F \cap L = \{x\}$. Now we are in Case 0 or Case 1, since every maximal induced forest must contain a vertex of $L \setminus \{x\}$ or C in addition to x. By similar arguments as above, we get that $\mu(G, F) \leq (|L| - 1)\mu_0 + |C|\mu_1$. Now we can conclude that if $|F \cap C| = 0$ and $|F \cap L| \in \{1, 2\}$ then $\mu(G, F) \leq \max\{1, |L| - 1\}\mu_0 + |C|\mu_1$.

The only possibility for F that remains is the situation where both $|F \cap C| = 0$ and $|F \cap L| = 0$. This means exactly that $F \cap X_\ell = \emptyset$. If $F \cap X_\ell = \emptyset$ and $F \cap X_p \neq \emptyset$, then no maximal induced forest containing F can contain two vertices of C as this will lead to a cycle on three vertices, so Case 2 is excluded. Thus we are in Case 0 or in Case 1. As a direct consequence of the analyses of these two cases we obtain that $\mu(G, F) \leq \max\{1, \binom{|L|}{2}\}\mu_0 + |L||C|\mu_1$. If $F \cap X_\ell = \emptyset$ and $F \cap X_p = \emptyset$, then we might be in any of the Cases 0, 1, and 2, as a maximal induced forest containing F can contain zero, one, or two vertices of C. As a direct consequence of the analyses of Cases 0, 1, and 2, we obtain the following.

$$\mu(G, F) \leq \max\left\{1, \binom{|L|}{2}\right\}\mu_0 + |L||C|\mu_1 + \binom{|C|}{2}\mu_2 \tag{1}$$

Now, observe that, since $|C| \geq 1$, $|L| \geq 1$, and $|X_\ell| \geq 3$, the following relations are immediate.

$$\begin{aligned}
(|C|-1)\mu_2 &\leq \max\left\{1, \binom{|L|}{2}\right\}\mu_0 + |L||C|\mu_1 + \binom{|C|}{2}\mu_2 \\
|L|\mu_1 + (|C|-1)\mu_2 &\leq \max\left\{1, \binom{|L|}{2}\right\}\mu_0 + |L||C|\mu_1 + \binom{|C|}{2}\mu_2 \\
\max\{1, |L|-1\}\mu_0 + |C|\mu_1 &\leq \max\left\{1, \binom{|L|}{2}\right\}\mu_0 + |L||C|\mu_1 + \binom{|C|}{2}\mu_2
\end{aligned}$$

We see that all upper bounds for $\mu(G, F)$ that we have obtained so far are upper bounded by the right hand side of inequality (1), which describes exactly the situation when $F \cap (X_\ell \cup X_p) = \emptyset$. Consequently, we can safely assume for the rest of the proof that $F \cap (X_\ell \cup X_p) = \emptyset$.

Let us now find upper bounds for μ_0, μ_1, and μ_2 under this assumption. Since the graphs involved in the definitions of these numbers do not contain any vertex of L, each of them has a clique tree with at most $k-1$ nodes, as argued previously. Therefore we can apply our induction hypothesis to obtain upper bounds on μ_0, μ_1, and μ_2. For μ_0, since $|V \setminus X_\ell| = n - |X_\ell|$ and $F \setminus X_\ell = F$, we get the following.

$$\mu_0 \leq 10^{\frac{n-|X_\ell|-|F|}{5}}$$

For μ_1, since $|(V \setminus X_\ell) \cup \{u\}| = n - |X_\ell| + 1$ and $(F \setminus X_\ell) \cup \{u\} = F \cup \{u\}$ for every $u \in C$, we conclude the following.

$$\mu_1 \leq 10^{\frac{n-|X_\ell|+1-(|F|+1)}{5}} \leq 10^{\frac{n-|X_\ell|-|F|}{5}}$$

For μ_2, observe that X_p contains at least one vertex outside of X_ℓ, since it is a maximal clique, and hence $|(V \setminus (X_\ell \cup X_p)) \cup \{u, v\}| \leq n - (|X_\ell|+1) + 2 = n - |X_\ell| + 1$. Since we in addition have $(F \setminus (X_\ell \cup X_p)) \cup \{u, v\} = F \cup \{u, v\}$ for every $u, v \in C$, we get the following bound.

$$\mu_2 \leq 10^{\frac{n-|X_\ell|+1-(|F|+2)}{5}} \leq 10^{\frac{n-|X_\ell|-|F|-1}{5}}$$

As a direct consequence of the above and of inequality (1), we have the following.

$$\mu(G, F) \leq \left(\max\left\{1, \binom{|L|}{2}\right\} + |L||C|\right) 10^{\frac{n-|X_\ell|-|F|}{5}} + \binom{|C|}{2} 10^{\frac{n-|X_\ell|-|F|-1}{5}} \qquad (2)$$

We will study the two cases $|L| = 1$ and $|L| \geq 2$ separately. Assume first that $|L| = 1$, and hence $\max\{1, \binom{|L|}{2}\} = 1$. Recall that $|X_\ell| \geq 3$, and thus $|C| \geq 2$. According to (2), we want to show that

$$(1 + |C|)\, 10^{\frac{n-|X_\ell|-|F|}{5}} + \binom{|C|}{2} 10^{\frac{n-|X_\ell|-|F|-1}{5}} \leq 10^{\frac{n-|F|}{5}} = 10^{\frac{n-|X_\ell|-|F|+|X_\ell|}{5}}.$$

Dividing both sides by $10^{\frac{n-|X_\ell|-|F|}{5}}$ gives us that what we want to show is the next inequality.

$$1 + |C| + \binom{|C|}{2} 10^{-\frac{1}{5}} \leq 10^{\frac{|X_\ell|}{5}} \tag{3}$$

If $|C| = 2$ then we are done, since we have that $3 + 10^{-\frac{1}{5}} \approx 3.631 < 3.981 \approx 10^{\frac{3}{5}}$. If $|C| \geq 3$, then since $|L| = 1$ and thus $|C| = |X_\ell| - 1$, we have that $|C| = \binom{|X_\ell|}{2} - \binom{|C|}{2}$. Consequently, inequality (3) becomes the following.

$$1 + \binom{|X_\ell|}{2} + \binom{|C|}{2}(10^{-\frac{1}{5}} - 1) \leq 10^{\frac{|X_\ell|}{5}}$$

Since $\binom{|X_\ell|}{2} \leq 10^{\frac{|X_\ell|}{5}}$ for all values of $|X_\ell| \geq 3$, this last inequality is equivalent to the following one.

$$1 \leq \binom{|C|}{2}(1 - 10^{-\frac{1}{5}})$$

As $|C| \geq 3$, $\binom{|C|}{2} \geq 3$, and $1 - 10^{-\frac{1}{5}} > 0.36$, we have that $1 < 3 \cdot 0.36 = 1.08$ and we can conclude that $\mu(G, F) \leq 10^{\frac{n-|F|}{5}}$, as desired.

The rest of the proof is devoted to the case where $|L| \geq 2$. Since $\max\{1, \binom{|L|}{2}\} = \binom{|L|}{2}$, inequality (2) immediately implies the following.

$$\mu(G, F) \leq \left(\binom{|L|}{2} + |L||C|\right) 10^{\frac{n-|X_\ell|-|F|}{5}} + \binom{|C|}{2} 10^{\frac{n-|X_\ell|-|F|-1}{5}}$$

We can rewrite this last inequality in the following way.

$$\mu(G, F) \leq 10^{\frac{n-|X_\ell|-|F|}{5}} \left(\binom{|L|}{2} + |L||C| + \binom{|C|}{2}\right) \tag{4}$$

Observe that $\binom{|L|}{2} + |L||C| + \binom{|C|}{2}$ is equal to the number of ways we can select two vertices from L, or one vertex from L and one from C, or two from C. This is equal to the number of ways we can select two vertices from $L \cup C = X_\ell$, i.e., $\binom{|L|}{2} + |L||C| + \binom{|C|}{2} = \binom{|X_\ell|}{2}$. Therefore, inequality (4) implies the following.

$$\mu(G, F) \leq 10^{\frac{n-|X_\ell|-|F|}{5}} \binom{|X_\ell|}{2}$$

Since $\binom{|X_\ell|}{2} \leq 10^{\frac{|X_\ell|}{5}}$ when $|X_\ell| \geq 3$, we get the desired bound as follows.

$$\mu(G, F) \leq 10^{\frac{n-|X_\ell|-|F|}{5}} \cdot 10^{\frac{|X_\ell|}{5}} = 10^{\frac{n-|F|}{5}}$$

This concludes the proof of Theorem 2. $\square$

The following corollary is now immediate from Theorem 2.

Corollary 1. *Every chordal graph on n vertices has at most $10^{\frac{n}{5}}$ minimal feedback vertex sets.*

Although our main aim was to prove the statement of Corollary 1, note that the stronger statement of Theorem 2 makes it more useful than Corollary 1. Suppose we want to bound the number of minimal feedback vertex sets in other graph classes or in general graphs using a branching algorithm or a combinatorial argument. If at some stage of the algorithm we end up with subgraphs of the input graph that are chordal, in which some vertices have been preselected to belong to the solution, then Theorem 2 can be applied directly to these subproblems, allowing their solutions to be combined into a solution for the whole graph.

4 A tight bound for cographs

We will now prove an upper bound of $10^{\frac{n}{5}}$, matching the lower bound given in Section 2, also on the number of minimal feedback vertex sets in cographs. As in the previous section, we will work with maximal induced forests. Even though the upper bound for cographs is the same as the upper bound for chordal graphs, the two proofs are completely different. In particular, the proof for cographs requires in addition the use of an upper bound on the number of maximal independent sets.

We start by defining a function f, which will ease the notation in the rest of this section:

$$f(\alpha, \beta, i, j) = \alpha^i + \alpha^j + i\beta^j + j\beta^i$$

We will use a property of f described in the next lemma.

Lemma 2. *For $\alpha = 10^{\frac{1}{5}}$, $\beta = 3^{\frac{1}{3}}$, $i \geq i' \geq 1$, and $j \geq j' \geq 1$, such that $(i', j') \in \{(1, 10), (2, 8), (3, 7), (4, 5), (5, 4), (7, 3), (8, 2), (10, 1)\}$, the following holds:*

$$f(\alpha, \beta, i, j) \leq \alpha^{i+j}.$$

Proof. Let $\mathcal{B} = \{(1, 10), (2, 8), (3, 7), (4, 5), (5, 4), (7, 3), (8, 2), (10, 1)\}$. The proof is by induction on (i, j). The base case is $(i, j) \in \mathcal{B}$. In this case, it is straightforward to verify that the statement of the lemma holds for all $(i, j) \in \mathcal{B}$ by checking that

$$\alpha^{-j}\left(1+j\left(\frac{\beta}{\alpha}\right)^{i}\right)+\alpha^{-i}\left(1+i\left(\frac{\beta}{\alpha}\right)^{j}\right)\leq 1. \tag{5}$$

For all $(i,j)\in\mathcal{B}$, we get the following sequence of upper bounds for the left hand side of inequality (5): $\{0.978, 0.965, 0.891, 0.997, 0.997, 0.891, 0.965, 0.978\}$.

Assume now that the statement of the lemma is true for $(i-1,j)$, and let us prove the statement for (i,j), where $i>i'$ and $j\geq j'$ for some pair $(i',j')\in\mathcal{B}$. We distinguish between two cases: $i\geq 3$ and $i\leq 2$.

First assume that $i\geq 3$. By the induction assumption, the claim holds for $(i-1,j)$, and thus $\alpha^{i-1}+\alpha^{j}+(i-1)\beta^{j}+j\beta^{i-1}\leq\alpha^{i+j-1}$. Hence, in order to prove that $f(\alpha,\beta,i,j)\leq\alpha^{i+j}$, it suffices to prove that $\alpha^{i}+\alpha^{j}+i\beta^{j}+j\beta^{i}\leq\alpha(\alpha^{i-1}+\alpha^{j}+(i-1)\beta^{j}+j\beta^{i-1})$. This can be rewritten as

$$\alpha^{j}(1-\alpha)+i\beta^{j}\left(1-\frac{i-1}{i}\alpha\right)+j\beta^{i}\left(1-\frac{\alpha}{\beta}\right)\leq 0. \tag{6}$$

We will show that the claim holds by arguing that the value of every expression appearing between a pair of parentheses on the left hand side of inequality (6) is negative. Thus we will obtain that the left side is negative and the claim holds. As $\alpha>1$ and $\alpha>\beta$, it is clear that $(1-\alpha)<0$ and $(1-\frac{\alpha}{\beta})<0$. The remaining expression is $(1-\frac{i-1}{i}\alpha)$, which we can rewrite as $(1-\alpha+\frac{\alpha}{i})$. Since $i\geq 3$ and $\alpha=10^{\frac{1}{5}}\approx 1.585$, we get $(1-\alpha+\frac{\alpha}{i})<0$.

Now let $i\leq 2$. In this case, $i'=1$ and thus $j'\geq 10$, since $(1,10)$ is the only pair in $\mathcal{B}$ that has a 1 as its first element. Since $j\geq j'$ by assumption, we have that $j\geq 10$. If $j>10$, then by the symmetric case (i'',j''), where $i''=j$ and $j''=i$, the previous argument can be applied. After all, for every such pair (i'',j'') with $i''>10$ and $j''\leq 2$, there is a pair $(i',j')\in\mathcal{B}$ with $i''>i'$ and $j''\geq j'$. Consequently, the only missing case is when $(i,j)=(2,10)$. Note that this time we cannot consider the symmetric case $(10,2)$ to determine the value for the pair $(2,10)$, since there is no pair $(i',j')\in\mathcal{B}$ such that $10>i'$ and $2\geq j'$. Instead, like in the base case, we simply use inequality (5) and notice that the left hand side is smaller than 0.81. □

We are now ready to prove our result on cographs.

Theorem 3. *Every cograph on n vertices has at most $10^{\frac{n}{5}}$ maximal induced forests.*

Proof. We will prove the theorem by induction on the number of vertices of a cograph. The base case concerns cographs on at most 5 vertices. All cographs on at most 5 vertices [28] are listed in Figure 3. Examining this figure, one can verify that none of these graphs has more than $10^{\frac{n}{5}}$ maximal induced forests.

Assume now that the statement of the theorem is true for all cographs on at most $n-1$ vertices, and let $G=(V,E)$ be a cograph on $n\geq 6$ vertices. By the definition of

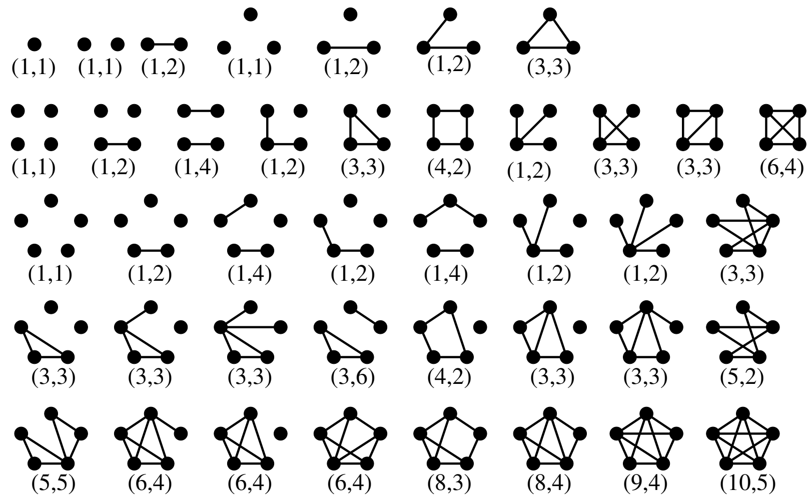

Fig. 3. All cographs on at most five vertices [28]. Each graph is labeled with a pair of numbers: the first number is the number of maximal induced forests, and the second number is the number of maximal independent sets in the graph.

cographs, G is either disconnected or the complete join of two cographs. As explained in Section 2, if G is disconnected, then it suffices to prove the upper bound on each connected component. Hence we can assume that G is connected. Let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be the two cographs such that G is the complete join of G_0 and G_1. Let $n_i = |V_i|$ for $i \in \{0, 1\}$. Observe that $n = n_0 + n_1$, that $n_0 \geq 1$ and $n_1 \geq 1$, and that every vertex in V_0 is adjacent to every vertex in V_1.

Let $A \subseteq V$ be such that $G[A]$ is a maximal induced forest of G. For $i, j \in \{0, 1\}$ and $i \neq j$, we can observe that

- if $|A \cap V_i| \geq 2$, then $|A \cap V_j| \leq 1$, since otherwise the forest would contain a cycle of length 4,
- if $|A \cap V_i| = 1$, then $A \cap V_j$ is a maximal independent set, since otherwise the forest would contain a cycle of length 3, and
- if $A \cap V_i = \emptyset$, then $A \subseteq V_j$ and $G[A]$ is a maximal induced forest in G_j.

Let γ_i denote the number of maximal independent sets in G_i, for $i \in \{1, 2\}$, and let μ, μ_0, and μ_1 denote the number of maximal induced forests in G, G_0, and G_1, respectively. The three observations above show that a maximal induced forest in G is either a maximal induced forest in G_i, or a single vertex in V_i and a maximal independent set in G_j, for $i, j \in \{0, 1\}$ with $i \neq j$. This gives us the following formula.

$$\mu \leq \mu_0 + \mu_1 + n_0\gamma_1 + n_1\gamma_0 \tag{7}$$

By Theorem 1, every graph on n vertices has at most $3^{\frac{n}{3}}$ maximal independent sets. Inserting this in (7) and using our induction assumption, we obtain the following.

$$\mu \leq 10^{\frac{n_0}{5}} + 10^{\frac{n_1}{5}} + n_0 3^{\frac{n_1}{3}} + n_1 3^{\frac{n_0}{3}} = f(10^{\frac{1}{5}}, 3^{\frac{1}{3}}, n_0, n_1) \tag{8}$$

Now, our aim is to apply Lemma 2 on inequality (8) for values of n_0 and n_1 that satisfy the premises of that lemma, which will immediately imply that $\mu \leq 10^{\frac{n}{5}}$ for large enough values of n. However, before we can do that, for the soundness of our induction, we need to show that for all smaller values of n_0 and n_1, it holds that $\mu \leq 10^{\frac{n_0+n_1}{5}} = 10^{\frac{n}{5}}$. We use Table 1 to help us keep track of the possible cases. For all values of n_0 and n_1 such that $n_0+n_1 \leq 5$, we know by the induction base case that $\mu \leq 10^{\frac{n}{5}}$. Each combination of pairs of such values is marked with "≤ 5" in Table 1. We now examine all pairs of values of n_0 and n_1 with $n_0+n_1 > 5$, and $n_0 \leq i$ and $n_1 \leq j$ for some pair $(i, j) \in \mathcal{B} = \{(1, 10), (2, 8), (3, 7), (4, 5), (5, 4), (7, 3), (8, 2), (10, 1)\}$. Once we have shown that $\mu \leq 10^{\frac{n}{5}}$ for all such values, we can apply Lemma 2 for all larger values. Pairs of small n_0, n_1 values for which Lemma 1 can be applied are marked with "OK" in Table 1. Clearly we can also apply the lemma on all larger values of n_0 and n_1.

We will now analyse, in the correct induction order, the values of n_0 and n_1 in the area between those entries that are marked "≤ 5" and those that are marked "OK" in Table 1. Let us start with $n_0 = 1$ and $n_1 = 5$. The single vertex of G_0 is either included in a maximal induced forest F of G, in which case F must be a maximal independent set in G_1, or it is not, in which case F is a maximal induced forest in G_1. Hence the number of maximal induced forests in G is at most the number of maximal induced forests in G_1 plus the number of maximal independent sets in G_1, i.e., $\mu \leq \mu_1 + \gamma_1$. Using Figure 3, we can verify that the number of maximal induced forests plus the number of maximal independent sets is at most 15 for any cograph on 5 vertices. Since $15 < 10^{\frac{6}{5}}$, the statement of the theorem holds for this case. This case is marked with "$C1$" in Table 1.

Now consider the case where $n_0 = 2$ and $n_1 = 4$. Observe first that if a vertex of G_0 is universal, then it can be moved to G_1, and G is still a complete join of the modified graphs G_0 and G_1. By this operation n_0 becomes 1, n_1 becomes 5, and we are back in Case $C1$. Therefore, assume now that the two vertices of G_0 are not adjacent. Let F be a maximal induced forest in G. If F contains 0 vertices from G_0, then F is a maximal induced forest in G_1. If F contains exactly 1 vertex from G_0, then F is a maximal independent set in G_1. Finally, if F contains both vertices of G_0, then F contains exactly 1 vertex from G_1. This yields the formula $\mu \leq \mu_1 + 2\gamma_1 + n_1$. For any cograph G_1 on 4 vertices in which the number of maximal induced forests plus 2 times the number of maximal independent sets is at most 11 we are fine, as

n_0 \ n_1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	≤ 5	≤ 5	≤ 5	≤ 5	$C1$	$R1$	$R2$	$R3$	$R4$	OK
2	≤ 5	≤ 5	≤ 5	$C2$	$C4$	$C6$	$C7$	OK	OK	OK
3	≤ 5	≤ 5	$C3$	$C5$	$C8$	$C9$	OK	OK	OK	OK
4	≤ 5	$C2$	$C5$	$C10$	OK	OK	OK	OK	OK	OK
5	$C1$	$C4$	$C8$	OK	OK	OK	OK	OK	OK	OK
6	$R1$	$C6$	$C9$	OK	OK	OK	OK	OK	OK	OK
7	$R2$	$C7$	OK	OK	OK	OK	OK	OK	OK	OK
8	$R3$	OK	OK	OK	OK	OK	OK	OK	OK	OK
9	$R4$	OK	OK	OK	OK	OK	OK	OK	OK	OK
10	OK	OK	OK	OK	OK	OK	OK	OK	OK	OK

Table 1. This table aids in the analysis of the maximum number of maximal induced forests in a cograph that is the join of two cographs on n_0 and n_1 vertices, respectively.

$11+4 < 10^{\frac{6}{5}}$. The only graph violating this condition is K_4 (see Figure 3). When G_1 is K_4, we notice that a maximal independent set of size 1 in G_1 will not be maximal induced forest when joined with only one vertex of G_0. Thus the term $2\gamma_1$ can be ignored, and we get $\mu \leq \mu_1 + n_1 \leq 6 + 4 \leq 10$. This case is marked with "$C2$" in Table 1.

The next case is when $n_0 = 3$ and $n_1 = 3$. Universal vertices can again be moved between G_0 and G_1, and due to symmetry this argument holds in both directions. We see in Figure 3 that cographs on 3 vertices without a universal vertex have at most one maximal induced forest and at most two maximal independent sets. Thus we get that $\mu \leq \mu_0 + \mu_1 + n_0\gamma_1 + n_1\gamma_0 \leq 1 + 1 + 3 \cdot 2 + 3 \cdot 2 = 14 < 10^{\frac{6}{5}}$. This case is marked with "$C3$" in Table 1.

Consider next a more general argument for the cases where $n_0 = 1$ and $n_1 = n - 1 \geq 6$. For the same reason as in Case $C1$, we have that $\mu \leq \mu_1 + \gamma_1$. Using the induction assumption, $\mu_1 \leq 10^{\frac{n_1-1}{5}} \leq 10^{\frac{n-1}{5}}$, we can obtain the desired upper bound in this case by proving that $10^{\frac{n-1}{5}} + 3^{\frac{n-1}{3}} \leq 10^{\frac{n}{5}}$. We can rewrite this as $(3^{\frac{1}{3}}/10^{\frac{1}{5}})^{n-1} \leq 10^{\frac{1}{5}} - 1$, which is true for every $n \geq 7$. However, this can be used for each such n only after our proof has covered all (n_0, n_1) pairs such that $n_0 + n_1 \leq n - 1$. Thus at this point, it can only be used to cover $(1, 6)$ and $(6, 1)$. Accordingly, we mark cells $(1, 6)$ and $(6, 1)$ with "$R1$" in Table 1.

The next case is when $n_0 = 2$ and $n_1 = 5$. As $n_0 = 2$, we have the formula $\mu \leq \mu_1 + 2\gamma_1 + n_1$ as we saw in Case $C2$. When G_1 is the disjoint union of a triangle and an edge, we get that $\mu \leq 3+2\cdot 6+5 = 20 < 10^{\frac{7}{5}}$. By considering the numbers for the remaining graphs on 5 vertices in Figure 3, we get $\mu \leq 10 + 2 \cdot 5 + 5 = 25 < 10^{\frac{7}{5}}$. This case is marked with "$C4$" in Table 1.

The next case is when $n_0 = 3$ and $n_1 = 4$. Note that G_0 does not contain universal vertices, as otherwise these can be moved to G_1 and we get case $(2, 5)$, which is already covered. Examining Figure 3, G_0 has at most one maximal induced forest and at most two maximal independent sets. We will use the formula $\mu \leq \mu_0+\mu_1+n_1\gamma_0+n_0\gamma_1$. If G_1 is K_4, then no maximal induced forest in G consists of only one vertex from G_0 and a maximal independent set of G_1, so the term $n_0\gamma_1$ can be ignored in this case. Consequently, we get $\mu \leq \mu_0+\mu_1+n_1\gamma_0 \leq 1+6+4\cdot 2 = 15 < 10^{\frac{7}{5}}$. For any remaining cograph on 4 vertices, the maximum number of maximal induced forests is 4, and the maximum number of maximal independent sets is 4, and we get that $\mu \leq \mu_0 + \mu_1 + n_0\gamma_1 + n_1\gamma_0 \leq 1 + 4 + 4 \cdot 2 + 3 \cdot 4 = 25 < 10^{\frac{7}{5}}$. In Table 1 this case is marked with "$C5$".

Recalling Case $R1$, by the arguments so far, the cells of Table 1 marked with "$R2$" have also been covered.

At this point, we have verified that the statement of the theorem holds for all cographs on at most 7 vertices. The two cases where $n_0 = 2$ and $n_1 \in \{6, 7\}$ are considered next. As the numbers of maximal induced forests and maximal independent sets are integers, using the induction assumption, we get the formula for $n_0 = 2$ that we saw in Case $C2$, i.e., $\mu \leq \mu_1+2\gamma_1+n_1 \leq \lfloor 10^{\frac{n_1}{5}} \rfloor+2\cdot\lfloor 3^{\frac{n_1}{3}} \rfloor+n_1$. We want to argue that this is at most $10^{\frac{n_1+2}{5}}$. For $n_1 = 6$, we have $\lfloor 10^{\frac{6}{5}} \rfloor+2\cdot\lfloor 3^{\frac{6}{3}} \rfloor+6 \leq 15+2\cdot 9+6 \leq 39 < 10^{\frac{8}{5}}$. For $n_1 = 7$, we have $\lfloor 10^{\frac{7}{5}} \rfloor + 2 \cdot \lfloor 3^{\frac{7}{3}} \rfloor + 7 \leq 25 + 2 \cdot 12 + 7 \leq 56 < 10^{\frac{9}{5}}$. These cases are marked with "$C6$" and "$C7$" in Table 1.

We continue now with the case where $n_0 = 3$ and $n_1 \in \{5, 6\}$. Like for the case $n_0 = 2$, we can notice that any universal vertex in G_0 can be moved to G_1, yielding a situation covered by either Case $C6$ or $C7$. Hence we assume that G_0 does not contain a universal vertex. For $n_1 = 5$, we use the upper bounds for graphs on 5 vertices given in Figure 3 to get $\mu \leq \mu_0 + \mu_1 + n_0\gamma_1 + n_1\gamma_0 \leq 1 + 10 + 5 \cdot 2 + 3 \cdot 6 = 39 < 10^{\frac{8}{5}}$. This case is marked with "$C8$" in Table 1. For $n_1 = 6$, the argument is quite similar: $\mu \leq \mu_0 + \mu_1 + n_1\gamma_0 + n_0\gamma_1 \leq 1 + 10^{\frac{6}{5}} + 6 \cdot 2 + 3 \cdot 3^{\frac{6}{3}} < 55 < 10^{\frac{9}{5}}$. This case is marked with "$C9$" in Table 1.

The only remaining case is when $n_0 = 4$ and $n_1 = 4$. If either G_0 or G_1 has a universal vertex, then we can move one vertex and reach a case that is already covered. Examining Figure 3, G_0 has at most 4 maximal induced forests and at most 4 maximal independent sets, and G_1 is either a cycle of length 4 or has at most 3 maximal induced forests and 4 maximal independent sets. If G_1 is a cycle of length 4, we get that $\mu \leq \mu_0 + \mu_1 + n_0\gamma_1 + n_1\gamma_0 \leq 4 + 4 + 4 \cdot 4 + 4 \cdot 2 = 32 < 10^{\frac{8}{5}}$. For the remaining case, we get that $\mu \leq \mu_0+\mu_1+n_0\gamma_1+n_1\gamma_0 \leq 4+3+4\cdot 4+4\cdot 4 = 39 < 10^{\frac{8}{5}}$. This last case is marked with "$C10$" in Table 1.

By the arguments so far, the cells of Table 1 marked with "$R3$" and "$R4$" have also been covered. We can now safely apply Lemma 2 on all other values of n_0 and n_1, and the statement of the theorem follows. □

5 Circular arc graphs and concluding remarks

We have shown that the maximum number of minimal feedback vertex sets in chordal graphs and cographs is at most $10^{\frac{n}{5}}$, and that this bound is tight. Recall that the example giving the lower bound in Lemma 1 is a family of disconnected graphs. This raises the following question: are there infinite families of *connected* chordal graphs or cographs that have $10^{\frac{n}{5}}$ minimal feedback vertex sets? Although we leave open this question, we show below that the gap can be made arbitrarily small above zero.

Lemma 3. *There are infinite families of connected chordal graphs, cographs, and circular arc graphs on n vertices that have $10^{\frac{n-1}{5}} + 5^{\frac{n-1}{5}}$ minimal feedback vertex sets.*

Proof. To create such a graph, we take arbitrarily many copies of K_5 and an additional vertex v. Now we add an edge between v and every vertex of each copy of K_5. The resulting graph G is chordal, cograph, and circular arc. The number of maximal induced forests of G is equal to the number of such forest that do not contain v plus the number of such forests that do contain v. The first number is $10^{\frac{n-1}{5}}$ by the proof of Lemma 1. The second number is $5^{\frac{n-1}{5}}$ since every maximal induced forest of this type contains exactly one vertex of each K_5 in addition to v. □

Fomin et al. [10] give an example of an infinite family of graphs having $105^{\frac{n}{10}} \approx 1.593^n$ minimal feedback vertex sets, providing the best known lower bound for general graphs. Interestingly, their example is a disjoint union of copies of a particular proper circular arc graph. This proper circular arch graph has 10 vertices and 105 minimal feedback vertex sets [10], and it is shown in Fig. 4. An infinite family of graphs having $105^{\frac{n}{10}}$ minimal feedback vertex sets is obtained by taking $\frac{n}{10}$ copies of this graph. Note, however, that disjoint unions of circular arc graphs are not necessarily circular arc graphs. In particular, this obtained family does not belong to the class of circular arc graphs. Consequently, two interesting questions emerge: What is the upper bound for disjoint unions of circular arc or proper circular arc graphs; can it be that it matches the lower bound? What is the maximum number of minimal feedback vertex sets in circular arc graphs?

Using our upper bound on chordal graphs, we show here that the maximum number of minimal feedback vertex sets of a circular arc graph is $O(1.585^n)$. Let G be a circular arc graph and let $\mathcal{C}$ be a circular arc representation of it. It is well known that we can assume that no two arcs share an endpoint, see e.g. [19]. Kloks, Kratsch, and Wong [19] place $2n$ points on the circle of $\mathcal{C}$ that they call *scanpoints* as follows: place a scanpoint between every two consecutive arc-endpoints, traversing the circle clock-wise. For each scanpoint p of $\mathcal{C}$, let $B(p)$ be the set of arcs that contain point p. We will call the vertex set corresponding to $B(p)$ a *breaker* of G,

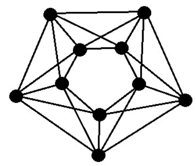

Fig. 4. A proper circular arc graph on 10 vertices that has 105 minimal feedback vertex sets [10].

for each scanpoint p. It is clear that every breaker B is a clique in G. Furthermore, for each breaker B in G, removing B from G results in a chordal graph, due to the well-known characterization of chordal graphs being intersection graphs of subtrees of a tree [4]. We will first show a bound for proper circular arc graphs using the following result by Golumbic [16].

Theorem 4 ([16]). *If G is a proper circular arc graph, then G has a proper circular arc representation in which no two arcs share a common endpoint and no to arcs together cover the whole cycle.*

Theorem 5. *Every proper circular arc graph on n vertices has at most $2n \cdot 10^{\frac{n-1}{5}}$ maximal induced forests.*

Proof. Let $G = (V, E)$ be a proper circular arc graph and let $\mathcal{C}$ be a proper circular arc representation of it with scanpoints. We claim that for every set $A \subseteq V$ such that $G[A]$ is a maximal induced forest of G, there is a breaker B of G such that $A \cap B = \emptyset$. To see this, assume the opposite for a contradiction: let A be such that $G[A]$ is a maximal induced forest of G, and A has a non-empty intersection with every breaker of G. This means in particular that A contains a set C of vertices whose arcs cover all the scanpoints of $\mathcal{C}$. By the way the scanpoints are defined, the arcs of the vertices in C cover the whole circle. By Theorem 4, C contains at least three vertices, and hence $G[C]$ contains a cycle, contradicting that $G[A]$ is a forest.

Since there are at most $2n$ breakers, every breaker contains at least one vertex, the removal of a breaker results in a chordal graph, and for every maximal induced forest there is a breaker it does not intersect, we get by Theorem 2 that G has at most $2n \cdot 10^{\frac{n-1}{5}}$ maximal induced forests. $\square$

For circular arc graphs that are not proper, we cannot use the nice property given by Theorem 4, and we have to deal with the case where two arcs cover the whole circle in the circular arc model.

Theorem 6. *Every circular arc graph on n vertices and m edges has at most $3m \cdot 10^{\frac{n-1}{5}}$ maximal induced forests.*

Proof. Let $G = (V, E)$ be a proper circular arc graph and let $\mathcal{C}$ be a proper circular arc representation of it with scanpoints. By the arguments in the proof of Theorem 5, if a maximal induced forest has a non-empty intersection with all the breakers of G, then we know that it contains two vertices whose arcs cover the whole circle in $\mathcal{C}$. Let A be such that $G[A]$ is a maximal induced forest, and let u, v be two vertices in A such that the arcs of u, v cover the whole circle in $\mathcal{C}$. This means that $uv \in E$ and $N_G(\{u, v\}) \cup \{u, v\} = V$. Consequently, $A \setminus \{u, v\}$ must be a maximal independent set in G, as otherwise there would be a cycle in $G[A]$. Hence the number of maximum induced forests of G of this type is at most m times the number of maximal independent sets of $G[V \setminus \{u, v\}]$. By Theorem 1 this number is thus $m \cdot 3^{\frac{n-2}{3}}$. Adding this to the number of maximal induced forests of the type studied in the proof of Theorem 5, we get that the total number is at most $m \cdot 3^{\frac{n-2}{3}} + 2n \cdot 10^{\frac{n-1}{5}} \leq 3m \cdot 10^{\frac{n-1}{5}}$. $\square$

Theorem 6 implies that the maximum number of minimal feedback vertex sets in a circular arc graph is at most $O(n^2)\, 10^{\frac{n}{5}} = O(1.585^n)$. A combinatorial upper bound without the use of O-notation remains an open question both for this class and for proper circular arc graphs.

As a consequence of our results and the result of Schwikowski and Speckenmeyer [25], all minimal feedback vertex sets, or equivalently all maximal induced forests, of a chordal graph, cograph, or circular arc graph can be listed in time $O(1.585^n)$. As a final question, we ask whether the exact number of minimal feedback vertex sets of a given graph can be computed in polynomial time for these three graph classes. For cographs, arguments along the lines of the results by Bui-Xuan et al. [3] are likely to work for a polynomial-time algorithm, since cographs have bounded clique-width. We would not be surprised if also for chordal graphs the counting problem could be solved in polynomial time.

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