Broadcast Domination on block graphs in linear time*

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Abstract. A broadcast domination on a graph assigns an integer value $f(u) \geq 0$ to each vertex u, such that every vertex u with $f(u) = 0$ is within distance $f(v)$ from a vertex v with $f(v) > 0$. The Broadcast Domination problem seeks to compute a broadcast domination where the sum of the assigned values is minimized. We show that Broadcast Domination can be solved in linear time on block graphs. For general graphs the best known algorithm runs in time $\mathcal{O}(n^6)$. For trees and interval graphs, linear-time algorithms are known. As block graphs form a superclass of trees, our result extends the classes of graphs on which this problem is solvable in linear time.

1 Introduction

Dominating Set and its variations are some of the most studied problems in graph theory; the number of papers on domination in graphs is in the thousands and there are several well known surveys and books on the topic, e.g., [7–9]. In 2002, Erwin [6] introduced the notion of *broadcast domination*. A *broadcast* on an undirected unweighted graph is a function from the vertices of the graph to non-negative integers. Such a function f is a *dominating broadcast* if every vertex v either has $f(v) > 0$ or is at distance at most $f(u)$ from another vertex u. The problem of computing a dominating broadcast that minimizes the sum of the function values of all the vertices is referred to as Broadcast Domination.

A few years after Erwin introduced the problem, it was discovered that it can be solved in polynomial time on all graphs [10]. However the running time of the algorithm for arbitrary graphs on n vertices is $\mathcal{O}(n^6)$. Algorithms for Broadcast Domination that run in time linear in the number of vertices and edges of the graph are known for interval graphs [2] and for trees [4]. Trees are also studied combinatorially with respect to their optimal dominating broadcast functions [11, 3].

In this paper, we extend the results on trees by presenting a linear-time algorithm for Broadcast Domination on block graphs. Block graphs are the graphs in which each cycle is a clique, and they are thus a superclass of trees. Interestingly, our approach is substantially different and simpler than the one used in the best known algorithm for trees [4]. However, the way to our algorithm

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goes through a number of new structural results on dominating broadcasts which we prove first.

2 Definitions and notation

We work on undirected, unweighted, connected graphs, denoted by $G = (V, E)$. The set of vertices and the set of edges of a graph G are also denoted by $V(G)$ and $E(G)$, respectively. Throughout the paper we will use $n = |V(G)|$ and $m = |E(G)|$. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$, is the minimum number of edges on a path between u and v. The *eccentricity* of a vertex v, denoted by $e(v)$, is the largest distance from v to to any vertex of G. The *radius* of G, denoted by $rad(G)$, is the smallest eccentricity in G. The *diameter* of G, denoted by $diam(G)$, is the largest eccentricity in G. The *length* of a path is the number of edges it contains. A path is a *shortest path* in G if its length is equal to the distance between its end vertices. We define a *diametral path* in G to be a shortest path of length $diam(G)$.

Given a vertex u in G and a positive integer k, the *ball* $B_G(u, k)$ is the set $\{x \in V(G) \mid d_G(u, x) \leq k\}$. The *neighbourhood* of a vertex v in G is the set of all the vertices adjacent to v, denoted by $N_G(v)$. The *degree* of a vertex v is defined by $deg_G(v) = |N_G(v)|$. We also define $N_G[v] = N_G(v) \cup \{v\}$. The neighbourhood of a set of vertices S, denoted $N_G(S)$, is the set of all vertices adjacent to vertices of S, but which are not in S themselves. Thus, $N_G(B_G(v, k)) = B_G(v, k+1) \setminus B_G(v, k)$. If G is clear from context, we omit the subscript in these definitions. Two sets S_1 and S_2 of vertices of a graph G are *adjacent* if there exists a vertex $v_1 \in S_1$ and a vertex $v_2 \in S_2$ such that v_1 and v_2 are adjacent in G. A set of vertices $S \subset V(G)$ is a *separator* if $G[V(G) \setminus S]$ is disconnected. If a single vertex is a separator then it is called a *cut-vertex*. If P is a path then xPy refers to the subpath in P from x to y, i.e., x_3Px_5 is the path x_3, x_4, x_5 if $P = x_1, x_2, x_3, x_4, x_5, x_6, x_7$.

A function $f : V \to \{0, 1, \cdots, diam(G)\}$ is a *broadcast* on $G = (V, E)$. The set of *broadcast dominators* defined by f is the set $V_f = \{v \mid f(v) > 0\}$. A vertex u is *dominated* if there is a vertex $v \in V_f$ such that $d(u, v) \leq f(v)$. In that case we say that u is dominated by v. A broadcast on G is *dominating* if every vertex in $V(G)$ is dominated. In this case f is also called a *broadcast domination*. The *cost* of a dominating broadcast f on a subset $S \subseteq V$ is $\sigma_f(S) = \sum_{v \in S} f(v)$. We denote $\sigma_f(V)$ also as $\sigma_f(G)$, and refer to it as *the cost of* f *on* G. Note that there is always a dominating broadcast of cost $rad(G)$.

The *broadcast domination number*, $\gamma_b(G)$ is the smallest cost of a dominating broadcast on G. If $\sigma_f(G) = \gamma_b(G)$ then we call f an *optimal* broadcast on G. For a disconnected graph, an optimal dominating broadcast is the union of optimal dominating broadcasts of its connected components. This justifies that it is sufficient to study the problem on connected graphs. A dominating broadcast is *efficient* if every vertex is dominated by exactly one vertex. Dunbar et al. [5] proved that for any non-efficient dominating broadcast f on a graph $G = (V, E)$, there is an efficient dominating broadcast f' on G such that $|V_{f'}| < |V_f|$ and $\sigma_{f'}(G) = \sigma_f(G)$. An efficient dominating broadcast f on $G = (V, E)$ defines a *domination graph* $G_f = (V_f, E_f)$ such that there is an edge between two vertices $u, v \in V_f$ if and only if $B_G(u, f(u))$ is adjacent to $B_G(v, f(v))$ in G. See Fig. 1 for an example. Heggernes and Lokshtanov [10] proved that for any efficient dominating broadcast f on G, if G_f has a vertex of degree more than 2 then there is an efficient dominating broadcast f' on G such that $|V_{f'}| < |V_f|$ and $\sigma_{f'}(G) = \sigma_f(G)$.

We define a *sparse* broadcast on G to be an optimal broadcast f such that $|V_f|$ is minimized. As a consequence of the above, if a given optimal broadcast f on G is sparse then f is efficient and G_f is a path or a cycle.

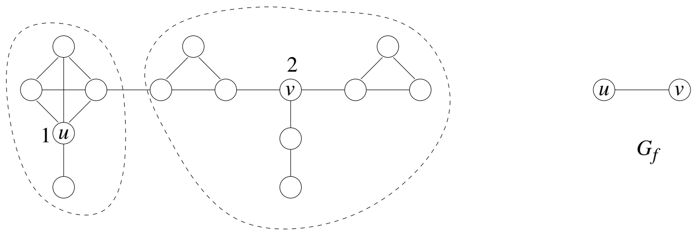

Fig. 1: A block graph G with a sparse broadcast f is shown on the left, where the broadcast dominators are u and v, with $f(u) = 1$ and $f(v) = 2$. The balls $B(u, 1)$ and $B(v, 2)$ are indicated on G. On the right G_f is shown.

3 General properties of dominating broadcasts

We start by a simple observation that will allow us to replace the broadcast dominators of an induced subgraph with others, without affecting the rest of the broadcast dominators.

Observation 1 *Let $G = (V, E)$ be a graph, $A \subseteq V$, and $G' = G[A]$. If G has an optimal broadcast f such that $\bigcup_{a \in V_f \cap A} B(a, f(a)) = A$, and f' is an optimal broadcast on G', then G has an optimal broadcast f^* defined as follows:*

$$f^*(x) = \begin{cases} f'(x) & \text{if } x \in A \\ f(x) & \text{if } x \in V \setminus A \end{cases}$$

Proof. All the vertices that were dominated by vertices in $V_f \setminus A$, are dominated by the same vertices in V_{f^*}, and all the vertices that were dominated by vertices in A are dominated by vertices in $V_{f'}(G')$. So f^* is a dominating broadcast.

Since $\bigcup_{a \in V_f \cap A} B(a, f(a)) = A$, the restriction of f to A is a dominating broadcast on G'. Since f' is an optimal dominating broadcast on G', we have $\sum_{x \in A} f'(x) \leq \sum_{x \in A} f(x)$. Since the function values of the vertices outside of A did not change, we obtain that $\sigma_{f^*}(G) \leq \sigma_f(G)$ and hence f^* is an optimal dominating broadcast on G. $\square$

The next lemmas and corollaries provide us with tools to decide whether a given dominating broadcast is sparse.

Lemma 1. *Let $B_1 = B(v, k_v)$ and $B_2 = B(u, k_u)$ be two balls in a graph G. If $k_v \leq k_u$, then there exists a vertex z in G such that $B_1 \cup B_2 \subseteq B(z, \ell)$ where*

$$\ell = \begin{cases} k_u & \text{if } B_1 \subseteq B_2 \\ \left\lceil \frac{d_G(u,v)+k_v+k_u}{2} \right\rceil & \text{otherwise} \end{cases}$$

Proof. If B_1 is a subset of B_2, then let $z = u$ and observe that $B_1 \cup B_2 \subseteq B(z, \ell) = B_2$. Otherwise, let z be a vertex on a shortest path from u to v such that $d_G(u, z) = \lceil (d_G(u, v) + k_v - k_u)/2 \rceil$. Then $d_G(z, x) \leq k_u + d_G(z, u) = \ell$ for every vertex x in B_2, and hence $B_2 \subseteq B(z, \ell)$. For every vertex y in B_1, $d_G(z, y) \leq k_v + d_G(v, z) = k_v + d_G(v, u) - d_G(u, z) \leq \ell$. Consequently, also $B_1 \subseteq B(z, \ell)$. □

By excluding the cases from the above formula when $\left\lceil \frac{d_G(u,v)+k_v+k_u}{2} \right\rceil < k_u$, we can simplify the result to the following corollary.

Corollary 1. *Let $B_1 = B(v, k_v)$ and $B_2 = B(u, k_u)$ be two balls in a graph G. If $k_u + k_v \geq d_G(u, v) + 2k$ for an integer $k \leq \min\{k_u, k_v\}$, then there exists a vertex $z \in G$ such that $B_1 \cup B_2 \subseteq B(z,\ k_u + k_v - k)$.*

Lemma 2. *Let x, y, z be three vertices in a graph G. Let P_y be a shortest path from x to y in G, and let P_z be a shortest path from x to z in G. If $P_y \cap P_z$ contains more vertices than x, and the integers k_x, k_y, k_z are such that $B(x, k_x)$ is adjacent to both $B(y, k_y)$ and $B(z, k_z)$, then there exists a vertex v in G such that $B(x, k_x) \cup B(y, k_y) \cup B(z, k_z) \subseteq B(v, k_x + k_y + k_z)$.*

Proof. If there exists any vertex other than x in $P_y \cap P_z$, then any shortest path from x to that other vertex is a subpath of a shortest path to both y and z from x. Consequently, the vertex u adjacent to x in P_y must be 1 closer to both z and y than x is. Since u and x are adjacent, we notice that $B(x, k_x) \subseteq B(u, k_x + 2)$. We observe that $B(u, k_x + 2)$ overlaps $B(z, k_z)$ in such a way that we can apply Corollary 1. Therefore, there exists a vertex w such that $B(u, k_x + 2) \cup B(z, k_z) \subseteq B(w, k_z + k_x + 1)$. And this ball overlaps $B(y, k_y)$ in such a way that, again by Corollary 1, there exists a vertex v such that $B(w, k_z + k_x + 1) \cup B(y, k_y) \subseteq B(v, k_z + k_x + k_y)$. Hence,

$$\begin{aligned} B(x, k_x) \cup B(y, k_y) \cup B(z, k_z) &\subseteq B(y, k_y) \cup B(u, k_x + 2) \cup B(z, k_z) \\ &\subseteq B(y, k_y) \cup B(w, k_z + k_x + 1) \\ &\subseteq B(v, k_z + k_x + k_y) \end{aligned}$$

□

Corollary 2. *Let f be a sparse broadcast on a graph $G = (V, E)$, and let x, y and z be distinct vertices in V_f such that y and z are adjacent to x in G_f. Then the intersection of a shortest path from x to y in G and a shortest path from x to z in G is exactly the set $\{x\}$.*

4 Structural properties of dominating broadcasts on block graphs

A graph is a *block graph* if the vertices of every cycle form a clique. The following theorem will be used in our proofs.

Theorem 1 ([1, 12]). *Given four vertices x, y, z and w in a block graph G, the two largest of the three following sums are equal:*

1. $d(x, y) + d(z, w)$
2. $d(x, z) + d(y, w)$
3. $d(x, w) + d(y, z)$

Since every cycle in a block graph is a clique and any shortest path contains at most two vertices of a clique, it follows that a shortest path between any pair of vertices is unique in a block graph. In particular, every vertex on a shortest path between two vertices s and t is a cut-vertex, separating s from t.

Lemma 3. *If G is a block graph then $rad(G) = \lceil diam(G)/2 \rceil$.*

Proof. If G is a complete graph then the statement trivially follows. Assume that G is not complete and let $P = s, \ldots, t$ be a diametral path in G. Then there is a vertex $x \in P$ such that $\min\{d(s, x), d(t, x)\} = \lfloor diam(G)/2 \rfloor$. Recall that x is a cut-vertex. We show that $e(x) \leq \lceil diam(G)/2 \rceil$. Assume for contradiction that there is a vertex y such that $d(x, y) > \lceil diam(G)/2 \rceil$. Since x disconnects s and t, x must also disconnect y from either s or t. Without loss of generality, let x disconnect y and t. In this case, $d(y, t) = d(y, x) + d(x, t)$. Since $d(y, x) > \lceil diam(G)/2 \rceil$ and $d(t, x) \geq \lfloor diam(G)/2 \rfloor$, we reach the contradiction that $d(y, t) > diam(G)$. Hence we conclude that $e(x) \leq \lceil diam(G)/2 \rceil$. Consequently $rad(G) \leq \lceil diam(G)/2 \rceil$. Since $rad(G) \geq \lceil diam(G)/2 \rceil$, the radius must be exactly $\lceil diam(G)/2 \rceil$. □

Lemma 4. *Let G be a block graph and s be a vertex of G. Then $e(s) = diam(G)$ if and only if G has a vertex x such that $d(x, s) \geq d(x, y)$ for all $y \in V(G)$.*

Proof. One direction is trivial as we can take x to be the end vertex of the diametral path starting from s. For the other direction, let x be a vertex such that $d(x, s) \geq d(x, y)$ for all $y \in V(G)$. Let $P = a, \ldots, b$ be a diametral path in G, and let $d_1 = d(a, b) + d(x, s)$, $d_2 = d(a, x) + d(b, s)$, $d_3 = d(a, s) + d(b, x)$. Since $d(x, s)$ is greater than or equal to both $d(a, x)$ and $d(b, x)$, and since $d(a, b)$ is greater than or equal to every distance in G, we see that the sum d_1 can be no less than either of d_2 or d_3. Without loss of generality, let $d_2 \leq d_3$. By Theorem 1 we know $d_1 = d_3$. Since $d(x, s) \geq d(x, b)$ we have that $d(a, b) \leq d(a, s)$. And since $d(a, b) = diam(G)$, we also have that $d(s, a) = diam(G)$, and hence $e(s) = diam(G)$. □

Observation 2 *Let G be a block graph and let f be a sparse broadcast on G. Then G_f is a path.*

Proof. As mentioned in the introduction, G_f is either a path or a cycle. Assume for contradiction that G_f is a cycle. Since f is efficient, V_f is an independent set in G. However, in combination with Corollary 2, this means we must have an induced cycle of length at least 6 in G, which contradicts that G is a block graph. □

Lemma 5. *Let G be a block graph and let f be a sparse broadcast on G. For every clique C of size at least* 3*, all vertices in C are dominated by the same broadcast dominator in V_f.*

Proof. Since G_f is a path and f is efficient by the above, the vertices of C can be dominated by at most two distinct vertices. C contains more than two vertices, therefore at least two vertices x and y in C must be dominated by the same vertex z in V_f. Since the distances in G between the vertices in C are all 1, we have by Theorem 1 that $d_G(w, z) \leq d_G(x, z) = d_G(y, z)$ for every vertex $w \in C$. Therefore, every vertex w in C is dominated by z. □

5 Algorithmic properties of dominating broadcasts on block graphs

Recall that any clique intersects with a diametral path in at most two vertices. Given a diametral path P, we define $C(P)$ to be the union of all cliques that intersect with P in exactly two vertices. A set $C \subseteq V(G)$ is called a *core* of G if $C = C(P)$ for a diametral path P in G. Through the series of lemmas of this section we will prove that in every block graph $G = (V, E)$ there is an optimal broadcast f and a core C, such that every dominator in V_f belongs to C. Finally Lemma 11 will enable us to find the dominators and their respective weights in f, one by one, as will be described in the resulting algorithm in the next section. We start by defining two operations. These operations are illustrated in Fig. 2.

Definition 1. *Given a ball $B = B(a_0, k)$ and a path $P = a_0, a_1, \ldots, a_p$ in a graph G, and a positive integer $l \leq p$, an* increase of l vertices along P for B *is the ball $B(a_l, k + l)$. This will also be referred to as the ball* $\textsc{Increase}(B, P, l)$.

Definition 2. *Given a ball $B = B(a_0, k)$ and a path $P = a_0, a_1, \ldots, a_p$ in a graph G, and a positive integer $l \leq \min\{k - 1, p\}$, a* decrease of l vertices along P for B *is the ball $B(a_l, k - l)$. This will also referred to as the ball* $\textsc{Decrease}(B, P, l)$.

For all graphs we can make the following observations, the first of which is immediate.

Observation 3 *Given a ball B_1, if $B_2 = \textsc{Increase}(B_1, aPb, l)$, then $B_1 = \textsc{Decrease}(B_2, bPa, l)$.*

Observation 4 *Given a ball $B = B(v, k)$ in a graph G, a path P of length ℓ in G from v to any vertex, and a positive integer $l \leq \ell$, $B \subseteq \textsc{Increase}(B, P, l)$.*

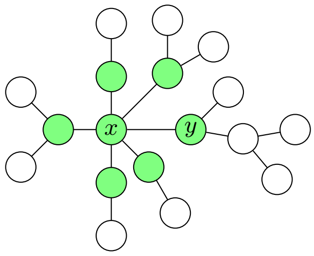

Fig. 2: A ball $B_1 = B(x, 1)$ in green to the left, and the ball $\textsc{Increase}(B_1, (x, y), 1) = B(y, 2)$ in blue to the right in the same graph.

Proof. Let $B(v', k') = \textsc{Increase}(B, P, l)$. Since $d_G(v, v') \leq l$, $d_G(v', x) \leq d_G(v', v) + d_G(v, x) \leq l + k = k'$, for every $x \in B$. Thus every vertex in B is also in $B(v', k')$. □

Lemma 6. *Let P be a shortest path between two vertices s and t in a block graph G. Let u be a vertex of P and let k_1 and k_2 be two positive integers such that $k_1 < k_2$ and $k_1 \leq d_G(u, s)$. Given the balls $B_1 = B(u, k_2)$ and $B_2 = \textsc{Decrease}(B_1, uPs, k_1)$, if $t \in B_2$ then $B_1 \cap B(s, d(s, t)) = B_2 \cap B(s, d(s, t))$.*

Proof. Let u' be the vertex on uPs such that $B(u', k_2 - k_1) = B_2$. It should be clear that uPs is the shortest path from u to s. Recall that in this case u' is a cut-vertex separating u and s. Thus, there is a set of vertices $S \subseteq B(s, d(s, t))$ such that every path from u to a vertex in S goes through u', and every path from s to a vertex in $T = B(s, d(s, t)) \setminus S$ goes through u'. The path uPu' is a shortest path since it is a sub path of P. Hence, $d(u, u') = k_1$. For every vertex $v \in S$, we thus have $d(u', v) = d(u, v) - k_1$. Consequently, every vertex in $B_1 \cap S$ is at distance at most $k_2 - k_1$ from u'. Hence $B_1 \cap S \subseteq B_2$.

Recall that u' is on every path from s to a vertex of T. Since every vertex $v \in T$ is at distance at most $d(s, t)$ from s, we get the following: $d(v, u') + d(u', s) = d(v, s) \leq d(s, t) = d(s, u') + d(u', t)$. Thus, each vertex in T is within distance $d(u', t) \leq k_2 - k_1$ from u'. Hence $T \subseteq B_2$. Hence $B_1 \cap B(s, d(s, t)) = B_2 \cap B(s, d(s, t))$. □

Lemma 7. *Let G be a block graph, let $P = s_o, \ldots, s_k, \ldots, s_p$ be a diametral path in G, and let $v \neq s_k$ be a vertex of G. If $2k \leq p$ and $s_0 \in B(v, k)$ then $B(v, k) \subset B(s_k, k)$.*

Proof. Since $s_0 \in B(v, k)$, we see that $d(s_0, x) \leq 2k$ for any vertex $x \in B(v, k)$. Hence, $B(v, k) \subseteq B(s_0, 2k)$. Note that the ball $B(s_k, k)$ is the same as the ball $\textsc{Increase}(B(s_0, 2k), P, k)$. If we let $B(s_p, p) = V(G)$, then by Lemma 6 we get that $B(s_k, k) = B(s_{2k}, 2k)$. We have thus proved that $B(v, k) \subseteq B(s, 2k) = B(s_k, k)$. However, since s_k is on the shortest path from s_0 to s_{2k} (the path s_0Ps_{2k}), s_k is a cut-vertex separating the two vertices. That means s_k is on the shortest path from v to either s_0 or s_{2k}. Since $d(v, s_0) \leq d(s_k, s_0)$, s_k must be on the path from v to s_{2k}. The distance from v to s_{2k} must therefore be larger than $d(s_k, s_{2k}) = k$. Consequently, $B(s_k, k) \neq B(v, k)$. Combining this with the above, we get that $B(v, k) \subset B(s_k, k)$. $\square$

Lemma 8. *Let G be a block graph, and let P be a diametral path $s, \ldots, t$. If f is a sparse broadcast on G, and $|V_f| > 1$, then s and t are dominated by two distinct vertices of P.*

Proof. Let s' be the vertex dominating s. Since $|V_f| > 1$ and f is optimal, the weight of every dominator must be less than $rad(G)$. So, $f(s') \leq rad(G) - 1 = \lceil d(s,t)/2 \rceil - 1$, by Lemma 3. The distance $d(s,t)$ is larger than $2(\lceil d(s,t)/2 \rceil - 1)$, so s' cannot dominate both s and t. We have by Lemma 7 that, unless s' is in P, there exists a vertex x such that $B(s', f(s')) \subset B(x, f(s'))$. However, in that case there must exist a non-efficient optimal broadcast g such that $|V_g| = |V_f|$. However, this makes g a sparse broadcast on G, and thus g must be efficient as argued in the preliminaries. This is a contradiction, and hence s' must be in P. The same proof is applicable for t. $\square$

Lemma 9. *Let G be a block graph and let C be a core of G. If f is a sparse broadcast on G such that $|V_f| > 1$, then each vertex in C is dominated by a dominator in C.*

Proof. Let $P = s_0, s_1 \ldots, s_p$ be a diametral path such that $C = C(P)$. Note that if each vertex on P is dominated by a vertex in C, then each vertex in $C \setminus P$ must also be dominated by a vertex in C by Lemma 5. By Lemma 8, vertices s_0 and s_p are dominated by vertices in $P \subseteq C$. Let s'_0 and s'_p be the vertices dominating s_0 and s_p, respectively. Let $s_i, s_{i+1}, \ldots, s_j$ be the vertices on P not dominated by s'_0 or s'_p and let x be the vertex dominating a vertex $s_d \in s_i P s_j$. Since either of s_i and s_j (alone) separates s'_0 and s'_p, the dominators of the vertices in $s_i P s_j$ must be between s'_0 and s'_p in G_f. Therefore, the degree of x in G_f must be exactly 2. Let x_l and x_r be the vertices in G_f to the left and right of x, respectively. Notice that C forms a union of some maximal cliques. By the definition of block graphs, each component of $G - C$ must therefore be adjacent to only one vertex in C. So, if x is not in C, then there must exist a single vertex $u \in C$ that disconnects x from C in G. However, then the shortest path from x to both x_l and x_r must contain u. By Lemma 2, this implies that u and x must be the same vertex, a contradiction. Therefore, if x is dominating a vertex in P, x itself must be in C. $\square$

Lemma 10. *Let G be a block graph and let C be a core of G. If f is a sparse broadcast on G and $|V_f| > 1$, then $V_f \subseteq C$.*

Proof. Recall that since f is sparse, it is efficient and G_f is a path. Let $P = s_0, s_1, \ldots, s_p$ be a diametral path in G, such that $C = C(P)$. Let s'_0 and s'_p be the vertices dominating s_0 and s_p, respectively. By Lemma 8, s'_0 and s'_p are on P, and by Lemma 9, the vertices between s'_0 and s'_p in G_f are in C. Therefore, if $V_f \not\subseteq C$, there must exist a vertex $x \in V_f \setminus C$ which is adjacent to s'_0 or s'_p in G_f. Without loss of generality, let x be adjacent to s'_0 in G_f. Let y be the vertex adjacent to s'_0 in G on the path $s'_0 P s_p$. We have two possibilities: $d_G(y,x) < d_G(s'_0, x)$ or $d_G(y,x) \geq d_G(s'_0, x)$.

In the first case y separates s'_0 and x in G. However, since y also separates in G s'_0 and the other neighbour of s'_0 in G_f, we know by Lemma 2 that f is not sparse, giving a contradiction. In the second case, y separates x and s_p. Note that $d_G(s_p, y) = d_G(s_p, s'_0) - 1$. Since $d_G(s'_0, s_0) \leq f(s'_0)$ and f is efficient, with $f(x) > 0$, we have $d_G(s'_0, x) = f(s'_0) + f(x) + 1 > d_G(s'_0, s_0) + 1$. This leads to the following contradiction: $d_G(s_p, x) = d_G(s_p, y) + d_G(y, x) > d_G(s_p, s'_0) + d_G(s'_0, s_0) = diam(G)$.

Consequently, there can be no vertex x in $G_f \setminus C$ if f is sparse. □

Lemma 11. *Let $P = s_0, s_1, \ldots, s_k, \ldots, s_p$ be a diametral path in a block graph G, let f be a sparse broadcast where $|V_f| > 1$, and assume that all vertices in $B(s_k, k)$ are dominated by the same dominator in V_f for $k \leq p/2$.*

If both of the following statements are true then there exists an optimal broadcast f' on G such that $f'(s_k) = k$, and if either of them is false then all vertices in $B(s_{k+1}, k+1)$ have the same dominator in V_f.

1. $d(s_0, x) = 2k + 1 \ \Rightarrow\ x = s_{2k+1}$
2. $d(s_p, x) = d(s_p, s_{2k}) + 1 \ \Rightarrow\ x \in N(s_{2k})$

Proof. Let s_i be the vertex that dominates $B(s_k, k)$. Since f is sparse, $f(s_i) = i$. Consequently, $i \geq k$. By Lemma 5, if s_{2k} and s_{2k+1} have a neighbour in common, they must have the same dominator, which implies that $i > k$, since $s_{2k+1} \notin B(s_k, k)$.

Assume that (1) is false, i.e., there is a vertex x such that $d(s_0, x) = 2k + 1$ and $x \neq s_{2k+1}$. If x is a neighbour of s_{2k+1} and s_{2k}, then $i > k$ as argued above, and hence all vertices in $B(s_{k+1}, k+1)$ are dominated by s_i. Assume that x is not adjacent to both of them. Hence s_{2k} separates s_{2k+1} and x, and $i = k$. But then the degree of s_i in G_f is larger than 1, which contradicts that G_f is a path.

Assume that (2) is false, i.e., there is a vertex x such that $d(s_p, x) = d(s_p, s_{2k}) + 1$ and $x \notin N(s_{2k})$. Let $x' \in C(P)$ be the vertex dominating x in f. We have $d(s_p, x') + d(x', x) \geq d(s_p, x) = d(s_p, s_{2k}) + 1 \geq d(s_p, x') + d(x', s_{2k})$. Hence, s_{2k} is dominated by x'. However, this is also true for s_{2k+1}, so $i > k$, as above.

Assume that both (1) and (2) are true. If $i = k$ then we are done. Assume that $i > k$. Let $R = B(s_p, d(s_p, s_{2k}))$ and let $B = B(s_i, i)$. Note that $d(s_{2k}, s_{i+k}) = i - k$, and thus, $s_{2k} \in B' = \textsc{Decrease}(B, s_i P s_p, k)$. By Lemma 6, this implies that $B' \cap R = B \cap R$. Since $B \setminus R \subset B(s_k, k)$, we now see that $B = B' \cup B(s_k, k)$. We can construct a new broadcast f' as follows:

$$f'(x) = \begin{cases} k & \text{if } x = s_k \\ 0 & \text{if } x = s_i \\ i - k & \text{if } x = s_{i+k} \\ f(x) & \text{otherwise} \end{cases}$$

By the above arguments, f' dominates the same vertices as f, and clearly $\sigma_{f'}(G) = \sigma_f(G)$. (Observe that now s_{2k} is dominated by both s_k and s_{i-k}, so f' is not efficient.) □

The above result can be used to construct a simple greedy algorithm for calculating an optimal broadcast domination on block graphs:

As long as $\gamma_b(G) < rad(G)$ there will always be a sparse broadcast f such that $|V_f| > 1$, and a diametric path $s_0, s_1, \ldots, s_p$ such that every vertex in $B(s_1, 1)$ are dominated by the same vertex. That creates the basis for Lemma 11. An algorithm can thus iterate through the possible values of k until both statements (1) and (2) of Lemma 11 are satisfied. When this happens, simply remove the dominated vertices $B(s_k, k)$, and repeat. Even though the graph is shrinking for each iteration, we remove non-overlapping induced subgraphs, and by Observation 1, all the individual optimal broadcast dominations for the subgraphs add up to an optimal dominating broadcast for the entire graph.

By the proof of Lemma 3, finding the optimal broadcast for G' when $\gamma_b(G') = rad(G)$, is as simple as picking the middle vertex in a diametral path. The knowledge that $\gamma_b(G') = rad(G)$ is obtained when the algorithm reaches a k such that $B(s_k, k) = V(G')$.

6 A linear-time algorithm for optimal broadcasts on block graphs

We have now all the results we need to present our algorithm.

Theorem 2. Broadcast Domination *can be solved in linear time on block graphs.*

Proof. As explained at the end of the previous section, the algorithm finds the leftmost dominator in an efficient optimal broadcast for G, according to Lemma 11. It then removes all the vertices dominated by this vertex, and finds the leftmost dominator in an efficient optimal broadcast for the remaining subgraph. This is repeated until there are no vertices left, and thus every vertex is dominated. By Observation 1 and Lemma 11, the resulting broadcast is optimal.

The algorithm starts by finding a vertex t such that $e(t) = diam(G)$, using Lemma 4. A diametral path between t and a vertex s is used in the algorithm. However, we do not need to compute every vertex in the path for each iteration. We can calculate each vertex s_i on the path $s = s_0, s_1, \cdots, s_i, \cdots, s_p = t$ as we need it without knowing the rest of the path: The vertex s_i is equivalent the vertex x such that $d_G(s_0, x) = i$ and $d_G(x, s_p) = p - i$.

We need not change t for each iteration either: Each time the graph G changes into a smaller graph G', every vertex to the left of a vertex s_j in the old diametral path are removed. However, since $diam(G) = d(s, t) = d(s, s_j) + d(s_j, t)$ for a vertex s to the left of s_j, no vertex to the right can be further away from s_j than t, and hence $e_{G'}(t) = diam(G')$, by Lemma 4.

The last *if*-statement in the algorithm, and the condition of the inner *while*-loop are both for the base case when there is only one dominator left.

Algorithm BLOCK GRAPH BROADCAST
Input: A block graph $G = (V, E)$
Output: An optimal broadcast f on G

```
for every v in V do
    f(v) ← 0
end for
x ← an arbitrary vertex in V
t ← a vertex at maximum distance from x
for i ← 0 to |V| − 1 do
    T[i] ← the set of vertices at distance i from t
    for v ∈ T[i] do
        D[v] ← i
    end for
end for
while V ≠ ∅ do
    s ← a vertex at maximum distance from t in G
    S[0] ← {s}
    S[1] ← N(s)
    S[2] ← N(S[1]) \ {s}
    k ← 0
    while |S[2k + 2]| > 0 do
        k ← k + 1
        S[2k + 1] ← the vertices at distance 2k + 1 from s
        S[2k + 2] ← the vertices at distance 2k + 2 from s
        s_{2k} ← the vertex in S[2k] such that D[s_{2k}] = D[s] − 2k
        if |S[2k + 1]| > 1 then continue
        if T[D[s] − 2k + 1] ⊂ N(s_{2k}) then break
    end while
    v ← the vertex in S[k] such that D[v] = D[s] − k
    if |S[2k + 2]| = 0 then k ← k + 1
    f(v) ← k
    V ← V \ B(v, f(v))
    G ← G[V]
end while
return f
```

The running time of the algorithm is $\mathcal{O}(n + m)$: We start with a $\mathcal{O}(n)$ loop initiating the values of f, and then populate T and D in $\mathcal{O}(n+m)$ by a breadth first search. The main loop might look like a bottle neck because of the time needed to populate entries of S in each iteration. However, each entry $S[i]$ can be calculated in time proportional to the cardinality of $N(S[i-1])$. Therefore, since each vertex appear in one and only one entry, the total running time to calculate all sets in S, is in fact no more than $\mathcal{O}(n+m)$. Hence, the total running time of the entire algorithm is also $\mathcal{O}(n+m)$. □

7 Concluding remarks

A graph is *chordal* if it does not contain an induced cycle of length 4 or more. Although block graphs and interval graphs are unrelated to each other, they are both subclasses of chordal graphs. It is easy to see that every chordal graph G has an optimal broadcast domination f such that G_f is a path. Following the results of [10], this immediately gives an $\mathcal{O}(n^4)$-time algorithm for BROADCAST DOMINATION on chordal graphs. Is there a linear-time or an $\mathcal{O}(n^2)$-time algorithm for BROADCAST DOMINATION on chordal graphs?

We believe that BROADCAST DOMINATION can be solved in general in $\mathcal{O}(n^5)$ time. Is there a faster algorithm for BROADCAST DOMINATION on arbitrary graphs?

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