An incremental polynomial time algorithm to enumerate all minimal edge dominating sets*

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Abstract

We show that all minimal edge dominating sets of a graph can be generated in incremental polynomial time. We present an algorithm that solves the equivalent problem of enumerating minimal (vertex) dominating sets of line graphs in incremental polynomial, and consequently output polynomial, time. Enumeration of minimal dominating sets in graphs has recently been shown to be equivalent to enumeration of minimal transversals in hypergraphs. The question whether the minimal transversals of a hypergraph can be enumerated in output polynomial time is a fundamental and challenging question; it has been open for several decades and has triggered extensive research. To obtain our result, we present a flipping method to generate all minimal dominating sets of a graph. Its basic idea is to apply a flipping operation to a minimal dominating set D^* to generate minimal dominating sets D such that $G[D]$ contains more edges than $G[D^*]$. We show that the flipping method works efficiently on line graphs, resulting in an algorithm with delay $O(n^2m^2|\mathcal{L}|)$ between each pair of consecutively output minimal dominating sets, where n and m are the numbers of vertices and edges of the input graph, respectively, and $\mathcal{L}$ is the set of already generated minimal dominating sets. Furthermore, we are able to improve the delay to $O(n^2m|\mathcal{L}|)$ on line graphs of bipartite graphs. Finally we show that the flipping method is also efficient on graphs of large girth, resulting in an incremental polynomial time algorithm to enumerate the minimal dominating sets of graphs of girth at least 7.

1 Introduction

Enumerating, i.e., generating or listing, all vertex or edge subsets of a graph that satisfy a specified property plays a central role in graph algorithms; see e.g. [1, 3, 10, 11, 12, 19, 25, 26, 27, 29, 34, 35, 37]. Enumeration algorithms with running time that is polynomial in the size of the input plus the size of the output are called output polynomial time algorithms.

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For various enumeration problems it has been shown that no output polynomial time algorithm can exist unless P = NP [25, 27, 29]. A potentially better behavior than output polynomial time is achieved by so called incremental polynomial time algorithms, which means that the next set in the list of output sets is generated in time that is polynomial in the size of the input plus the size of the already generated part of the output. Incremental polynomial time immediately implies output polynomial time.

One of the most classical and widely studied enumeration problems is that of listing all minimal transversals of a hypergraph, i.e., minimal hitting sets of its set of hyperedges. This problem has applications in areas like database theory, machine learning, data mining, game theory, artificial intelligence, mathematical programming, and distributed systems; extensive lists of corresponding references are provided by e.g., Eiter and Gottlob [12], and Elbassioni, Makino, and Rauf [13]. Whether or not all minimal transversals of a hypergraph can be listed in output polynomial time has been identified as a fundamental challenge in a long list of seminal papers, e.g., [10, 11, 12, 13, 14, 19, 32], and it remains unresolved despite continuous attempts since the 1980's.

Recently Kanté, Limouzy, Mary, and Nourine [21] have proved that enumerating the minimal transversals of a hypergraph is equivalent to enumerating the minimal dominating sets of a graph. In particular, they show that an output polynomial time algorithm for enumerating minimal dominating sets in graphs implies an output polynomial time algorithm for enumerating minimal transversals in hypergraphs. Dominating sets form one of the best studied notions in computer science; the number of papers on domination in graphs is in the thousands, and several well known surveys and books are dedicated to the topic (see, e.g., [17]).

Given the importance of the hypergraph transversal enumeration problem and the failed attempts to resolve whether it can be solved in output polynomial time, efforts to identify tractable special cases have been highly appreciated [5, 6, 8, 9, 10, 12, 13, 30, 31]. The newly proved equivalence to domination allows for new ways to attack this long-standing open problem. In fact some results on output polynomial algorithms to enumerate minimal dominating sets in graphs already exist for graphs of bounded treewidth and of bounded clique-width [7], interval graphs [10, 23], strongly chordal graphs [10], planar graphs [12], degenerate graphs [12], graphs of girth at least 5 [24][1], split graphs [20], line graphs [22], path graphs [22], and permutation graphs [23].

In this paper we show that all minimal dominating sets of line graphs and of graphs of large girth can be enumerated in incremental polynomial time. More precisely, we give algorithms where the time delay between two consecutively generated minimal dominating sets is $O(n^2m^2|\mathcal{L}|)$ on line graphs, $O(n^2m|\mathcal{L}|)$ on line graphs of bipartite graphs, and $O(n^2m|\mathcal{L}|^2)$ on graphs of girth at least 7, where $\mathcal{L}$ is the set of already generated minimal dominating sets of an input graph on n vertices and m edges. Line graphs form one of the oldest and most studied graph classes [16, 18, 28, 38] and they can be recognized in linear time [33]. Our results, in addition to proving tractability for two substantial cases

[1] We are grateful to the anonymous referee who pointed to us that an output polynomial time algorithm for graph of girth at least 5 can be derived from the results of Khachyan et al. [24] about hypergraphs with bounded edge-intersections.

of the hypergraph transversal enumeration problem, imply incremental polynomial time enumeration of minimal edge dominating sets in *arbitrary* graphs. In particular, we obtain an algorithm with delay $O(m^6|\mathcal{L}|)$ to enumerate all minimal edge dominating sets of any graph on m edges, where $\mathcal{L}$ is the set of already generated edge dominating sets. For bipartite graphs, we are able to reduce the delay to $O(m^4|\mathcal{L}|)$.

Our algorithms are based on the supergraph technique for enumerating vertex subsets in graphs [3, 26, 34, 37]. As a central tool in our algorithms, we present a new *flipping* method to generate the out-neighbors of a node of the supergraph, in other words, to generate new minimal dominating sets from a parent dominating set. Given a minimal dominating set D^*, our flipping operation replaces an isolated vertex of $G[D^*]$ with a neighbor outside of D^*, and, if necessary, supplies the resulting set with additional vertices to obtain new minimal dominating sets D, such that $G[D]$ has more edges compared to $G[D^*]$. Each of our algorithms starts with enumerating all maximal independent sets of the input graph G using the algorithm of Johnson, Papadimitriou, and Yannakakis [19], which gives the initial set of minimal dominating sets. Then the flipping operation is applied to every appropriate minimal dominating set found, to find new minimal dominating sets inducing subgraphs with more edges. We show that on all graphs, the flipping method enables us to identify a unique parent for each minimal dominating set. On line graphs and graphs of girth at least 7, we are able to prove additional (different) properties of the parents, which allow us to obtain the desired running time on these graph classes.

Regarding the algorithm for enumerating the minimal dominating sets of line graphs by Kanté, Limouzy, Mary, and Nourine [22] listed above, we would like to mention that our work has been simultaneous with their work, and their method is completely different from ours. In particular, their main result is that line graphs (and path graphs) have closed neighborhood hypergraphs of bounded conformality. Their algorithm is then a direct consequence of the results by Boros, Elbassioni, Gurvich, and Khachiyan [4].

2 Definitions and Preliminary Results

As input graphs to our enumeration problem, we consider finite undirected graphs without loops or multiple edges. Given such a graph $G = (V, E)$, its vertex and edge sets, V and E, are also denoted by $V(G)$ and $E(G)$, respectively. The subgraph of G induced by a subset $U \subseteq V$ is denoted by $G[U]$. For a vertex v, we denote by $N(v)$ its *(open) neighborhood*, that is, the set of vertices that are adjacent to v. The *closed neighborhood* of v is the set $N(v) \cup \{v\}$, and it is denoted by $N[v]$. If $N(v) = \emptyset$ then v is *isolated*. For a set $U \subseteq V$, $N[U] = \cup_{v \in U} N[v]$, and $N(U) = N[U] \setminus U$. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G; if G has no cycles, then $g(G) = +\infty$. A set of vertices is a *clique* if it induces a complete subgraph of G. A clique is *maximal* if no proper superset of it is a clique.

Two edges in E are adjacent if they share an endpoint. The *line graph* $L(G)$ *of* G is the graph whose set of vertices is $E(G)$, such that two vertices e and e' of $L(G)$ are adjacent if and only if e and e' are adjacent edges of G. A graph H is a *line graph* if

H is isomorphic to $L(G)$ for some graph G. Equivalently, a graph is a line graph if its edges can be partitioned into maximal cliques such that no vertex lies in more than two maximal cliques. This implies in particular that the neighborhood of every vertex can be partitioned into at most two cliques. A graph that is isomorphic to $K_{1,3}$ is called a *claw*. Graphs that do not contain a claw as an induced subgraph are called *claw-free*. It is well known that line graphs are claw-free.

A vertex v *dominates* a vertex u if $u \in N(v)$; similarly v dominates a set of vertices U if $U \subseteq N[v]$. For two sets $D, U \subseteq V$, D dominates U if $U \subseteq N[D]$. A set of vertices D is a *dominating set* of $G = (V, E)$ if D dominates V. A dominating set is *minimal* if no proper subset of it is a dominating set. Let D be a dominating set of G, and let $v \in D$. Vertex u is a *private vertex*, or simply *private*, for vertex v (with respect to D) if u is dominated by v but is not nominated by $D \setminus \{v\}$. Clearly, D is a minimal dominating set if and only if each vertex of D has a private vertex. We denote by $P_D[v]$ the set of all private vertices for v. Notice that a vertex of D can be private for itself. Vertex u is a *private neighbor* of $v \in D$ if $u \in N(v) \cap P_D[v]$. The set of all private neighbors of v is denoted by $P_D(v)$. Note that $P_D[v] = P_D(v) \cup \{v\}$ if v is isolated in $G[D]$, and otherwise $P_D[v] = P_D(v)$.

A set of edges $A \subseteq E$ is an *edge dominating* set if each edge $e \in E$ is either in A or is adjacent to an edge in A. An edge dominating set is *minimal* if no proper subset of it is an edge dominating set. It is easy to see that A is a (minimal) edge dominating set of G if and only if A is a (minimal) dominating set of $L(G)$.

Let $\phi(X)$ be a property of a set of vertices or edges X of a graph, e.g., "X is a minimal dominating set". The *enumeration problem for property* $\phi(X)$ for a given graph G on n vertices and m edges asks for the set $\mathcal{C}$ of all subsets of vertices or edges X of G that satisfy $\phi(X)$. An *enumeration algorithm* is an algorithm that solves this problem, i.e., that lists the elements of $\mathcal{C}$ without repetitions. An enumeration algorithm $\mathcal{A}$ is said to be *output polynomial time* if there is a polynomial $p(x, y)$ such that all elements of $\mathcal{C}$ are listed in time bounded by $p((n+m), |\mathcal{C}|)$. Assume now that $X_1, \ldots, X_\ell$ are the elements of $\mathcal{C}$ enumerated in the order in which they are generated by $\mathcal{A}$. The *delay* of $\mathcal{A}$ is the maximum time $\mathcal{A}$ requires between outputting X_{i-1} and X_i, for $i \in \{1, \ldots, \ell\}$. Algorithm $\mathcal{A}$ is *incremental polynomial time* if there is a polynomial $p(x, i)$ such that for each $i \in \{1, \ldots, \ell\}$, X_i is generated in time bounded by $p((n + m), i)$. Finally, $\mathcal{A}$ is a *polynomial delay* algorithm if there is a polynomial $p(x)$ such that for each $i \in \{1, \ldots, \ell\}$, the delay between outputting X_{i-1} and X_i is at most $p(n + m)$.

A set of vertices $U \subseteq V$ is an *independent set* if no two vertices of U are adjacent in G, and an independent set is *maximal* if no proper superset of it is an independent set. The following observation is folklore.

Observation 1. *Every maximal independent set of a graph G is a minimal dominating set of G. Furthermore, the set of all maximal independent sets of G is exactly the set of all its minimal dominating sets D such that $G[D]$ has no edges.*

Tsukiyama et al. [36] showed that maximal independent sets can be enumerated with polynomial delay. Johnson et al. [19] showed that such an enumeration can be done in lexicographic order.

Theorem 1 ([19])**.** *All maximal independent sets of a graph with n vertices and m edges can be enumerated in lexicographic order with delay $O(n(m + n \log |\mathcal{I}|))$, where $\mathcal{I}$ is the set of already generated maximal independent sets.*

Let $v_1, \ldots, v_n$ be the vertices of a graph G. Suppose that D' is a dominating set of G. We say that a minimal dominating set D is obtained from D' by *greedy removal of vertices* (*with respect to order* $v_1, \ldots, v_n$) if we initially let $D = D'$, and then recursively apply the following rule:

If D is not minimal, then find a vertex v_i with the smallest index i such that $D \setminus \{v_i\}$ is a dominating set in G, and set $D = D \setminus \{v_i\}$.

Clearly, when we apply this rule, we never remove vertices of D' that have private neighbors.

Finally, we give some definitions on directed graphs, as the supergraph technique that we use creates an auxiliary directed graph. To distinguish this graph from the input graph, we will call the vertices of a directed graph *nodes*. The edges of a directed graph have directions and are called *arcs*. An arc (u, v) has direction from node u to node v. The *out-neighbors* of a node u are all nodes v such that (u, v) is an arc. Similarly, the *in-neighbors* of a node v are all nodes u such that (u, v) is an arc. In this paper, an in-neighbor will sometimes be called a *parent*.

3 Enumeration by flipping: the general approach

In this section we describe the general scheme of our enumeration algorithms. Let G be a graph; we fix an (arbitrary) order of its vertices: $v_1, \ldots, v_n$. Observe that this order induces a lexicographic order on the set $2^{V(G)}$. Whenever greedy removal of vertices of a dominating set is performed further in the paper, it is done with respect to this ordering.

Let D be a minimal dominating set of G such that $G[D]$ has at least one edge uw. Then vertex $u \in D$ is dominated by vertex $w \in D$. Let $v \in P_D(u)$. Let $X_{uv} \subseteq P_D(u) \setminus N[v]$ be a maximal independent set in $G[P_D(u) \setminus N[v]]$ selected greedily with respect to ordering $v_1, \ldots, v_n$, i.e., we initially set $X_{uv} = \emptyset$ and then recursively include in X_{uv} the vertex of $P_D(u) \setminus (N[\{v\} \cup X_{uv}])$ with the smallest index as long as it is possible. Consider the set $D' = (D \setminus \{u\}) \cup X_{uv} \cup \{v\}$. Notice that D' is a dominating set in G, since all vertices of $P_D(u)$ are dominated by $X_{uv} \cup \{v\}$. Let Z_{uv} be the set of vertices that are removed to ensure minimality, and let $D^* = ((D \setminus \{u\}) \cup X_{uv} \cup \{v\}) \setminus Z_{uv}$. See Figure 1 for an example.

Lemma 1. *The set D^* is a minimal dominating set in G such that $X_{uv} \cup \{v\} \subseteq D^*$, $|E(G[D^*])| < |E(G[D])|$, and v is an isolated vertex of $G[D^*]$.*

Proof. Since vertices of $X_{uv} \cup \{v\}$ are privates for themselves, they are not removed by greedy removal of vertices. Notice that $E(G[D^*]) \subseteq E(G[(D']) \subseteq E(G[D]) \setminus \{uw\}$ as $G[X_{uv} \cup \{v\}]$ is an independent set and vertices $X_{uv} \cup \{v\}$ have no neighbors in $D \setminus \{u\}$. Finally v is an isolated vertex as there are no edges incident to v in $G[D']$. □

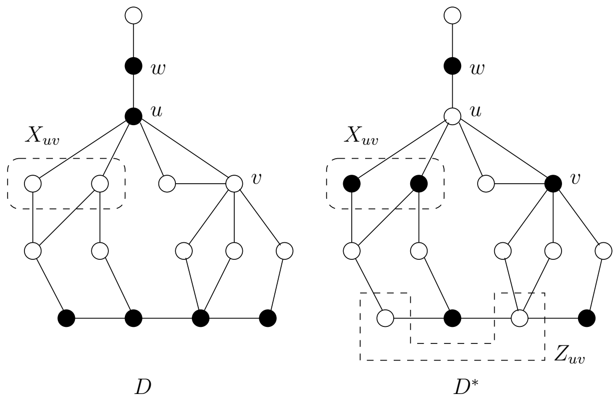

Figure 1: A minimal dominating set D and its parent D^*; the vertices of D and D^* are black.

Our main tool, the *flipping* operation is exactly the *reverse* of how we generated D^* from D; i.e., it replaces an isolated vertex v of $G[D^*]$ with a neighbor u in G to obtain D. In particular, we are interested in all minimal dominating sets D that can be generated from D^* in this way.

Given D and D^* as defined above, we say that D^* is a *parent of D with respect to flipping u and v*. We say that D^* is a *parent* of D if there are vertices $u, v \in V(G)$ such that D^* is a parent with respect to flipping u and v. It is important to note that each minimal dominating set D such that $E(G[D]) \neq \emptyset$ has a unique parent with respect to flipping of any vertices $u \in D \cap N[D \setminus \{u\}]$ and $v \in P_D(u)$, as both sets X_{uv} and Z_{uv} are lexicographically first sets selected by a greedy algorithm. Similarly, we say that D is a *child* of D^* (with respect to flipping u and v) if D^* is the parent of D (with respect to flipping u and v).

Assume that there is an enumeration algorithm $\mathcal{A}$ that, given a minimal dominating set D^* of a graph G such that $G[D^*]$ has isolated vertices, an isolated vertex v of $G[D^*]$, and a neighbor u of v in G, generates with polynomial delay a set of minimal dominating sets $\mathcal{D}$ with the property that $\mathcal{D}$ contains all minimal dominating sets D that are children of D^* with respect to flipping u and v. In this case we can enumerate all minimal dominating sets of the graph G with n vertices and m edges as follows.

Our method is a variant of the supergraph technique that has been applied for enumerating subsets with various properties in graphs [3, 26, 34, 37]. More precisely, we define a directed graph $\mathcal{G}$ whose nodes are minimal dominating sets of G, with an additional special node r, called the *root*, that has no in-neighbors. Recall that by Observation 1, maximal independent sets are minimal dominating sets, i.e., they are nodes of $\mathcal{G}$. We add an arc from the root r to every maximal independent set of G. For each minimal dominating set $D^* \in V(\mathcal{G})$, we add an arc from D^* to every minimal dominating set D such that $\mathcal{A}$ generates D from D^* for some choice of u and v.

Next we run Depth-First Search in $\mathcal{G}$ starting from r. Observe that we need not construct $\mathcal{G}$ explicitly to do this, as for each node $W \neq r$ of $\mathcal{G}$ we can use $\mathcal{A}$ to generate all out-neighbors of W, and we can generate the out-neighbors of r with polynomial delay by Theorem 1. Hence, we maintain a list $\mathcal{L}$ of minimal dominating sets of G sorted in lexicographic order that are already visited nodes of $\mathcal{G}$. Also we keep a stack $\mathcal{S}$ of records R_W for $W \in V(\mathcal{G})$ that are on the path from r to the current node of $\mathcal{G}$. These records are used to generate out-neighbors. The record R_r contains the last generated maximal independent set and the information that is necessary to proceed with the enumeration of maximal independent sets. Each of the records R_W, for $W \neq r$, contains the current choice of u and v, the last set D generated by $\mathcal{A}$ for the instance (W, u, v), and the information that is necessary for $\mathcal{A}$ to proceed with the enumeration.

Lemma 2. *Suppose that $\mathcal{A}$ generates the elements of $\mathcal{D}$ for a triple (D^*, u, v) with delay $O(p(n,m))$. Let $\mathcal{L}^*$ be the set of all minimal dominating sets. Then the algorithm described above enumerates all minimal dominating sets as follows:*

- *with delay $O((p(n,m)+n^2)m|\mathcal{L}|^2)$ and total running time $O((p(n,m)+n^2)m|\mathcal{L}^*|^2)$;*
- *if $|E(G[D])| > |E(G[D^*])|$ for every $D \in \mathcal{D}$, then the delay is $O((p(n,m)+n^2)m^2|\mathcal{L}|)$, and the total running time is $O((p(n,m)+n^2)m|\mathcal{L}^*|^2)$;*
- *if $\mathcal{D}$ contains only children of D^* with respect to flipping of u and v, then the delay is $O((p(n,m)+n^2)m|\mathcal{L}|)$, and the total running time is $O((p(n,m)+n^2)m|\mathcal{L}^*|)$.*

Proof. Recall that any minimal dominating set D with at least one edge has a parent D^* and $|E(G[D^*])| < |E(G[D])|$. Because $\mathcal{A}$ generates D from D^*, (D^*, D) is an arc in $\mathcal{G}$. It follows that for any minimal dominating set $D \in V(\mathcal{G})$ with at least one edge, there is a maximal independent set $I \in V(\mathcal{G})$ such that I and D are connected by a directed path in $\mathcal{G}$. As (r, I) is an arc in $\mathcal{G}$, D is reachable from r. We conclude that Depth-First Search visits, and thus enumerates all nodes of $\mathcal{G}$. It remains to evaluate the running time.

To get a new minimal dominating set, we consider the records in $\mathcal{S}$. For each record R_W for $W \neq r$, we have at most m possibilities for u and v to get a new set D. As soon as a new set is generated it is added to $\mathcal{L}$ unless it is already in $\mathcal{L}$. Hence, we generate at most $m|\mathcal{L}|$ sets for W in time $O((p(n,m)+n^2)m|\mathcal{L}|)$, as each set is generated with delay $O(p(n,m))$, and after its generation we immediately test whether or not it is already in $\mathcal{L}$, which takes $O(n \log |\mathcal{L}|) = O(n^2)$ time, because $|\mathcal{L}| \leq 2^n$. For R_r, we generate at most $|\mathcal{L}|$ sets. Because any isolated vertex of G belongs to every maximal independent set, each set is generated with delay $O(n'(m + n' \log |\mathcal{L}|))$, i.e., in time $O(n'(m + n'^2))$ by Theorem 1, where n' is the number of non-isolated vertices. As $n' \leq 2m$, these sets are generated in time $O(n^2 m|\mathcal{L}|)$. Since $|\mathcal{S}| \leq |\mathcal{L}|$, in time $O((p(n,m)+n^2)m|\mathcal{L}|^2)$ we either obtain a new minimal dominating set or conclude that the list of minimal dominating sets is exhausted.

To get the bound for the total running time, observe that $|E(\mathcal{G})| = O(|\mathcal{L}^*|^2)$ as $\mathcal{G}$ has a node for every minimal dominating set of G. In total, Depth-First Search is run once in the whole of $\mathcal{G}$; recall that it runs in time that is linear in $|E(\mathcal{G})|$. In addition, for each arc in $E(\mathcal{G})$ we perform $O((p(n,m)+n^2)m)$ operations, and consequently the total running time is $O((p(n,m)+n^2)m|\mathcal{L}^*|^2)$.

If for every $D \in \mathcal{D}$, $|E(G[D])| > |E(G[D^*])|$, then the delay is less. To see this, observe that the number of edges in any minimal dominating set is at most m. Hence, any directed path starting from r in $\mathcal{G}$ has length at most m, and therefore, $|\mathcal{S}| \leq m+1$. By the same arguments as above, we get that in time $O((p(n,m)+n^2)m^2|\mathcal{L}|)$ we either obtain a new minimal dominating set or conclude that the list of minimal dominating sets is complete. Note that the total running time remains the same as in the previous case, as $\mathcal{G}$ still has $O(|\mathcal{L}^*|^2)$ arcs and we still perform $O((p(n,m)+n^2)m)$ operations on each of them.

Assume finally that $\mathcal{D}$ contains only children of D^* with respect to flipping of u and v. Since each minimal dominating set D with $E(G[D]) \neq \emptyset$ has a unique parent with respect to flipping of any vertices $u \in D \cap N[D \setminus \{u\}]$ and $v \in P_D(u)$, each D has at most m parents. Hence, we generate at most $m|\mathcal{L}|$ sets until we obtain a new minimal dominating set or conclude that the list is exhausted. As to generate a set and check whether it is already listed we spend time $O(p(n,m)+n^2)$, the delay between two consecutive minimal dominating sets that are output is $O((p(n,m)+n^2)m|\mathcal{L}|)$. As $|E(\mathcal{G})| = O(|\mathcal{L}^*|)$ in this case, by the same arguments as above, the total running time is $O((p(n,m)+n^2)m|\mathcal{L}^*|)$. $\square$

To be able to apply our method, we have to show how to construct an algorithm, like algorithm $\mathcal{A}$ above, that produces $\mathcal{D}$ with delay $O(p(n,m))$, where $p(n,m)$ is a polynomial in n and m. We will use the following lemma for this purpose.

Lemma 3. *Let D be a child of D^* with respect to flipping u and v, i.e., $D^* = ((D \setminus \{u\}) \cup X_{uv} \cup \{v\}) \setminus Z_{uv}$. Then for every vertex $z \in Z_{uv}$, the following three statements are true:*

1. *$z \notin N[X_{uv} \cup \{v\}]$,*
2. *z is dominated by a vertex of $D^* \setminus (X_{uv} \cup \{v\})$,*
3. *there is a vertex $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ such that x and z are adjacent and $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$.*

Furthermore, for every $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ such that $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$, there is a vertex $z \in Z_{uv}$ such that x and z are adjacent.

Proof. 1. To show that $z \notin N[X_{uv} \cup \{v\}]$, it is sufficient to observe that v and the vertices of X_{uv} are private neighbors of $u \in D$, and therefore, no vertex of $X_{uv} \cup \{v\}$ is adjacent to a vertex of $D \setminus \{u\} \supseteq Z_{uv}$.

2. Vertices of Z_{uv} are removed from D' by greedy removal in order to obtain a minimal dominating set D^*. Thus, there is at least one vertex in $D^* = D' \setminus Z_{uv}$ that dominates z. By statement 1, this is not a vertex of $X_{uv} \cup \{v\}$.

3. By statement 2, vertex z has a neighbor in $D^* \cap D$, i.e., z is not a private for itself. As D is a minimal dominating set, there exists a vertex $x \in P_D(z)$. Vertex z is removed from D' by greedy removal, implying that $P_{D'}(z) = \emptyset$, thus we can conclude that $x \in N(X_{uv} \cup \{v\})$, as these are the only vertices added to D in order to obtain D'.

Finally, let $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ be a vertex such that $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$. Observe that x is not dominated by the set $(D^* \setminus (X_{uv} \cup \{v\})) \cup \{u\}$. This means that x is dominated by a vertex of Z_{uv} in the dominating set $D = (D^* \setminus (X_{uv} \cup \{v\})) \cup \{u\} \cup Z_{uv}$. $\square$

We use this lemma to construct an algorithm for generating $\mathcal{D}$. The idea is to generate $\mathcal{D}$ by considering all possible candidates for X_{uv} and Z_{uv}. It would be interesting to know whether this can be done efficiently in general. On line graphs and graphs of girth at least 7, we are able to prove additional properties of the parent minimal dominating sets which result in efficient algorithms for generating $\mathcal{D}$, as will be explained in the sections below.

4 Enumeration of minimal edge dominating sets

In this section we show that all minimal edge dominating sets of an *arbitrary* graph can be enumerated in incremental polynomial time. We achieve this by enumerating the minimal dominating sets in line graphs.

4.1 Enumeration of minimal dominating sets of line graphs

For line graphs, we construct an enumeration algorithm that, given a minimal dominating set D^* of a graph G such that $G[D^*]$ has isolated vertices, an isolated vertex v of $G[D^*]$, and a neighbor u of v in G, generates with polynomial delay a set of minimal dominating sets $\mathcal{D}$ that contains all children of D^* with respect to flipping u and v, and has the property that $|E(G[D])| > |E(G[D^*])|$, for every $D \in \mathcal{D}$.

This is possible because on line graphs we can prove additional properties of a parent in the flipping method. Let D be a minimal dominating set of a graph G such that $G[D]$ has at least one edge uw, and assume that $v \in P_D(u)$. Recall that D^* is defined by choosing a maximal independent set $X_{uv} \subseteq P_D(u) \setminus N[v]$ in $G[P_D(u) \setminus N[v]]$, then considering the set $D' = (D \setminus \{u\}) \cup X_{uv} \cup \{v\}$, and letting $D^* = D' \setminus Z_{uv}$ where $Z_{uv} \subseteq D \cap D'$.

Lemma 4. *If G is a line graph, then:*

- $X_{uv} = \emptyset$,
- *each vertex of Z_{uv} is adjacent to exactly one vertex of $P_{D^*}(v) \setminus N[u]$,*
- *each vertex of $P_{D^*}(v) \setminus N[u]$ is adjacent to exactly one vertex of Z_{uv}.*

Proof. Because G is a line graph, the neighborhood of u can be partitioned into two cliques K_1 and K_2. Vertex v is in $P_D(u)$, and for each $x \in P_D(u)$, $xw \notin E(G)$, since $w \in D$. Assume that $w \in K_1$. Then $P_D(u) \subseteq K_2 \subseteq N[v]$. Hence, $X_{uv} \subseteq P_D(u) \setminus N[v] = \emptyset$.

By definition, each vertex of Z_{uv} is adjacent to at least one vertex of $P_{D^*}(v) \setminus N[u]$. Assume that a vertex $z \in Z_{uv}$ is adjacent to at least two vertices $x, y \in P_{D^*}(v) \setminus N[u]$. By the construction of Z_{uv}, z is adjacent to a vertex $z' \in D^* \setminus \{v\}$. Notice that x and y are not adjacent to z', since $x, y \in P_{D^*}(v)$, and recall that x and y are not adjacent to u. Since G is a line graph, x and y are adjacent. Because v is a private vertex for u with respect to D, v and z' are not adjacent, since $z' \in D \cap D^*$ by Lemma 3. Also by Lemma 3, we know that v and z are not adjacent. If $uz, uz' \notin E(G)$, then we get the left graph in Figure 2, if $uz' \in E(G)$ and $uz \notin E(G)$, then we get the center graph in Figure 2, and finally, if the edge uz exists, we get the right graph in Figure 2. Beineke has shown that none of these graphs can be an induced subgraph of a line graph [2], and hence we obtain a contradiction.

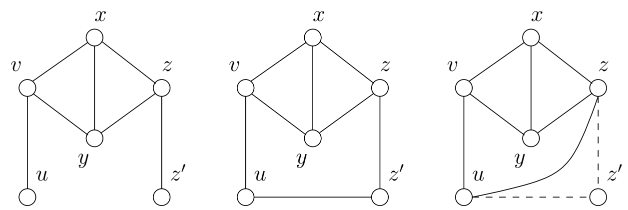

Figure 2: Subgraphs $G[\{u, v, x, y, z, z'\}]$ and $G[\{u, v, x, y, z\}]$ that are forbidden induced subgraphs of line graphs [2].

Finally, observe that each vertex of $P_{D^*}(v) \setminus N[u]$ is adjacent to at least one vertex of Z_{uv} by Lemma 3, because the set of all vertices $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ such that $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$ is exactly the set $P_{D^*}(v) \setminus N[u]$. To obtain a contradiction, assume that there is a vertex $x \in P_{D^*}(v) \setminus N[u]$ adjacent to two distinct vertices $y, z \in Z_{uv}$. Both vertices y and z belong to Z_{uv} that by definition is a subset of D, hence $x \notin P_D[y]$ and $x \notin P_D[z]$. By the second claim, x is the unique vertex of $P_{D^*}(v) \setminus N[u]$ adjacent to y and z. But since $y \in Z_{uv}$, we get that $P_D(y) \cap (P_{D^*}(v) \setminus N[u]) \neq \emptyset$ and y has a neighbor in $P_{D^*}(v) \setminus N[u]$ different from x, which gives the desired contradiction. □

Consider a line graph G with n vertices $v_1, \ldots, v_n$ and m edges. Let D^* be a minimal dominating set and let v be an isolated vertex of $G[D^*]$. Suppose that u is a neighbor of v. Let $\{x_1, \ldots, x_k\} = P_{D^*}(v) \setminus N[u]$. We construct minimal dominating sets from $(D^* \setminus \{v\}) \cup \{u\}$ by adding a set Z that contains a neighbor of each x_i from $N(x_i) \setminus N[v]$. Recall that the vertices $x_1, \ldots, x_k$ should be dominated by Z_{uv} for every child of D^* by Lemma 3, and by the same lemma each x_i is dominated by a vertex from $N(x_i) \setminus N[v]$.

Let $U = N[u] \cup (\bigcup_{i=1}^{k} (N[x_i] \setminus N[v]) \cup \{x_i\})$. We need the following observation that will be used also in the next section. To see its correctness, it is sufficient to notice that because G is claw-free, $N[x_i] \setminus N[v]$ is a clique.

Lemma 5. *For any choice of a set $Z = \{z_1, \ldots, z_k\}$ such that $z_i \in N(x_i) \setminus N[v]$ for $i \in \{1, \ldots, k\}$, U is dominated by $Z \cup \{u\}$.*

We want to ensure that during subsequent removal of vertices from $D^* \setminus \{v\}$ (which we do to guarantee minimality), the number of edges in the obtained minimal dominating set does not decrease. To do this, for each vertex $v_j \in V(G)$, we construct the sets of vertices R_j that cannot belong to Z_{uv} for any child D of D^*, where both D and D^* contain v_j. First, we set $R_j = \emptyset$ for every $v_j \notin D^* \setminus \{v\}$. Let v_j be a vertex of $D^* \setminus \{v\}$ that has a neighbor v_s such that either $v_s \in D^*$ or $v_s = u$. Since G is claw-free, $K = N(v_j) \setminus N[v_s]$ is a clique. Then we set $R_j = K$ in this case. Notice that we can have several possibilities for v_s. In this case v_s is chosen arbitrarily. For all other $v_j \in D^* \setminus \{v\}$, $R_j = \emptyset$. Denote by R the set $\cup_{j=1}^{n} R_j$. For each $i \in \{1, \ldots, k\}$, let

$$Z_i = \{z \in V \mid z \in N(D^* \setminus \{v\}) \cap (N(x_i) \setminus (N[v] \cup R)), N(z) \cap (P_{D^*}(v) \setminus N[u]) = \{x_i\}\}.$$

We generate a set $\mathcal{D}$ of minimal dominating sets as follows.

Case 1. If at least one of the following three conditions is fulfilled, then we set $\mathcal{D} = \emptyset$:

i) there is a vertex $x \in D^* \setminus \{v\}$ such that $N[x] \subseteq N[D^* \setminus \{v, x\}] \cup U$,

ii) $k \geq 1$ and there is an index $i \in \{1, \ldots, k\}$ such that $Z_i = \emptyset$,

iii) u is not adjacent to any vertex of $D^* \setminus \{v\}$ and $N(u) \cap (\cup_{i=1}^{k} Z_i) = \emptyset$.

Otherwise, we consider the following two cases.

Case 2. If u is adjacent to a vertex of $D^* \setminus \{v\}$, then we consecutively construct all sets $Z = \{z_1, \ldots, z_k\}$ where $z_i \in Z_i$, for $1 \leq i \leq k$ (if $k = 0$, then $Z = \emptyset$). For each Z, we construct the set $D' = (D^* \setminus \{v\}) \cup \{u\} \cup Z$. Notice that D' is a dominating set as all vertices of $P_{D^*}[v]$ are dominated by D', but D' is not necessarily minimal. Hence, we construct a minimal dominating set D from D' by greedy removal of vertices. The obtained set D is unique for a given set Z, and it is added to $\mathcal{D}$.

Recall that by the definition of the parent-child relation, u should be dominated by a vertex in a child dominating set. If u is not adjacent to a vertex of $D^* \setminus \{v\}$, it should be adjacent to at least one of the added vertices. This gives us the next case.

Case 3. If u is not adjacent to any vertex of $D^* \setminus \{v\}$, and $N(u) \cap (\bigcup_{i=1}^{k} Z_i) \neq \emptyset$, then we proceed as follows. Let j be the smallest index such that $N(u) \cap Z_j \neq \emptyset$, and let j' be the smallest index such that $j' \geq j$ and $Z_{j'} \setminus N(u) = \emptyset$. It is assumed that $j' = k$, if $Z_{j'} \setminus N(u) \neq \emptyset$ for all $j' \geq j$. For each t starting from $t = j$ and continuing until $t = j'$, we do the following. If $N(u) \cap Z_t = \emptyset$ then we go to next step $t = t + 1$. Otherwise, for each $w \in N(u) \cap Z_t$, we consider all possible sets $Z = \{z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k\} \cup \{w\}$ such that $z_i \in Z_i \setminus N(u)$ for $1 \leq i \leq t - 1$, and $z_i \in Z_i$ for $t + 1 \leq i \leq k$. As above, for each such set Z, we construct the set $D' = (D^* \setminus \{v\}) \cup \{u\} \cup Z$ and then create a minimal dominating set D from D' by greedy removal of vertices. The obtained set D is unique for a given set Z, and it is added to $\mathcal{D}$.

We summarize the properties of the above algorithm in the following lemma.

Lemma 6. *The set $\mathcal{D}$ is a set of minimal dominating sets such that $\mathcal{D}$ contains all children of D^* with respect to flipping u and v. Furthermore, $|E(G[D])| > |E(G[D^*])|$ for every $D \in \mathcal{D}$, and the elements of $\mathcal{D}$ are generated with delay $O(n + m)$.*

Proof. Notice that each set D constructed in Cases 2 or 3 is a minimal dominating set, and $\{u\} \cup Z \subseteq D \subseteq D'$. Furthermore every $y \in D'$ such that $y \in \{u\} \cup Z$ has a private: v is a private for u, each x_i is a private for z_i, and x_t is a private for w in Case 3. Observe also that all the constructed sets D are distinct, because they are constructed for distinct sets Z. We prove the following two claims.

Claim 1. *Each minimal dominating set D, that is a child of D^* with respect to flipping u and v, is in $\mathcal{D}$.*

Proof. Let D be a child of D^* with respect to flipping u and v. Then $D = (D^* \setminus \{v\}) \cup \{u\} \cup Z_{uv}$. Recall that by Lemma 4, $X_{uv} = \emptyset$. This means that $(D^* \setminus \{v\}) \cup \{u\}$ dominates all vertices of G except $P_{D^*}(v) \setminus N[u]$. Since $X_{uv} = \emptyset$, it is clear that each child of D^* with respect to flipping u and v, is obtained by adding vertices that dominate $P_{D^*}(v) \setminus N[u]$, by Lemma 3. Also by Lemma 3, no vertex of Z_{uv} is a neighbor of v, which means that $P_{D^*}(v) \setminus N[u]$ has to be dominated by $N(P_{D^*}(v) \setminus N[u]) \setminus N[v]$. Clearly, $(N[v] \setminus N[u]) \setminus N[D^* \setminus \{v\})] = P_{D^*} \setminus N[u] = \{x_1, \ldots, x_k\}$, and for each $z \in Z_{uv}$, there is x_i adjacent to z. By the last statement of Lemma 3, each x_i is adjacent to a vertex of Z_{uv}. Now by Lemma 4, each x_i is adjacent to exactly one vertex of Z_{uv}, and each vertex $z \in Z_{uv}$ is adjacent to a single vertex of $\{x_1, \ldots, x_k\}$. Denote by z_i the unique neighbor of x_i in Z_{uv}. Clearly, $Z_{uv} = \{z_1, \ldots, z_k\}$.

By Lemma 5, the vertices of the set U are dominated by $Z_{uv} \cup \{u\}$. Hence, for every vertex $x \in D^* \setminus \{v\}$, $N[x] \setminus (N[D^* \setminus \{v, x\}] \cup U) \neq \emptyset$, and we do not have Case 1 *i)*.

Now we show that $z_i \in Z_i$ for every $i \in \{1, \ldots, k\}$. To obtain a contradiction, assume that $z_i \notin Z_i$ for some $i \in \{1, \ldots, k\}$. Because $z_i \in N(x_i) \setminus N[v]$, $N(z_i) \cap (P_{D^*}(v) \setminus N[u]) = \{x_i\}$ and by Lemma 3, z_i is dominated by a vertex of $D^* \setminus \{v\}$, we have that $z_i \in R_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$. Hence, the vertex $v_j \in D^*$ is adjacent to some vertex $v_s \in D^*$, or v_j is adjacent to $v_s = u$ and $R_j = N(v_j) \setminus N[v_s]$. Suppose that v_j is adjacent to $v_s \in D^*$. Since v is an isolated vertex in D^*, $v_s \neq v$. Then v_j is dominated by v_s and all neighbors of v_j that are not dominated by v_s are in the clique R_j, but all these vertices are dominated by z_i. Hence, $N[v_j] \subseteq N[v_s] \cup N[z_i]$, i.e., $P_D[v_j] = \emptyset$, and we have a contradiction. Let now v_j be adjacent to u. Then we conclude that $N[v_j] \subseteq N[u] \cup N[z_i]$ and again get a contradiction. We obtain that $Z_i \neq \emptyset$ for $i \in \{1, \ldots, k\}$ and we do not have Case 1 *ii)*. Recall that for the child D of D^*, u is not an isolated vertex of $D \subseteq (D^* \setminus \{v\}) \cup (\cup_{i=1}^{k} Z_i)$. Hence, we do not have Case 1 *iii)* either. Finally, as $z_i \in Z_i$, Z_{uv} is listed in Case 2 or 3. We conclude that each child of D^* with respect to flipping u and v, is contained in $\mathcal{D}$. □

Claim 2. *For every $D \in \mathcal{D}$, $|E(G[D])| > |E(G[D^*])|$.*

Proof. Assume that a minimal dominating set $D \in \mathcal{D}$ is obtained by greedy removal vertices from $D' = (D^* \setminus \{v\}) \cup \{u\} \cup Z$ where $Z = \{z_1, \ldots, z_k\}$.

We show that only isolated vertices of $D^* \setminus \{v\}$ that are not adjacent to u can be removed. To obtain a contradiction, assume that some vertex $v_j \in D^* \setminus \{v\}$ such that v_j has a neighbor v_s where $v_s \in D^* \setminus \{v\}$ or $v_s = u$, and $R_j = N(v_j) \setminus N[v_s]$ is removed. As v_j is removed, v_j has no privates with respect to D'. Notice that by the construction of the sets $Z_1, \ldots, Z_k$, $R_j \cap Z = \emptyset$. Observe also that $u \notin R_j$ as we would have $N[v_j] \subseteq N[v_s] \cup N[u]$, and we would have Case 1 *i)*. Let $P = N(v_j) \setminus N[D^* \setminus \{v, v_j\}]$. Since v_j has no privates with respect to D', each vertex $y \in P$ is dominated by a vertex of Z or by u. Clearly, if y is dominated by u, then $y \in N[u] \subseteq U$. Suppose that y is dominated by $z_i \in Z$ but not u. Since x_i is not dominated by $D^* \setminus \{v\}$, $y \neq x_i$. We consider two cases.

Case a). The vertex v_j has a neighbor $v_s \in D^* \setminus \{v\}$ and $R_j = N(v_j) \setminus N[v_s]$. By the construction of Z_i, z_i is dominated by some vertex $z \in D^* \setminus \{v\}$. If $z = v_j$, then because $z_i \notin R_j$, z_i is adjacent to v_s. Hence, without loss of generality we can assume that $z \neq v_j$, as otherwise we can take $z = v_s$. Since $y \in P = N(v_j) \setminus N[D^* \setminus \{v, v_j\}]$, z is not adjacent to y, and because x_i is not dominated by $D^* \setminus \{v\}$, z is not adjacent to x_i. Since G is claw-free, $yx_i \in E(G)$. If y is adjacent to v, then $G[\{z, z_i, y, x_i, v, u\}]$ or $G[\{z_i, y, x_i, v, u\}]$ is isomorphic to one of the graphs shown in Fig. 2 and forbidden for line graphs [2]. Thus y is not adjacent to v. This means that $y \in N(x_i) \setminus N[v] \subseteq U$. Hence, we conclude that $y \in U$ in the considered case.

Case b). Some vertex v_j is adjacent to u, and $R_j = N(v_j) \setminus N[u]$. Suppose that z_i is adjacent to u. Recall that u is not adjacent to y and x_i. Hence, as G is claw-free, $yx_i \in E(G)$. If y is adjacent to v, then $G[\{z_i, y, x_i, v, u\}]$ is forbidden for line graphs [2]. Thus y is not adjacent to v. This means that $y \in N(x_i) \setminus N[v] \subseteq U$. Suppose now that z_i is not adjacent to u. By the construction of Z_i, z_i is dominated by some vertex $z \in D^* \setminus \{v\}$. If $z = v_j$, then because $z_i \notin R_j$, z_i is adjacent to u and we obtain a contradiction. Thus $z \neq v_j$. Since $y \in P$, z is not adjacent to y, and because x_i is not dominated by $D^* \setminus \{v\}$, z is not adjacent to x_i. Since G is claw-free, $yx_i \in E(G)$. If y is adjacent to v, then $G[\{z, z_i, y, x_i, v, u\}]$ or $G[\{z_i, y, x_i, v, u\}]$ is forbidden for line graphs [2]. Thus y is not adjacent to v. This means that $y \in N(x_i) \setminus N[v] \subseteq U$. Hence, we conclude that $y \in U$ in this case, too.

We have that $P \subseteq U$, but in this case $N[v_j] \subseteq N[D^* \setminus \{v, v_j\}] \cup U$, and we have Case 1 *i)* of our algorithm.

We proved that only isolated vertices of $D^* \setminus \{v\}$ that are not adjacent to u may be removed by greedy removal of vertices. If u is adjacent to a vertex of D^*, then $|E(G[D])| > |E(G[D^*])|$, because we do not destroy edges between the vertices of $D^* \setminus \{v\}$, and we add at least one edge incident with u to the constructed set. Suppose that u is not adjacent to the vertices of $D^* \setminus \{v\}$. Then we have Case 3, and Z contains at least one vertex adjacent to u, i.e., we increase the number of edges. This observation concludes the proof that $|E(G[D])| > |E(G[D^*])|$. □

To complete the proof of Lemma 6, it remains to evaluate the running time. Observe that the sets $Z_1, \ldots, Z_k$ can be constructed in time $O(n + m)$. The set Z can clearly be generated with delay $O(n)$, and greedy removal of vertices on each generated set can be done in time $O(n + m)$. Note that each set is generated exactly once. □

Combining Lemmas 2 and 6, we obtain the following theorem.

Theorem 2. *All minimal dominating sets of a line graph can be enumerated in incremental polynomial time. On input graphs with n vertices and m edges, the delay is $O(n^2 m^2 |\mathcal{L}|)$, and the total running time is $O(n^2 m |\mathcal{L}^*|^2)$, where $\mathcal{L}$ is the set of already generated minimal dominating sets and $\mathcal{L}^*$ is the set of all minimal dominating sets.*

This theorem immediately gives us the following corollary.

Corollary 1. *All minimal edge dominating sets of an arbitrary graph be enumerated in incremental polynomial time. On input graphs with m edges, the delay is $O(m^6 |\mathcal{L}|)$ and the total running time is $O(m^4 |\mathcal{L}^*|^2)$, where $\mathcal{L}$ is the set of already generated minimal edge dominating sets and $\mathcal{L}^*$ is the set of all minimal edge dominating sets.*

4.2 Enumeration of minimal dominating sets of line graphs of bipartite graphs

We can improve the dependence of the total running time on the size of the output if we restrict our attention to edge dominating sets of bipartite graphs. Again, we work on the equivalent problem of generating minimal dominating sets of line graphs of bipartite graphs. The following observation is not difficult to verify.

Lemma 7. *Let H be a non-empty bipartite graph and let $G = L(H)$. Then*

- *for each vertex v in G, either $N(v)$ is a clique or $N(v)$ is a union of two disjoint cliques K_1 and K_2 such that no vertex of K_1 is adjacent to a vertex of K_2, and*
- *G has no induced cycle on $2k+1$ vertices for any $k > 1$.*

Furthermore, if u and v are adjacent vertices of G, then for every $x \in N(v) \setminus N[u]$, $N(x) \setminus N[v]$ is a clique, and for any distinct $x, y \in N(v) \setminus N[u]$, $N(x) \setminus N[v]$ and $N(y) \setminus N[v]$ are disjoint.

Let G be the line graph of a bipartite graph H, such that n is the number of vertices and m is the number of edges of G. Let $V(G) = \{v_1, \ldots, v_n\}$. Let D^* be a minimal dominating set of G that has an isolated vertex v in $G[D^*]$. Suppose that u is a neighbor of v in G. Let $\{x_1, \ldots, x_k\} = P_{D^*}(v) \setminus N[u]$.

As with general line graphs, we construct minimal dominating sets from $(D^* \setminus \{v\}) \cup \{u\}$ by adding a set of vertices Z that contains a neighbor of each x_i. Now we have to ensure that neighbors are selected in such a way that the obtained sets are minimal dominating sets that are children of D^*. To do this, for each vertex $v_j \in V(G)$, we construct the sets of vertices that cannot belong to Z_{uv} for any child D of D^*, where both D and D^* contain v_j. For each v_j, we define sets R_j and S_j, where the sets R_j are used to ensure minimality, and the sets S_j are used to guarantee that the obtained set is a child of D^*. First, we set $R_j = S_j = \emptyset$ for every $v_j \notin D^* \setminus \{v\}$. Recall that by Lemma 5, the set $U = N[u] \cup (\bigcup_{i=1}^{k} ((N[x_i] \setminus N[v]) \cup \{x_i\})$ is dominated for every choice of $Z \subseteq V(G) \setminus N[v]$ that dominates $\{x_1, \ldots, x_k\}$, and $\{x_1, \ldots, x_k\}$ should be dominated by Z_{uv} for every child of D^*, by Lemma 3.

Let v_j be a vertex of $D^* \setminus \{v\}$. By Lemma 7, $N(v_j)$ is a union of at most two disjoint cliques with no edges between them. We define sets R_j as follows:

i) if $P_{D^*}[v_j] \setminus U = \{v_j\}$ and $N(v_j) \cap (N(v) \setminus N[u]) = \emptyset$, then $R_j = N(v_j)$;

ii) if $P_{D^*}[v_j] \setminus U \neq \{v_j\}$, and

- for every vertex[2] $x \in N(v_j) \cap (N(v) \setminus N[u])$, x is dominated by at least two vertices of $D^* \setminus \{v\}$, and
- $P_{D^*}(v_j) \setminus U \subseteq K$, where K is a maximal clique in $N(v_j)$,

[2]Note that $N(v_j) \cap (N(v) \setminus N[u])$ might be empty.

then $R_j = K$;

iii) in all other cases $R_j = \emptyset$.

Let v_j be a vertex of $D^* \setminus \{v\}$. If v_j is adjacent to a vertex $x \in N(v) \setminus N[u]$ that has no other neighbors in $D^* \setminus \{v\}$, and $(P_{D^*}[v_j] \cap N(x)) \setminus U \subseteq \{v_j\}$, then we define $S_j = \{v_\ell \in V(G) \mid v_j v_\ell \in E(G), \ell > j\}$. Otherwise, we set $S_j = \emptyset$.

Now for each $i \in \{1, \ldots, k\}$, we construct the set Z_i. A vertex $v_s \in N(x_i) \setminus N[v]$ is included in Z_i if and only if v_s is adjacent to a vertex of $D^* \setminus \{v\}$, and $v_s \notin R_j$ and $v_s \notin S_j$ for all $j \in \{1, \ldots, n\}$. Observe that by Lemma 7, the sets Z_i are disjoint cliques, and notice that $Z_i \cap D^* = \emptyset$ as x_i is in $P_{D^*}(v) \setminus N[u]$. We generate a set $\mathcal{D}$ of minimal dominating sets as follows.

Case 1. If at least one of the following three conditions is fulfilled, then we set $\mathcal{D} = \emptyset$:

i) there is a vertex $x \in D^* \setminus \{v\}$ such that $N[x] \subseteq N[D^* \setminus \{v, x\}] \cup U$,

ii) $k \geq 1$, and there is an index $i \in \{1, \ldots, k\}$ such that $Z_i = \emptyset$,

iii) u is not adjacent to any vertex of $D^* \setminus \{v\}$, and $N(u) \cap (\cup_{j=1}^{k} Z_j) = \emptyset$.

Otherwise, we consider the following two cases.

Case 2. If u is adjacent to a vertex of $D^* \setminus \{v\}$, then one after another we consider all possible sets $Z = \{z_1, \ldots, z_k\}$ such that $z_i \in Z_i$ for $1 \leq i \leq k$ (if $k = 0$ then $Z = \emptyset$). For each Z, we construct the set $D = (D^* \setminus \{v\}) \cup \{u\} \cup Z$ and add it to $\mathcal{D}$.

Recall that by the definition of the parent-child relation, u should be dominated by a vertex of D. If u is not adjacent to a vertex of $D^* \setminus \{v\}$, it should be adjacent to at least one of the added vertices. This gives us the next case.

Case 3. If u is not adjacent to any vertex of $D^* \setminus \{v\}$, and $N(u) \cap (\bigcup_{i=1}^{k} Z_i) \neq \emptyset$, then we proceed as follows. Let j be the smallest index such that $N(u) \cap Z_j \neq \emptyset$, and let j' be the smallest index such that $j' \geq j$ and $Z_{j'} \setminus N(u) = \emptyset$. It is assumed that $j' = k$, if $Z_{j'} \setminus N(u) \neq \emptyset$ for $j' \geq j$. For each t starting from $t = j$ and continuing until $t = j'$, we do the following. If $N(u) \cap Z_t = \emptyset$ then we go to next step $t = t + 1$. Otherwise, for each $w \in N(u) \cap Z_t$, we consider all possible sets $Z = \{z_1, \ldots, z_{t-1}, z_{t+1}, \ldots, z_k\} \cup \{w\}$ such that $z_i \in Z_i \setminus N(u)$ for $1 \leq i \leq t - 1$, and $z_i \in Z_i$ for $t + 1 \leq i \leq k$. As above, for each such set Z, we construct the set $D = (D^* \setminus \{v\}) \cup \{u\} \cup Z$ and add it to $\mathcal{D}$.

The correctness of the described algorithm is proved in Lemma 8 below. The next observation follows immediately from the first claim of Lemma 7. The graph obtained from the complete graph on four vertices by the deletion of one edge is called a *diamond*. A graph that has no induced subgraph isomorphic to a diamond is said to be *diamond-free*.

Observation 2. *Line graphs of bipartite graphs are diamond-free.*

Lemma 8. *The set $\mathcal{D}$ is the set of all children of D^* with respect to flipping u and v, and the elements of $\mathcal{D}$ can be generated with delay $O(n + m)$.*

Proof. We prove the following three claims.

Claim 1. *Each $D \in \mathcal{D}$ is a minimal dominating set.*

Proof. Let $D' = (D^* \setminus \{v\}) \cup \{u\}$. Recall that by Case 1 *i)*, if there is a vertex $x \in D^* \setminus \{v\}$ such that $N[x] \subseteq N[(D^* \setminus \{v, x\}) \cup \{u\}] \cup U$, then $\mathcal{D} = \emptyset$. Hence, for each $x \in D'$ such that $x \neq u$, at least one vertex of $N[x]$ is not dominated by $D' \setminus \{x\}$ and is not included in U. Therefore, it is possible to extend D' to a minimal dominating set, and we do it by including Z. Let $D = D' \cup Z$. D is a dominating set of G, because $P_{D^*}[v]$ is dominated by $Z \cup \{u\}$. To obtain a contradiction, assume that D is not minimal. Then there is a vertex $v_j \in D$ such that v_j has no privates with respect to D. Vertex v is a private for u, each vertex x_i is a private for z_i, and x_t is a private for w in Case 3. Hence, $v_j \in D^* \setminus \{v\}$. As U is dominated by every minimal dominating set D generated by our algorithm from D^*, we can conclude that at least one vertex of $N[v_j]$ is not dominated by $D' \setminus \{v_j\}$ and is not included in U, and $P = P_{D^*}[v_j] \setminus U \neq \emptyset$. We have that $P \subseteq N[Z]$ as $P_D[v_j] = \emptyset$. There is a vertex $z_i \in Z$ such that z_i is adjacent to a vertex $y \in P$, and let us assume for now that $z_i v_j \notin E(G)$. By the construction of the sets Z_i, z_i is adjacent to a vertex $z \in D^* \setminus \{v\}$. Because $z_i v_j \notin E(G)$, $z \neq v_j$. Since x_i is dominated only by z_i, x_i is not dominated by z. Then either $zy \in E(G)$ or $yx_i \in E(G)$, since G is a line graph. Since $y \notin U$ by definition, $yx_i \notin E(G)$ as $(N[x_i] \setminus N[v]) \cup \{x_i\} \subseteq U$. Thus, we can conclude that $zy \in E(G)$, but then y is dominated by at least two vertices of $D^* \setminus \{v\}$, contradicting that $y \in P^*_D(v_j)$. As this argument holds for every vertex $y \in P$, we get that P is dominated by $Z \cap N(v_j)$.

If $P = \{v_j\}$ and $N(v_j) \cap (N(v) \setminus N[u]) = \emptyset$, then $R_j = N(v_j)$, but this contradicts the definition of Z_i, as $Z_i \cap R_i = \emptyset$. Hence if $P = \{v_j\}$ then there is a vertex $x \in N(v_j) \cap (N(v) \setminus N[u])$. In this case, because x is not a private for v_j with respect to D, x is dominated by a vertex $y \in D$. As x is adjacent to both $v, v_j \in D^*$, it is clear that $x \notin \{x_1, x_2, \ldots, x_k\}$, and by Lemma 4.2 we can conclude that $\{x, x_1, x_2, \ldots, x_k\}$ is a clique and that $N(x) \cap N(x_i) = \emptyset$ for $i \in [1, \ldots, k]$ and thus $y \notin Z_i$. Hence, $y \in D^* \setminus \{v, v_j\}$. By Lemma 7, $yv_j \in E(G)$ and $v_j \notin P$, contradicting $P = \{v_j\}$. It follows that P has at least one vertex in $N(v_j)$. By Lemma 7, $N(v_j)$ is a union of at most two cliques. Suppose that $N(v_j)$ is a union of two non-empty cliques K_1, K_2 such that there are no edges between K_1 and K_2, and suppose that P intersects both K_1 and K_2. Then there are vertices $z_i, z_{i'} \in Z$ such that $z_i \in K_1$ and $z_{i'} \in K_2$. Then we conclude that G has an induced cycle $v_j, z_i, x_i, x'_i, z'_i$ on 5 vertices, which contradicts Lemma 7. Therefore, there is a maximal clique K in $N(v_j)$ such that $P \subseteq K \cup \{v_j\}$, and there is $z_i \in Z$ such that $z_i \in K$. Since $z_i \notin R_j$, v_j is adjacent to a vertex $x \in N(v) \setminus N[u]$ such that x is not adjacent to a vertex of $D^* \setminus \{v, v_j\}$. Again, as $x \in N(v) \cap N(v_j)$, the vertex $x \notin P^*_D(v)$, and by Lemma 4.2 we have that $N(x) \cap N(x_i) = \emptyset$ for $1 \leq i \leq k$. Thus, we can conclude that x is a private for v_j with respect to D. The obtained contradiction shows that D is minimal. □

Claim 2. *Each $D \in \mathcal{D}$ is a child of D^* with respect to flipping u and v.*

Proof. To obtain a contradiction, assume that there is $D \in \mathcal{D}$ that is not a child of D^*. Let Z be the set associated with D when D was generated. Denote by D^{**} the parent of D

with respect to flipping u and v. Hence, $D = (D^{**} \setminus \{v\}) \cup \{u\} \cup Z_{uv}$ and $Z \neq Z_{uv}$. Clearly, $D^* \setminus D^{**} \subseteq Z_{uv}$, $D^* \setminus D^{**} \neq \emptyset$, $D^{**} \setminus D^* \subseteq Z$, and $D^{**} \setminus D^* \neq \emptyset$. Suppose that $W = Z \cap Z_{uv}$ and $D' = ((D \setminus \{u\}) \cup \{v\}) \setminus W$. Observe that D' is a dominating set of G, however it is not minimal. Let v_i be the vertex with the smallest index i in $(D^{**} \setminus D^*) \cup (D^* \setminus D^{**})$. Since Z_{uv} is chosen by greedy removal of vertices, $v_i \in Z_{uv}$. Notice also that v_i has no privates with respect to D' but v_i has privates with respect to $D^* = D' \setminus (Z \setminus Z_{uv})$, and v_i is in $N(x)$ for some $x \in N(v) \setminus N[u]$, where $x \notin P_{D^*}(v) \setminus N[u] = \{x_1, \ldots, x_k\}$. We consider two cases.

Case a). Assume that v_i is not adjacent to a vertex of $Z \setminus W$. Then because v_i has no privates with respect to D', v_i is adjacent to another vertex $v_{i'} \in D^* \setminus \{v\}$. Let $y \in P_{D^*}[v_i]$ and assume that $v_{j_1}, \ldots, v_{j_s} \in Z \setminus Z_{uv}$ are the vertices adjacent to y. Clearly, these are the only candidates to dominate y. By the construction of Z, each vertex v_{j_ℓ} is a unique element of D in the neighborhood of some vertex $x_{p_\ell} \in P_{D^*}(v) \setminus N[u]$. Vertex $y \neq v_i$ and $y \notin \{v_{j_1}, \ldots, v_{j_s}\}$ as there are no edges between v_i and vertices of $Z \setminus W$. Recall that $v_{j_1}, \ldots, v_{j_s}$ are dominated by D^*, and each vertex v_{j_ℓ} is the unique element of $D \cap N(x_{p_\ell})$. Since G is the line graph of a bipartite graph, y is adjacent either to x or to some vertices x_{p_ℓ}, since otherwise $xv_iyv_{j_\ell}x_{p_\ell}$ would be an induced cycle on 5 vertices, which contradicts Lemma 7. Note that y cannot be adjacent to both x and some vertex x_{p_ℓ}, because otherwise the vertices v_i, y, x, x_ℓ induce a diamond, contrary to Observation 2.

Suppose that y is adjacent to x, and hence not to any vertex of $\{x_{p_1}, \ldots, x_{p_s}\}$. Then $\{y\} \cup \{v_{j_1}, \ldots, v_{j_s}\}$ is a clique, because if some $v_{j_\ell}, v_{j_{\ell'}}$ were not adjacent, then $N(y)$ would contain three pairwise non-adjacent vertices $x, v_{j_\ell}, v_{j_{\ell'}}$ for $s > 1$, inducing a claw and contradicting that G is claw-free. By the construction of Z, each v_{j_ℓ} is dominated by a vertex of D^*. Then every vertex that dominates v_{j_ℓ}, is in the maximal clique that contains $\{y\} \cup \{v_{j_1}, \ldots, v_{j_s}\}$, and y remains dominated after the removal of $v_{j_1}, \ldots, v_{j_s}$, giving a contradiction. Hence, y is not adjacent to x but adjacent to some vertices of $\{x_{p_1}, \ldots, x_{p_s}\}$. Assume that $yx_{p_1} \in E(G)$. The neighborhood of y is a union of at most two disjoint cliques K_1 and K_2 such that there are no edges between vertices of K_1 and vertices of K_2, by Lemma 7. We conclude that $s = 1$. Since G is a line graph and hence claw-free, either $v_{i'}x \in E(G)$ or $v_{i'}y \in E(G)$, as otherwise $y, v_{i'}, x, v_i$ induces a claw. If $v_{i'}x \in E(G)$, then $N[v_i] \subseteq N[(D^* \setminus \{v, v_i\}) \cup \{u\}] \cup U$, and we would set $\mathcal{D} = \emptyset$ by Case 1 *i)* of the algorithm. If $v_{i'}y \in E(G)$, then y cannot become a private neighbor of v_i with respect to D^* which is obtained from D' by the removal of Z, again giving a contradiction.

Case b). Assume that v_i is adjacent to at least one vertex of $Z \setminus W$. Denote by $v_{j_1}, \ldots, v_{j_s}$ these vertices. Observe that $i < j_1, \ldots, j_s$, because otherwise greedy removal would remove these vertices first. It follows that $v_{j_1}, \ldots, v_{j_s} \notin S_i$, and hence for each vertex v_{j_t}, at least one of the conditions in the definition of S_i does not apply. Recall that $v_i \in D^* \setminus \{v\}$ and v_i is adjacent to $x \in N(v) \setminus N[u]$. If (1) x has no other neighbors in $D^* \setminus \{v\}$, and (2) $(P_{D^*}[v_i] \cap N(x)) \setminus U \subseteq \{v_i\}$, then $v_{j_1}, \ldots, v_{j_s}$ would be in S_i. Hence, at least one of the conditions (1) and (2) is not fulfilled.

Suppose that condition (1) does not hold and there is a vertex $v_{i'} \in D^* \setminus \{v\}$ such that $v_{i'} \neq v_i$ and $v_{i'} \in N(x) \setminus N[v]$ as v have no neighbors in D^*. Since G is a line graph, $v_i v_{i'} \in E(G)$. The set $N(v_i)$ is a union of at most two cliques and $v_{i'}, v_{j_1}$ are in distinct cliques contained in $N(v_i)$. Hence, either $N[v_i] \subseteq N[(D^* \setminus \{v, v_i\}) \cup \{u\}] \cup U$, and we would set $\mathcal{D} = \emptyset$ according to Case 1 *i)*, or $v_{j_1} \in R_i$. To see this, observe that if $N[v_i] \setminus (N[(D^* \setminus \{v, v_i\}) \cup \{u\}] \cup U) \neq \emptyset$, then $P_{D^*}[v_i] \setminus U \neq \{v_i\}$ as $v_{i'} v_i \in E(G)$. Also x is the unique vertex of $N(v_i) \cap (N(v) \setminus N[u])$, because G is diamond-free by Observation 2. Then $P_{D^*}(v_i) \setminus U \subseteq K$, where K is the maximal clique in $N(v_i) \setminus N(x)$, and $v_{j_1} \in K$. But by definition, $R_i = K$, which gives a contradiction.

Therefore, we can assume now that condition (1) holds but not condition (2), and $(P_{D^*}[v_i] \cap N(x)) \setminus U$ is not a subset of $\{v_i\}$. Then there is a vertex $y \in P_{D^*}[v_i]$ such that $y \neq v_i$ and $yx \in E(G)$. Let $v_{p_1}, \ldots, v_{p_l}$ be the vertices of $Z \setminus Z_{uv}$ adjacent to y. By the construction of Z, each vertex v_{p_r} is a unique element of D in the neighborhood of some vertex $x_{q_r} \in P_{D^*}(v) \setminus N[u]$. Recall that $v_{p_1}, \ldots, v_{p_l}$ are dominated by D^* and each vertex v_{p_r} is a single element of D in $N(x_{q_r})$. Since G is the line graph of a bipartite graph, by Observation 2, y is not adjacent to $x_{p_1}, \ldots, x_{p_l}$. By the construction of Z, each v_{p_r} is dominated by a vertex of D^*. Then every vertex that dominates v_{p_r} is in the maximal clique that contains $\{y\} \cup \{v_{p_1}, \ldots, v_{p_l}\}$ and y remains dominated after the removal of $v_{p_1}, \ldots, v_{p_l}$, resulting in a contradiction. □

Claim 3. *If D is a child of D^* with respect to flipping u and v, then $D \in \mathcal{D}$.*

Proof. Let D be a child of D^* with respect to flipping u and v. Then $D = (D^* \setminus \{v\}) \cup \{u\} \cup Z_{uv}$. By Lemmas 3, 4 and 7, $Z_{uv} = \{z_1, \ldots, z_k\}$, where z_i is in the clique $N(x_i) \setminus N[v]$ for $\{x_1, \ldots, x_k\} = P_{D^*}(v) \setminus N[u]$, and the cliques $N(x_i) \setminus N[v]$ are disjoint. Also by Lemma 3, each z_i is adjacent to a vertex of $D^* \setminus \{v\}$. We show that each $z_i \in Z_i$, and thus $Z_i \neq \emptyset$.

We prove that $z_i \notin R_j$ for any $j \in \{1, \ldots, n\}$. To obtain a contradiction, assume that $z_i \in R_j$. Particularly, it means that $R_j \neq \emptyset$. Recall that in this case $v_j \in D^* \setminus \{v\}$. Furthermore, if $P_{D^*}[v_j] \setminus U = \{v_j\}$ and $N(v_j) \cap (N(v) \setminus N[u]) = \emptyset$, then $R_j = N(v_j)$. Moreover, if $P_{D^*}[v_j] \setminus U \neq \{v_j\}$ and *(i)* for any vertex $x \in N(v_j) \cap (N(v) \setminus N[u])$, x is dominated by at least two vertices of $D^* \setminus \{v\}$, and *(ii)* $P_{D^*}(v_j) \setminus U \subseteq K$, where K is a maximal clique in $N(v_i)$, then $R_i = K$. We consider two cases.

Assume first that $P_{D^*}[v_j] \setminus U = \{v_j\}$ and $N(v_j) \cap (N(v) \setminus N[u]) = \emptyset$. Then all neighbors of v_j that are not in U are dominated by $D^* \setminus \{v, v_j\}$. Therefore, all the neighbors of v_j are dominated by $D \setminus \{v_j\}$. As $z_i \in R_j$ and $R_j = N(v_j)$, in this case vertex v_j has no private neighbors, contradicting the minimality of D.

Now let $P_{D^*}(v_j) \setminus U \neq \emptyset$. If $N(v_j) \cap (N(v) \setminus N[u]) = \emptyset$ and $P_{D^*}(v_j) \setminus U \subseteq K$, where K is a maximal clique in $N(v_j)$. Then $z_i \in K$ as $R_j = K$, and we can again conclude that D is not a minimal dominating set. Suppose that $N(v_i) \cap (N(v) \setminus N[u]) \neq \emptyset$ and each $x \in N(v_i) \cap (N(v) \setminus N[u])$ is dominated by at least two vertices of $D^* \setminus \{v\}$. If $P_{D^*}(v_j) \setminus U \subseteq K$, where K is a maximal clique in $N(v_j)$, then $z_i \in K$. As $P_D[v_j] \neq \emptyset$ and there is a vertex $z_i \in K$, we can conclude that $P_D[v_j] \subseteq N(v_j) \cap (N(v) \setminus N[u])$, but all these vertices are dominated by $D^* \setminus \{v, v_j\}$, which contradicts the minimality of D.

The next step is to show that $z_i \notin S_j$ for $j \in \{1, \ldots, n\}$. To obtain a contradiction, let $z_i \in S_j$ for some $j \in \{1, \ldots, n\}$. Clearly, $S_j \neq \emptyset$ in this case, and recall that v_j is a vertex of $D^* \setminus \{v\}$ such that (a) v_j is adjacent to a vertex $x \in N(v) \setminus N[u]$ that has no other neighbors in $D^* \setminus \{v\}$, and (b) $(P_{D^*}[v_j] \cap N(x)) \setminus U \subseteq \{v_j\}$. Then $z_i = v_s$ for $s > j$ and $v_s v_j \in E(G)$. It follows that v_j is in the clique $K = N(x) \setminus N[v]$, and $K \cap D^* = \{v_j\}$, as $N(x) \cap D^* = \{v, v_j\}$. Consider $D' = D^* \cup \{z_i\}$. Vertex v_j has a private with respect to this set, as otherwise $v_j \in D^* \setminus \{v\}$ would be removed by greedy removal of vertices when reaching parent D^* from D. Notice that v_j is not a private for itself because it is dominated by z_i. Assume that there is a private neighbor $y \in P_{D'}(v_j)$ such that $y \in K$. By condition (b), $P_{D'}(v_j) \cap K \subseteq U$. Hence, y is either dominated by some vertex z_t or y is adjacent to u. Suppose first that there is a vertex z_t adjacent to y and notice that z_t dominates a vertex x_t where $x \neq x_t$. Vertices x and x_t are adjacent as $N[v] \setminus N(u)$ is a clique by Lemma 7. By Lemma 3, z_t is not adjacent to v, and z_t is not adjacent to x, as otherwise the vertices v, x_t, z_t, x would induce a diamond and violate Observation 2. Furthermore, y is not adjacent to x_t, as otherwise the vertices x, y, z_t, x_t would induce a diamond. Vertex z_t is dominated by some vertex $y' \in D^*$. Observe also that y' is not adjacent to x_t, because x_t is a private for v with respect to D^*. Since G is a line graph and hence claw-free, and $yx_t, x_t y' \notin E(G)$ which implies that $y' = v_j$ as y is private for v_j with respect to D', we conclude that $yy' \in E(G)$, which contradicts our assumption that y is private for v_j with respect to D'. Now the vertices x, v_j, z_t, y induce a diamond which violates Observation 2. Suppose now that y is not dominated by Z_{uv}. Thus $yu \in E(G)$. Since D is a child of D^*, u is adjacent to a vertex $w \in D$. Assume that $w \in D^* \setminus \{v\}$. Notice that y is not adjacent to v as it is private for v_j for with respect to D'. Observe also that w is not adjacent to v, as v is private for u with respect to D. Since G is claw-free, we conclude that $yw \in E(G)$. As y is private for v_j with respect to D', this means that $v_j = w$. Now we have a diamond induced by the vertices u, v_j, x, y, which violates Observation 2. Hence, $w \notin D^* \setminus \{v\}$, which means that $w \in Z_{uv}$. By the previous case $wy \notin E(G)$. Now we have that u is adjacent to v, w, y, but the vertices v, w, y are pairwise non-adjacent, so G contains a claw as an induced subgraph, which is a contradiction. Thus for every $y \in P_{D'}[v_j]$, $y \notin K$. It follows that $yz_i \in E(G)$ as G is a line graph. Hence, y is not a private for v_j with respect to D'. The obtained contradiction proves that $z_i \notin S_j$ for $j \in \{1, \dots, n\}$.

We have thus shown that $z_i \in Z_i$ and $Z_i \neq \emptyset$ for $i \in \{1, \dots, k\}$. Now we show that none of the three conditions of Case 1 applies, and thus we can conclude that $\mathcal{D} \neq \emptyset$. *(i)* Since $(D^* \setminus \{v\}) \cup \{u\}$ is a subset of minimal dominating set D, then for every $x \in D^* \setminus \{v\}$, $N[x]$ has a vertex that is not in U and not dominated by other vertices of $D^* \setminus \{v\}$. *ii)* If $k \geq 1$, then by the argument above $Z_i \neq \emptyset$ as $z_i \in Z_i$ for $i \in \{1, \dots, k\}$. *iii)* Because D is a child of D^*, u is adjacent to some vertex of $D^* \setminus \{v\}$ or u is adjacent to some vertex of $Z_{uv} \subseteq \cup_{j=1}^{k} Z_j$.

It remains to observe that $Z = Z_{uv}$ should be considered for the inclusion in $(D^* \setminus \{v\}) \cap \{u\}$. If u is adjacent to some vertex of $D^* \setminus \{v\}$, D is included in $\mathcal{D}$ in Case 2, and if u is not adjacent to the vertices of $D^* \setminus \{v\}$, but u is adjacent to some vertex $w \in Z_{uv} \subseteq \cup_{j=1}^{k} Z_j$, then D is included in $\mathcal{D}$ when we consider Case 3. □

To conclude the proof of the first statement of Lemma 8, it is sufficient to observe that the three claims above immediately imply that $\mathcal{D}$ is the set of all children of D^* with respect to flipping u and v.

For the correctness of the running time statement, observe that the sets $Z_1, \ldots, Z_k$ can be constructed in $O(n+m)$ time. Within the same time we can also compute $Z_i \cap N(u)$ and $Z_i \setminus N(u)$ for $1 \leq i \leq k$, which gives delay $O(n+m)$ before the first minimal dominating set is generated. In Case 2, we can trivially generate the next set Z in $O(n)$ time. In Case 3, every considered vertex w results in the generation of a new set Z. Hence, in both Cases 2 and 3, we can construct the next set Z with delay $O(n)$. Note that by the way we generate sets Z, each minimal dominating set is generated exactly once. $\square$

By Lemmas 2 and 8, we have the following theorem and corollary.

Theorem 3. *All minimal dominating sets of the line graph of a bipartite graph can be enumerated in incremental polynomial time. On input graphs with n vertices and m edges, the delay is $O(n^2m|\mathcal{L}|)$, and the total running time is $O(n^2m|\mathcal{L}^*|)$, where $\mathcal{L}$ is the set of already generated minimal dominating sets, and $\mathcal{L}^*$ is the set of all minimal dominating sets.*

Corollary 2. *All minimal edge dominating sets of a bipartite graph edges can be enumerated in incremental polynomial time. On input graphs with m edges, the delay is $O(m^4|\mathcal{L}|)$, and the total running time is $O(m^4|\mathcal{L}^*|)$, where $\mathcal{L}$ is the set of already generated minimal dominating sets, and $\mathcal{L}^*$ is the set of all minimal edge dominating sets.*

5 Enumeration of minimal dominating sets of graphs of large girth

On line graphs we were able to observe properties of the parent relation in addition to uniqueness, which made it possible to apply the flipping method and design efficient algorithms for enumerating the minimal dominating sets. In this section we show that the flipping method can also be applied to graphs of girth at least 7. To do this, we observe other desirable properties of the parent relation on this graph class.

Let D be a minimal dominating set of a graph G such that $G[D]$ has at least one edge. Let also $u \in D$ be a vertex dominated by another vertex $w \in D$ and assume that $v \in P_D(u)$. Recall that its parent D^* is defined by choosing a maximal independent set $X_{uv} \subseteq P_D(u) \setminus N[v]$ in $G[P_D(u) \setminus N[v]]$, considering the set $D' = (D \setminus \{u\}) \cup X_{uv} \cup \{v\}$, and then letting $D^* = D' \setminus Z_{uv}$ where $Z_{uv} \subseteq D \cap D'$. We can easily observe the following.

Lemma 9. *If G is a graph of girth at least 7, then $X_{uv} = P_D(u) \setminus \{v\}$, and each vertex of Z_{uv} dominates at most one vertex of $\cup_{x \in X_{uv} \cup \{v\}} P_{D^*}(x) \setminus N[u]$.*

Let D^* be a minimal dominating set of a graph G of girth at least 7 with n vertices and m edges, and let v be an isolated vertex of $G[D^*]$. Suppose that u is a neighbor of v. Let $\{y_1, \ldots, y_k\} = P_{D^*}(v) \setminus N[u]$. For each $i \in \{1, \ldots, k\}$, denote by $Z_i = N(y_i) \setminus \{v\}$. We generate a set $\mathcal{D}$ of minimal dominating sets as follows.

Case 1. If $k \geq 1$ and there is an index $i \in \{1, \ldots, k\}$ such that $Z_i = \emptyset$, then $\mathcal{D} = \emptyset$.

Case 2. If $k \geq 1$ and $Z_i \neq \emptyset$ for all $i \in \{1, \ldots, k\}$, then we successively consider all sets $Z = \{z_1, \ldots, z_k\}$ where $z_i \in Z_i$. If $k = 0$, then $Z = \emptyset$. Observe that $D' = (D^* \setminus \{v\}) \cup \{u\} \cup Z$ is a dominating set. Let W be the set of isolated vertices of $G[D^* \setminus \{v\}]$ belonging to $N(u)$. We construct a partition of W into three sets X_0, X_1, X_2 (which can be empty) as follows. A vertex $x \in W$ is included in X_0 if $P_{D^*}(x) = \emptyset$, x is included in X_1 if $P_{D^*}(x)$ contains a vertex of degree one in G, and otherwise x is included in X_2.

Case 2.1. If $X_2 = \emptyset$, then let $D'' = D' \setminus X_0$. Observe that D'' is a dominating set, since vertices of X_0 are dominated by u, vertices of X_0 have no private neighbors outside of D'', and a vertex of X_0 does not dominate any neighbor of another vertex of X_0 since $g(G) \geq 7$. Observe also that $X_0 \cap Z = \emptyset$, as otherwise this would make a cycle of length 5. We construct a minimal dominating set D from D'' by greedy removal of vertices and add it to $\mathcal{D}$.

Case 2.2. If $X_2 \neq \emptyset$, then we consider all subsets $X \subseteq X_2$. Let $X = \{x_1, \ldots, x_p\}$. Recall that for each $x_j \in X$, $P_{D'}(x_j) \neq \emptyset$. Let $P_{D'}(x_j) = \{x_j^1, \ldots, x_j^{s_j}\}$. Denote by $R_{j,t}$ the set $N(x_j^t) \setminus \{x_j\}$ for $j \in \{1, \ldots, p\}$ and $t \in \{1, \ldots, s_j\}$. Note that $N(x_j^t) \setminus \{x_j\} \neq \emptyset$ because x_j^t has degree at least 2.

By Lemma 9, for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, p\}$, and $t \in \{1, \ldots, s_j\}$, we have that $Z_i \cap R_{j,t} = \emptyset$, since otherwise we get a cycle of length 6. Furthermore, for any $j, j' \in \{1, \ldots, p\}$ and $t, t' \in \{1, \ldots, s_j\}$ such that $(j,t) \neq (j',t')$, we have that $R_{j,t} \cap R_{j',t'} = \emptyset$, since otherwise we get a cycle of length 5. Now we consecutively consider all sets $R = \{w_{j,t} \mid 1 \leq j \leq p, 1 \leq t \leq s_j\}$, where each vertex $w_{j,t}$ is chosen from $R_{j,t}$. For each choice of X and R, we construct the set $D'' = (D' \setminus (X_0 \cup X)) \cup R$. The set D'' is a dominating set, because u dominates $X_0 \cup X$, no vertex of X_0 has a private neighbor with respect to D', and each private vertex for every vertex of X with respect to D' is dominated by a vertex of R. We construct a minimal dominating set D from D'' by greedy removal of vertices and add it to $\mathcal{D}$.

Lemma 10. *The set $\mathcal{D}$ is a set of minimal dominating sets such that $\mathcal{D}$ contains all children of D^* with respect to flipping u and v, and elements of $\mathcal{D}$ are generated with delay $O(n+m)$.*

Proof. First, observe that if $\mathcal{D} \neq \emptyset$, then each D is a minimal dominating set. Moreover, each set D constructed in Case 2 is a minimal dominating set that contains $\{u\} \cup Z \cup R$. This is because each vertex in $\{u\} \cup Z \cup R$ has a private with respect to D'', due to the fact that v is a private for u, each y_i is a private for z_i, and each x_{j,t_j} is a private for $w_{j,t}$. We claim that all sets in $\mathcal{D}$ are distinct. For Case 2.1, the claim is straightforward, because the sets D are constructed for distinct sets Z. For Case 2.2, we can observe that the vertices of $X_2 \setminus X$ cannot be deleted by greedy removal, because they have private neighbors with respect to D^*, and these private neighbors are not dominated by Z, X, or R, as $g(G) \geq 7$. Hence, the sets constructed for distinct X are distinct. Therefore, the sets in $\mathcal{D}$ are distinct in this case as well.

Now we prove that $\mathcal{D}$ contains all children of D^* with respect to flipping u and v. Let $D = (D^* \setminus (\{v\} \cup X_{uv})) \cap \{u\} \cap Z_{uv}$.

We claim that $X_0 \subseteq X_{uv} \subseteq X_0 \cup X_2$. Recall that for each vertex x of W, $x \in X_0$ if $P_{D^*}(x) = \emptyset$, $x \in X_1$ if $P_{D^*}(x)$ contains a vertex of degree one in G, and $x \in X_2$ otherwise, where W is the set of isolated vertices of $G[D^* \setminus \{v\}]$ in $N(u)$. We have $X_{uv} \subseteq W$, since the vertices of X_{uv} are privates for u in D. Suppose that $x \in X_1$. Then x is adjacent to a vertex y of degree one such that $y \in P_{D^*}(x)$. Vertex y can be dominated either by itself or by its unique neighbor x. If $x \in X_{uv}$, then y has to be dominated by itself in D, but then x is not a private for u with respect to D. It follows that $X_{uv} \subseteq X_0 \cup X_2$. Let now $x \in X_0$. We have that $P_{D^*}(x) = \emptyset$, and since the neighbors of x different from u cannot be dominated by $X_{uv} \setminus \{x\}$, we conclude that $x \in X_{uv}$.

Let $X = X_{uv} \setminus X_0$. Clearly, our algorithm considers this set. By Lemma 3, for each $z \in Z_{uv}$, there is a vertex $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ adjacent to z such that $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$, and for any $x \in N[X_{uv} \cup \{v\}] \setminus N[u]$ such that $x \notin N[D^* \setminus (X_{uv} \cup \{v\})]$, there is an adjacent $z \in Z_{uv}$. To see this, it is sufficient to observe that all neighbors of the vertices of X_0 except u are dominated by $D^* \setminus (X_{uv} \cup \{v\})$, because $g(G) \geq 7$.

By Lemma 9, for every $x \in N[X \cup \{v\}] \setminus N[u]$ such that $x \notin N[D^* \setminus (X \cup \{v\})]$, Z_{uv} contains the unique vertex adjacent to x. Since we consider all possible ways to dominate such vertices in our algorithm, there are sets Z and R such that $Z_{uv} = Z$ in Case 2.1 or $Z_{uv} = Z \cup R$ in Case 2.2. Consequently, D is in $\mathcal{D}$.

To complete the proof, we consider the running time. Initially all sets Z_i can be generated in $O(n+m)$ time. Likewise, W, X_0, X_1, X_2 can be generated within the same time bound. Every set X can be generated with $O(n)$ delay from the previous set X. For every set X, we need to generate the sets R. Before we can start generating the sets R, we need to generate a list of sets $R_{j,t}$. These sets $R_{j,t}$ have empty intersections with each other, hence the sum of the sizes of all sets $R_{j,t}$ is $O(n)$ for each X. Thus, after generating X, we can generate the list of all $R_{j,t}$ in $O(n+m)$ time. Now, as long as X is fixed, we can generate every R in time $O(n)$ using the list of $R_{i,j}$. Note that every R gives us a new dominating set D. This means that the delay between each generated dominating set D is $O(n+m)$. □

Combining Lemmas 2 and 10, we obtain the following theorem.

Theorem 4. *All minimal dominating sets of a graph of girth at least 7 can be enumerated in incremental polynomial time. On input graphs with n vertices and m edges, the delay is $O(n^2m|\mathcal{L}|^2)$, and the total running time is $O(n^2m|\mathcal{L}^*|^2)$, where $\mathcal{L}$ is the set of already generated minimal dominating sets and $\mathcal{L}^*$ is the set of all minimal dominating sets.*

6 Concluding remarks

The flipping method that we have described in this paper has the property that each generated minimal dominating set has a unique parent. It would be very interesting to know whether this can be used to obtain output polynomial time algorithms for enumerating minimal dominating sets in general. For the algorithms that we have given in this paper, on the studied graph classes we were able to give additional properties of the parents to obtain the desired running times. Are there additional properties of parents in general graphs that can result in efficient algorithms?

As a first step towards resolving these questions, on which other graph classes can the flipping method be used to enumerate the minimal dominating sets in output polynomial time? Another interesting question is whether the minimal dominating sets of line graphs or graphs of large girth can be enumerated with polynomial delay.

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