Elsevier Editorial System(tm) for Information Processing Letters
Manuscript Draft

Manuscript Number:

Title: Enumerating Minimal Dominating Sets in Chordal Graphs

Article Type: Regular Submission

Keywords: Input sensitive enumeration; branching algorithm; maximum number of minimal dominating sets; chordal graphs.

Corresponding Author: Prof. Faisal N. Abu-Khzam, PhD

Corresponding Author's Institution: Lebanese American University

First Author: Faisal N. Abu-Khzam, PhD

Order of Authors: Faisal N. Abu-Khzam, PhD; Pinar Heggernes, PhD

Abstract: The maximum number of minimal dominating sets in a chordal graph on n vertices is known to be at most 1.6181^n. However, no example of a chordal graph with more than 1.4422^n minimal dominating sets is known. In this paper, we narrow this gap between the known upper and lower bounds by showing that the maximum number of minimal dominating sets in a chordal graph is at most 1.5214^n.

Enumerating Minimal Dominating Sets in Chordal Graphs

Highlights:

- A structural theorem about the distribution of simplicial vertices in a chordal graph
- An algorithm for enumerating minimal dominating sets in a chordal graph
- Upper bound on number of minimal dominating sets improved from 1.6181^n to 1.5214^n

Enumerating Minimal Dominating Sets in Chordal Graphs

Faisal N. Abu-Khzam[a], Pinar Heggernes[b]

[a] *Department of Computer Science and Mathematics*
Lebanese American University, Beirut, Lebanon
[b] *Department of Informatics, University of Bergen, Norway*

Abstract

The maximum number of minimal dominating sets in a chordal graph on n vertices is known to be at most 1.6181^n. However, no example of a chordal graph with more than 1.4422^n minimal dominating sets is known. In this paper, we narrow this gap between the known upper and lower bounds by showing that the maximum number of minimal dominating sets in a chordal graph is at most 1.5214^n.

Keywords: Input sensitive enumeration, branching algorithm, maximum number of minimal dominating sets, chordal graphs

1. Introduction

Enumerating the minimal dominating sets of a graph is one of the most studied and important problems within enumeration algorithms, especially since it corresponds to the problem of enumerating the minimal transversals of a hypergraph [8]. Whether this enumeration can be performed in output polynomial time is one of the biggest and longest standing open problems within enumeration algorithms. Many special cases have been studied throughout the last decades, and recently a polynomial delay algorithm for chordal graphs has been given by Kanté et al. [9]. The problem has also been handled with exponential-time branching algorithms, which at the same time give an upper bound on the maximum number of minimal dominating sets a graph can have. On arbitrary graphs with n vertices, the upper bound is shown by Fomin et al. [5] to be 1.7159^n, whereas no example of a graph that has more than 1.5704^n minimal dominating sets is known

Email addresses: `faisal.abukhzam@lau.edu.lb` (Faisal N. Abu-Khzam), `Pinar.Heggernes@uib.no` (Pinar Heggernes)

[5]. This gap between the known upper and lower bounds has triggered a flow of research on special cases, and on several graph classes one has found the exact bound for the maximum number of minimal dominating sets [2, 3]. On chordal graphs, however, a gap has remained, with the best known upper and lower bounds being respectively 1.6181^n and 1.4422^n, shown by Couturier et al. [2].

In this paper, we improve the upper bound, and we show that a chordal graph has at most 1.5214^n minimal dominating sets. An improved upper bound is of particular interest in light of the above mentioned recent polynomial delay algorithm for enumerating the minimal dominating sets of chordal graphs. It shows that such a polynomial delay algorithm will never spend more than $O(1.5214^n)$ time. It should be noted that chordal graphs form one of the most famous and widely studied graph classes, with a large variety of application areas [1, 7].

2. Preliminaries

We work with undirected and simple graphs. Such a graph $G = (V, E)$ is *non-trivial* if it contains at least one edge. The set of neighbors of a vertex $v \in V$ is denoted by $N(v)$, and $N[v] = N(v) \cup \{v\}$. The subgraph of G induced by a set of vertices $U \subseteq V$ is denoted by $G[U]$, and we use $G - v$ to denote $G[V \setminus \{v\}]$. A *clique* in G is a set of vertices that are all pairwise adjacent. A vertex v is *simplicial* if $N(v)$ is a clique.

A graph is *chordal* if it contains no induced (chordless) cycle of length 4 or more as an induced subgraph. Observe that induced subgraphs of chordal graphs are also chordal. A chordal graph that is not complete contains at least two non-adjacent simplicial vertices [4]. If two simplicial vertices u, v of G are adjacent, then it is easy to see that $N[u] = N[v]$; in this case we call u and v *simplicial twins*. We call a non-simplicial vertex v of a chordal graph G *almost-simplicial* if v has exactly one simplicial neighbor u, such that v is simplicial in $G - u$.

Observation 1. *Let G be a chordal graph and let u be a simplicial vertex of degree at least 2. If v is an almost-simplicial neighbor of u, then $N[v] \subseteq N[w]$ for every vertex $w \in N(u) \cap N(v)$.*

Simplicial vertices in chordal graphs could be arbitrarily far from each other; a simple path is an example. We show that, under some conditions, we can always find pairs of simplicial vertices that are close to each other. Our enumeration algorithm is heavily based on the following theorem, which could be of interest by itself.

Theorem 2. *If G is a non-trivial chordal graph, then there exists a pair of vertices u and v satisfying at least one of the following:*

1. *u and v are both simplicial vertices, and $N(u) \cap N(v) \neq \emptyset$;*
2. *u and v are simplicial twins;*
3. *u is simplicial, v is an almost-simplicial neighbor of u, and the degree of v in G is at least 2.*

Proof. If one of the first two conditions is satisfied, we are done. Assume that neither of the first two condition is satisfied, i.e., every pair of simplicial vertices have disjoint neighborhoods, and no two simplical vertices are adjacent. Remove all simplicial vertices of G, which results in a non-empty chordal graph G'. Let v be a simplicial vertex of G'. Since the first condition was not satisfied, v was adjacent in G to at most one simplicial vertex. Since v was not simplicial in G, it was almost-simplicial, and hence it had exactly one simplicial neighbor u. Since the second condition was not satisfied, v cannot be of degree 0 in G', since otherwise it would be the simplicial twin of u in G. Thus the degree of v in G was at least 2. Consequently, u and v satisfy the third condition.

The following corollary of Theorem 2 will also be useful.

Corollary 3. *Let G be a non-trivial chordal graph without simplicial twins, in which every simplicial vertex is of degree 1, and no two vertices of degree 1 have a common neighbor. Then at least one of the following holds:*

1. *There exist two simplicial vertices u and u' with (distinct) almost-simplicial neighbors v and v', respectively, such that $N[v] \cap N[v'] \neq \emptyset$.*
2. *There exist three vertices u, v, and w, such that u is simplicial in G with an almost-simplicial neighbor v, and $w \in N(v)$ is simplicial in $G - v$ and has degree at least 2 in $G - u$.*

Proof. Since G is non-trivial and has no simplicial twins, G is not complete. Thus G has at least two non-adjacent simplicial vertices, say u and u'. By the premises of the corollary, $N(u) \cap N(u') = \emptyset$. Since every simplicial vertex has at least one non-simplicial neighbor, removing all simplicial vertices yields a non-trivial chordal graph G'. Furthermore, every simplicial vertex of G' was an almost-simplicial vertex in G. Consequently, applying Theorem 2 to G' completes the proof.

A vertex v is said to *dominate* the vertices in $N[v]$. A vertex set $S \subseteq V$ is a *dominating set* of G if $N[S] = V$. A dominating set S is *minimal* if no proper subset of S is a dominating set. If S is a minimal dominating set, then for every vertex $v \in S$, there is a vertex $u \in N[v]$ which is dominated only by v. We call such a vertex u a *private neighbor* of v; note that a vertex in S might be its own private neighbor.

In our algorithm, we will use the following observation by Couturier et al. [2] and some basic elements of their algorithm for enumerating the minimal dominating sets of chordal graphs. However, all our major branching rules are new, and they are based on Theorem 2.

Observation 4 ([2]). *A simplicial vertex u cannot belong to a minimal dominating set containing a neighbor of u.*

3. Enumeration Algorithm

For enumerating the minimal dominating sets of an input chordal graph $G = (V, E)$, we describe a recursive branching algorithm called $\textsc{mds}(U, D)$, with $U \subseteq V$ and $D \subseteq V \setminus U$. The algorithm generates all minimal dominating sets S of G satisfying $D \subseteq S$ and $S \cap (V \setminus U) = D$. The initial call $\textsc{mds}(V, \emptyset)$ will thus ensure that all minimal dominating sets of G are enumerated. By the description of the algorithm it will follow that every vertex of $V \setminus U$ is either dominated by D or guaranteed to be dominated in every following branch of the algorithm. Hence the vertices in $V \setminus U$ are not relevant for the further choices to be made by the algorithm. At every step of the algorithm we pick a simplicial vertex of $G[U]$ and branch on possibilities around this vertex. A vertex of $G[U]$ is said to be *dominated* if it is adjacent in G to a vertex of D.

The following branching rules are applied successively in the sense that a branching rule is applied only when all the conditions assumed in previous branching rules do not hold. When we speak about neighborhoods and degrees, we always refer to $G[U]$ unless otherwise stated. At any step of the algorithm, if a dominated vertex u of $G[U]$ has no neighbors, we simply delete it and proceed with the instance $\textsc{mds}(U \setminus \{u\},\ D)$. The algorithm stops when U is empty or all vertices in U are dominated.

3.1. Branching on a dominated simplicial vertex of degree at least 2

If $G[U]$ has a dominated simplicial vertex u of degree at least 2, then we branch by either placing u in D or deleting u. When we place u in D we can delete $N[u]$ by Observation 4. The resulting instances are $\textsc{mds}(U \setminus$

$N[u],\ D \cup \{u\})$ and $\textsc{mds}(U \setminus \{u\},\ D)$. The corresponding recurrence is $T(n) \le T(n-3) + T(n-1)$.

3.2. Branching on a common neighbor of two simplicial vertices

If $G[U]$ has two simplicial vertices u and v, such that $N(u) \cap N(v)$ contains at least one vertex w, then we proceed as follows, analyzing all possible cases.

1. If both u and v are dominated, then assuming the previous branching on dominated simplicial vertices is not applicable, both u and v are of degree 1. In this case we branch on u by either placing it in D or deleting it. When u is placed in D, we can delete v and w by Observation 4. The resulting instances are $\textsc{mds}(U \setminus \{u, v, w\},\ D \cup \{u\})$ and $\textsc{mds}(U \setminus \{u\},\ D)$, and the corresponding recurrence is $T(n) = T(n-3) + T(n-1)$.
2. In the case where u is not dominated and its degree is 1, we branch by placing either u or w in D. When u is placed in D we can delete both u and w by Observation 4. In the second case, similarly, we delete all three vertices since w dominates $N[v]$. The resulting instances are $\textsc{mds}(U \setminus \{u, w\},\ D \cup \{u\})$ and $\textsc{mds}(U \setminus \{u, v, w\},\ D \cup \{w\})$. The corresponding recurrence is $T(n) = T(n-2) + T(n-3)$.
3. It remains to consider the case where one of the two simplicial vertices, say u, is not dominated and its degree is at least 2. In this case we branch by either placing w in D or discarding it from future minimal dominating sets containing D. If w is placed in D then we delete vertices u, v and w by Observation 4, If w is discarded, we just delete w although it might not be dominated. The correctness of this is argued by Couturier et al. [2], since u is not deleted, $N[u] \subseteq N[w]$, and thus w will become dominated as soon as u gets dominated. The resulting instances are $\textsc{mds}(U \setminus \{u, v, w\},\ D \cup \{w\})$ and $\textsc{mds}(U \setminus \{w\},\ D)$. The corresponding recurrence is $T(n) = T(n-3) + T(n-1)$.

3.3. Branching on adjacent simplicial vertices

If $G[U]$ has adjacent simplicial vertices u and v, then u and v are simplicial twins and at most one of them can belong to a dominating set. In this case we proceed as follows.

If there is a vertex w in $N(u) = N(v)$ then we branch on w as in the previous case. Otherwise u and v are both of degree 1. If they are both dominated, we can simply delete them since they have no other neighbors. Otherwise we branch on placing either u or v in D. In each case, both vertices can be deleted, and we get the instances $\textsc{mds}(U\setminus\{u,v\},\ D\cup\{u\})$ and $\textsc{mds}(U\setminus\{u,v\},\ D\cup\{v\})$. The corresponding recurrence is $T(n) = 2\,T(n-2)$.

3.4. The case of a non-dominated simplicial vertex with an almost-simplicial neighbor

If $G[U]$ has a non-dominated simplicial vertex u with an almost-simplicial neighbor $v \in N(u)$, then we proceed as below, depeding on which of the following cases applies.

1. If v is the only neighbor of u, then either u or v must be added to D to dominate u, and the other one must be deleted since u and v cannot both belong to a minimal dominating set. Thus we get the instances $\textsc{mds}(U\setminus\{u,v\},\ D\cup\{u\})$ and $\textsc{mds}(U\setminus\{u,v\},\ D\cup\{v\})$. The corresponding recurrence is $T(n) = 2\,T(n-2)$.
2. If u has another neighbor w, then $N[v] \subseteq N[w]$ by Observation 1, and we branch on placing w in D or discarding it. If we take w, then we delete u, v and w. If we discard w, we delete it, since dominating v at a later step will lead to dominating w. We get the instances $\textsc{mds}(U\setminus\{u,v,w\},\ D\cup\{w\})$ and $\textsc{mds}(U\setminus\{w\},\ D)$. The corresponding recurrence is $T(n) = T(n-3) + T(n-1)$.

When all the rules above have been applied, then all the simplicial vertices of $G[U]$ are dominated and of degree 1. Furthermore, no two such vertices share a common neighbor. Thus for the rest, we only need to consider such simplicial vertices. Let u be such a simplical vertex, and let v be the unique neighbor of u. By Corollary 3, we can assume that v is almost-simplicial. If v is dominated then we can simply delete u and continue with the reduced instance, so we can assume that v is not dominated. Since previous rules have been applied, the degree of v is at least 2 in G.

3.5. The case of a dominated simplicial vertex u of degree 1, whose undominated almost-simplicial neighbor v has degree 2

If v has degree 2, then it has exactly one neighbor other than u; let $N(v) = \{u, w\}$. In order to dominate v, we need at least one of u, v, and w to be added to D. We branch on these three possibilities. If we place w in D then we delete all three vertices u, v, w from U, since otherwise we will not get a minimal dominating set. If we place v in D, again we delete all three vertices, for minimality. If we place u in D, then we delete v by Observation 4, but we cannot delete w if it is not dominated. We get the instances $\textsc{mds}(U\setminus\{u,v,w\},\ D\cup\{w\})$, $\textsc{mds}(U\setminus\{u,v,w\},\ D\cup\{v\})$, and $\textsc{mds}(U \setminus \{u,v\},\ D \cup \{u\})$. The corresponding recurrence is: $T(n) = 2\,T(n-3) + T(n-2)$.

Assuming that none of the previous branching conditions hold, we have now a chordal graph $G[U]$ whose simplicial vertices are all of degree 1, dominated, and each of them has an undominated neighbor with degree at least 3. By Corollary 3, we can find a simplicial vertex u with an almost-simplicial neighbor v.

3.6. The case of a dominated simplicial vertex u of degree 1, whose undominated almost-simplicial neighbor v has degree at least 3

By Corollary 3, we are left with exactly the following possibilites.

1. v has a neighbor w of degree at least 2, such that w is simplicial in $G[U \setminus \{v\}]$.

   This means that v has another neighbor t such that $N[v] \setminus \{u\} \subseteq N[w] \subseteq N[t]$. In this case, we branch by either placing t in D or discarding it. Because of the mentioned neighborhood inclusions, when we take t then we can delete u, v, w, and t. When we discard t, we can simply delete it by previous arguments, since it will be dominated whenever v is dominated. The resulting instances are $\textsc{mds}(U \setminus \{u,v,w,t\},\ D \cup \{t\})$ and $\textsc{mds}(U \setminus \{t\},\ D)$. The corresponding recurrence is $T(n) = T(n-4) + T(n-1)$.

2. There exists another simplicial vertex u' with exactly one almost-simplicial neighbor $v' \neq v$, such that $v' \in N(v)$.

   In this case, note first that since v and v' are almost-simplicial and adjacent, they are simplicial twins in $G[U \setminus \{u,u'\}]$. We branch on the possibilities of adding v to D, adding u to D, and adding neither of them to D.

   If we add v to D then we can delete all four vertices u, u', v and v' by Observation 4, since they are then all dominated, and v and v' are simplicial twins in the absence of u and u'. If we add u to D, then we can delete u and v by previous arguments. If we add neither u nor v to D, then it means that v will have to be dominated by a vertex in $N(v) \setminus \{u\}$. Since v and v' have the same neighborhoods outside of u and u', this means that v and v' will be dominated by the same vertex in this case. Consequently, there is no need for u' anymore, which is already dominated and can never be added to such a dominating set. Thus we can delete u, v and u'. Deleting v is safe since v' remains in the graph, v' is not yet dominated, and v will become dominated as soon as v' becomes dominated. The resulting instances are $\text{MDS}(U \setminus \{u, v, u', v'\},\ D \cup \{v\})$, $\text{MDS}(U \setminus \{u, v\},\ D \cup \{u\})$, and $\text{MDS}(U \setminus \{u, v, u'\},\ D)$. The corresponding recurrence is $T(n) = T(n-4) + T(n-2) + T(n-3)$.

3. There exists another simplicial vertex u' with exactly one almost-simplicial neighbor $v' \neq v$, such that v and v' have a common neighbor w.

   We branch on the possibilities of placing u in D, placing w in D, and placing neither of them in D.

   If we place u in D, then we can delete u and v by previous arguments. If we place w in D, then since v and v' both become dominated, we can delete u and u' that are already dominated and not anymore relevant. In the remaining graph, v and v' are simplicial, thus they can also be deleted, by Observation 4. If we do not place u or w in D, then we can delete both of them. This is safe because v will then become dominated by another neighbor (or by being placed in the dominating set itself), at which point w will also become dominated. The resulting instances are $\text{MDS}(U \setminus \{u, v\},\ D \cup \{u\})$, $\text{MDS}(U \setminus \{u, v, u', v', w\},\ D \cup \{w\})$, and $\text{MDS}(U \setminus \{u, w\},\ D)$. The corresponding recurrence is $T(n) = 2\,T(n-2) + T(n-5)$.

This completes the description of our branching algorithm for enumerating the minimal dominating sets of a chordal graph, and we are ready to state and prove the following theorem.

Theorem 5. *A chordal graph on n vertices has at most 1.5214^n minimal dominating sets, and these can be enumerated in time $\mathcal{O}(1.5214^n)$.*

Proof. Let $G = (V, E)$ be an arbitrary chordal graph on n vertices. Consider a run of Algorithm $\text{MDS}(V, \emptyset)$. Since the correctness of each rule is already argued for in the description of algorithm, by carefully checking that all cases are covered, we can conclude that every minimal dominating set of G will be generated and found as a leaf of the resulting search tree. Note that some non-minimal dominating sets could also be generated, so at each leaf we check in polynomial time whether the dominating set at hand is minimal, and discard it if not. Thus the number of leaves of the search tree of $\text{MDS}(V, \emptyset)$ gives an upper bound on the number of minimal dominating sets of the input graph G. The standard way to find this number is by analyzing the recurrences resulting from the branching rules of the algorithm, and take the largest number; see e.g., [2, 6].

Below we list all the recurrences that have been encountered during the description of the algorithm, and the corresponding upper bound on the number of leaves of the search tree, assuming that $T(n) = 1$.

$T(n) \leq T(n-3) + T(n-1)$ yields $T(n) < 1.4656^n$.
$T(n) = T(n-2) + T(n-3)$ yields $T(n) < 1.3248^n$.
$T(n) = 2\,T(n-2)$ yields $T(n) < 1.4143^n$.
$T(n) = 2\,T(n-3) + T(n-2)$ yields $T(n) < 1.5214^n$.
$T(n) = T(n-2) + T(n-3) + T(n-4)$ yields $T(n) < 1.4656^n$.
$T(n) = 2\,T(n-2) + T(n-5)$ yields $T(n) < 1.5129^n$.
$T(n) = T(n-4) + T(n-1)$ yields $T(n) < 1.3803^n$.
$T(n) = T(n-4) + T(n-2) + T(n-3)$ yields $T(n) < 1.4656^n$.
$T(n) = T(n) = 2\,T(n-2) + T(n-5)$ yields $T(n) < 1.5129^n$.

Consequently, the number of leaves of the search tree is striclty less than 1.5214^n. Since the total number of nodes of the search tree is not more than a polynomial times the number of leaves, and we spend polynomial time at each step, we can conclude that the running time of the algorithm is $\mathcal{O}(1.5214^n)$.

4. Concluding Remarks

We narrowed the gap between the known upper and lower bounds on the maximum number of minimal dominating sets of a chordal graph. We believe that our structural result on chordal graphs given in Theorem 2 can be useful for faster algorithms also for other problems on chordal graphs.

References

[1] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes: a survey*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[2] J. Couturier, P. Heggernes, P. van 't Hof, and D. Kratsch, *Minimal dominating sets in graph classes: Combinatorial bounds and enumeration*, Theor. Comput. Sci., 487 (2013), pp. 82–94.

[3] J. Couturier, R. Letourneur, and M. Liedloff, *On the number of minimal dominating sets on some graph classes*, Theor. Comput. Sci., 562 (2015), pp. 634–642.

[4] G. Dirac, *On rigid circuit graphs*, Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg, 25 (1961), pp. 71–76.

[5] F. V. Fomin, F. Grandoni, A. V. Pyatkin, and A. A. Stepanov, *Combinatorial bounds via measure and conquer: Bounding minimal dominating sets and applications*, ACM Transactions on Algorithms, 5 (2009).

[6] F. V. Fomin and D. Kratsch, *Exact exponential algorithms*, Texts in Theoretical Computer Science. An EATCS Series, Springer, Heidelberg, 2010.

[7] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*, vol. 57 of Annals of Discrete Mathematics, Elsevier Science B.V., Amsterdam, second ed., 2004.

[8] M. M. Kanté, V. Limouzy, A. Mary, and L. Nourine, *On the enumeration of minimal dominating sets and related notions*, SIAM J. Discrete Math., 28 (2014), pp. 1916–1929.

[9] M. M. Kanté, V. Limouzy, A. Mary, L. Nourine, and T. Uno, *A polynomial delay algorithm for enumerating minimal dominating sets in chordal graphs*, Proceedings of WG 2015. To appear.