Enumerating minimal connected dominating sets in graphs of bounded chordality[☆,☆☆]

Petr A. Golovach[a,*], Pinar Heggernes[a], Dieter Kratsch[b]

[a] *Department of Informatics, University of Bergen, N-5020 Bergen, Norway*
[b] *Université de Lorraine, LITA, Metz, France*

Abstract

Enumerating objects of specified type is one of the principal tasks in algorithmics. In graph algorithms one often enumerates vertex subsets satisfying a certain property. We study the enumeration of all minimal connected dominating sets of an input graph from various graph classes of bounded chordality. We establish enumeration algorithms as well as lower and upper bounds for the maximum number of minimal connected dominating sets in such graphs. In particular, we present algorithms to enumerate all minimal connected dominating sets of chordal graphs in time $O(1.7159^n)$, of split graphs in time $O(1.3803^n)$, and of AT-free, strongly chordal, and distance-hereditary graphs in time $O^*(3^{n/3})$, where n is the number of vertices of the input graph. Our algorithms imply corresponding upper bounds for the number of minimal connected dominating sets for these graph classes.

Keywords: Exact Exponential Algorithms, Enumeration, Minimal connected dominating sets

1. Introduction

Enumerating objects of specified type and properties has important applications in various domains of computer science, such as data mining, machine learning, and artificial intelligence, as well as in other sciences, especially biology. In particular, enumeration algorithms whose running time is measured in the size of the input have gained increasing interest recently. The reason for this is two-fold. Firstly, many exact exponential-time algorithms for the solution of NP-hard problems rely on such enumeration algorithms. Sometimes the fastest known algorithm to solve an optimization problem is by simply enumerating all minimal or maximal feasible solutions (e.g., for

[☆] A preliminary version of this paper appeared as an extended abstract in the proceedings of IPEC 2015.

[☆☆] This work is supported by the Research Council of Norway, the French National Research Agency (ANR project GraphEn / ANR-15-CE40-0009), and the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 267959.

[*] Corresponding author. University of Bergen, PB 7803, N-5020, Bergen, Norway. Phone: +47 55584385

Email addresses: petr.golovach@uib.no (Petr A. Golovach), pinar.heggernes@uib.no (Pinar Heggernes), dieter.kratsch@univ-lorraine.fr (Dieter Kratsch)

subset feedback vertex sets [16]), whereas other times the enumeration of some objects is useful for algorithms solving completely different problems (e.g., enumeration of maximal independent sets in triangle-free graphs for computing graph homomorphisms [15]). Secondly, the running times of such enumeration algorithms very often imply an upper bound on the maximum number of enumerated objects a graph can have. This is a field of research that has long history within combinatorics, and enumeration algorithms provide an alternative way to prove such combinatorial bounds. In fact, several classical examples exist in this direction, of which one of the most famous is perhaps that of Moon and Moser [28] who showed that the maximum number of maximal independent sets in a graph on n vertices is $3^{n/3}$. Although the arguments of [28] were purely combinatorial, the same bound is also achieved by an enumeration algorithm with running time $O^*(3^{n/3})$, where the O^*-notation suppresses polynomial factors.

The mentioned result on the number of maximal independent sets is tight, as there is a graph that has exactly $3^{n/3}$ maximal independent sets, namely a disjoint union of $n/3$ triangles. However, for many upper bounds, no such matching lower bound is known, and hence for the maximum number of many objects there is a gap between the known upper and lower bounds. This motivates the study of enumeration of objects in graphs belonging to various graph classes. For example, the maximum number of minimal dominating sets in graphs is known to be at most 1.7159^n [14], however no graph having more than 1.5704^n minimal dominating sets is known. On the other hand, on many graph classes matching upper and lower bounds can be shown on the maximum number of minimal dominating sets [8, 10]. Furthermore, even if the bound on general graphs is tight, a better bound might exist for graph classes, which might be useful algorithmically and interesting combinatorially. For example, the maximum number of maximal independent sets in triangle-free graphs is at most $2^{n/2}$ and they can be listed in time $O^*(2^{n/2})$ [6], which was used in the above mentioned algorithm for homomorphisms [15]. As a consequence, there has been extensive research in this direction recently, both on general graphs and in particular on graph classes. Examples of algorithms for the enumeration and combinatorial lower and upper bounds on graph classes exist for minimal feedback vertex sets, minimal subset feedback vertex sets, minimal dominating sets, minimal separators, and potential maximal cliques [8, 9, 10, 13, 16, 18, 20, 21, 23].

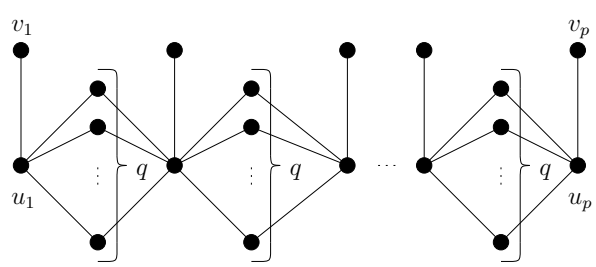

Figure 1: A graph with 2^p minimal dominating sets and q^{p-1} minimal connected dominating sets.

In this paper we initiate the study of the enumeration and maximum number of minimal *connected* dominating sets in a given graph. We refer to the book of Haynes, Hedetniemi and Slater [25] for the detailed introduction to various variants of domination in graphs and, in particular, for the survey of known results about the connected dominating set problem, its applications and relations to other types of domination problems. It can be noticed that the ratio between the numbers of minimal dominating sets and minimal connected dominating sets can be exponential. For example, the number of minimal dominating sets of the graph G shown in Figure 1 is 2^p, as for each $i \in \{1, \ldots, p\}$, exactly one vertex from the pair $\{u_i, v_i\}$ is included in any minimal dominating set, and for each selection of the elements of the pairs $\{u_i, v_i\}$, there is a unique minimal dominating set that contains these vertices. From the other side, D is a minimal connected dominating set if and only if D is the set of vertices of a (u_1, u_p)-path and, therefore, G has q^{p-1} minimal connected dominating sets. In particular, if $q = 1$, then G has $2^{(n+1)/3}$ minimal dominating sets and a single unique minimal connected dominating set, and if $q = 4$, then G has $2^{(n+4)/6}$ minimal dominating sets and $4^{(n-2)/6}$ minimal connected dominating sets. Interestingly, the best known upper bound for the maximum number of minimal connected dominating sets in an arbitrary graph is 2^n, i.e., the trivial one. The best lower bound we achieve in this paper is $3^{(n-2)/3}$, and thus the gap between the known lower and upper bounds is huge on arbitrary graphs. Furthermore, although connected dominating sets have been subject to extensive study when it comes to optimization and decision variants, their enumeration has been left completely unattended. In fact computing a minimum connected dominating set is one of the classical NP-hard problems already mentioned in the monograph of Garey and Johnson [19]. The best known running time of an algorithm solving this problem is $O(1.8619^n)$ [1], which is surprisingly larger than the best known lower bound $3^{(n-2)/3} \approx 1.4423^n$.

The results that we present in this paper are summarized in the following table, where n is the number of vertices and m is the number of edges of an input graph belonging to the given class.

Graph Class	Lower Bound	Upper Bound	Enumeration Algorithm
chordal	$3^{(n-2)/3}$	1.7159^n	$O(1.7159^n)$
split	1.3195^n [8]	$\max\{1.3803^n, n\}$	$O(1.3803^n)$
cobipartite	1.3195^n [8]	$2 \cdot 1.3803^n + n^2$	$O(1.3803^n)$
interval	$3^{(n-2)/3}$	$\max\{3^{(n-2)/3}, n\}$	$O^*(3^{n/3})$
AT-free	$3^{(n-2)/3}$	$3^{n/3} \cdot (n^{10} + n^9)$	$O^*(3^{n/3})$
strongly chordal	$3^{(n-2)/3}$	$3^{n/3}$	$O^*(3^{n/3})$
distance-hereditary	$3^{(n-2)/3}$	$3^{n/3} \cdot n$	$O^*(3^{n/3})$
cograph	m	m	$O(m)$

2. Preliminaries

We consider finite undirected graphs without loops or multiple edges. For each of the graph problems considered in this paper, we let $n = |V(G)|$ and $m = |E(G)|$ denote the number of vertices and edges, respectively, of the input graph G. For a graph G and a subset $U \subseteq V(G)$ of vertices, we write $G[U]$ to denote the subgraph of

G induced by U. We write $G - U$ to denote the subgraph of G induced by $V(G) \setminus U$, and $G - u$ if $U = \{u\}$. A set $U \subseteq V(G)$ is *connected* if $G[U]$ is a connected graph. For a vertex v, we denote by $N_G(v)$ the *(open) neighborhood* of v, i.e., the set of vertices that are adjacent to v in G. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For a set of vertices $U \subseteq V(G)$, $N_G[U] = \cup_{v \in U} N_G[v]$ and $N_G(U) = N_G[U] \setminus U$. We say that P is a (u, v)*-path* if P is a path that joins u and v. The vertices of P different from u and v are the *inner* vertices of P. The *distance* $\mathrm{dist}_G(u, v)$ between vertices u and v of G is the number of edges on a shortest (u, v)-path. A path (cycle) P is *induced* if it has no *chord*, i.e., there is no edge of G incident to any two vertices of P that are not adjacent in P. The *chordality* $\mathrm{chord}(G)$ of a graph G is the length of a longest induced cycle in G; if G has no cycles, then $\mathrm{chord}(G) = 0$. A vertex v is a *cut vertex* of a connected graph G with at least two vertices if $G - v$ is disconnected.

For a non-negative integer k, a graph G is k*-chordal* if $\mathrm{chord}(G) \leq k$. A graph is *chordal* if it is 3-chordal. A graph G is *strongly chordal* if G is chordal and every cycle C of even length at least 6 in G has an *odd chord*, i.e., an edge that connects two vertices of C that are an odd distance apart from each other in the cycle. A graph G is *distance-hereditary* if for any connected induced subgraph H of G, $\mathrm{dist}_H(u, v) = \mathrm{dist}_G(u, v)$ for $u, v \in V(H)$. Distance-it is trivialhereditary graphs are 4-chordal. An *asteroidal triple (AT)* is a set of three pairwise non-adjacent vertices such that between each pair of them there is a path that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. Consequently, AT-free graphs are 5-chordal.

A graph is a *split* graph if its vertex set can be partitioned into an independent set and a clique. Split graphs are chordal. A graph G is an *interval graph* if it is the intersection graph of a set of closed intervals on the real line, i.e., the vertices of G correspond to the intervals and two vertices are adjacent in G if and only if their intervals have at least one point in common. Notice that interval graphs are strongly chordal and AT-free; in fact a graph is interval if and only if it is chordal and AT-free. A graph is a *cobipartite* graph if its vertex set can be partitioned into two cliques. A graph is a *cograph* if it has no induced path on 4 vertices. Cobipartite graphs and cographs are AT-free as well as 4-chordal.

Each of the above-mentioned graph classes can be recognized in polynomial (in most cases linear) time, and they are closed under taking induced subgraphs [5, 24]. See the monographs by Brandstädt et al. [5] and Golumbic [24] for more properties and characterizations of these classes and their inclusion relationships.

A vertex v of a graph G *dominates* a vertex u if $u \in N_G[v]$; similarly v dominates a set of vertices U if $U \subseteq N_G[v]$. For two sets $D, U \subseteq V$, D dominates U if $U \subseteq N_G[D]$. A set of vertices D is a *dominating set* of G if D dominates $V(G)$. A set of vertices D is a *connected dominating set* if D is connected and D dominates $V(G)$. A (connected) dominating set is *minimal* if no proper subset of it is a (connected) dominating set. Let $D \subseteq V(G)$, and let $v \in D$. A vertex u is a *private vertex*, or simply *private*, for the vertex v (with respect to D) if u is dominated by v but is not dominated by $D \setminus \{v\}$. Clearly, a dominating set D is minimal if and only if each vertex of D has a private vertex. Notice also that a connected dominating set D is minimal if and only if for any $v \in D$, v has a private vertex or $D \setminus \{v\}$ is disconnected, i.e., v is a cut vertex of $G[D]$. Observe that a vertex can be private for itself with respect to a dominating set D, but if D is a connected dominating set of size at least two, then any

private u of v is a neighbor of v.

Since a disconnected graph has no connected dominating sets, we construct our enumeration algorithms and state all upper bounds for the number of minimal connected dominating sets for connected graphs without loss of generality.

For technical reasons, we also consider *red-blue domination*. Let $\{R, B\}$ form a bipartition of the vertex set of a graph G. We refer to the vertices of R as the *red* vertices, the vertices of B as the *blue* vertices, and we say that G is a *red-blue graph*. A set of vertices $D \subseteq R$ is a *red dominating set* if D dominates B, and D is *minimal* if no proper subset of it dominates B. It is straightforward to see that $D \subseteq R$ is a minimal red dominating set if and only if D dominates B and each vertex of D has a private blue vertex.

It is standard to use recursive branching algorithms for enumerations and obtaining upper bounds. We refer to the book of Fomin and Kratsch [17] for a detailed introduction. If a recursive branching algorithm $\mathcal{A}$ is used to enumerate all objects satisfying a certain property, then the total number of such objects is upper bounded by the number of leaves of the search tree produced by the algorithm if the algorithm outputs objects at the leaves and at most one object is produced for each leaf. To obtain an upper bound for the leaves of a search tree, we use a technique based on solving recurrences for branching steps. To do it, to each instance I of the considered enumeration problem, a *measure* (or *size*) $\mu(I)$ is assigned. If the algorithm branches on an instance I into t new instances, such that the measure decreases by $c_1, c_2, \ldots, c_t$ for each new instance respectively, we say that $(c_1, c_2, \ldots, c_t)$ is the *branching vector* of this step. If $L(s)$ is the maximum number of leaves of a search tree for an instance I of measure $s = \mu(I)$, then we obtain the recurrence $L(s) \leq L(s-c_1)+L(s-c_2)+\ldots+L(s-c_t)$ and the corresponding characteristic polynomial $p(x) = x^c - x^{c-c_1} - \ldots - x^{c-c_t}$ for $c = \max\{c_1, \ldots, c_t\}$. The unique positive real root α of $p(x)$ is called the *branching number* of the branching vector. Notice that if the algorithm has several branching steps with distinct branching vectors, then for the maximum value of the branching number λ, the standard analysis (see [17]) gives us that $L(s) = O^*(\lambda^s)$. If we consider a graph problem and the measure of the original instance is upper bounded by the number of vertices n, we obtain that the number of leaves is $O^*(\lambda^n)$. Observe that such a bound on the number of leaves of a search tree simultaneously gives us the running time of the algorithm in the form $O^*(\lambda^n)$ if each step of the algorithm can be executed in polynomial time. It is common to round λ to the fourth digit after the decimal point. By rounding the last digit up, we can use O-notation instead of O^*-notation. In many cases an easy induction allows us to obtain the upper bound for the number of leaves of the search tree in the form λ^n. The following sufficient conditions were explicitly stated in [22].

Lemma 1 ([22]). *Let $\mathcal{A}$ be a branching algorithm enumerating all objects of property P of an input graph G with a (possibly infinite) collection of branching vectors $\{(c_1^{(i)}, \ldots, c_{t_i}^{(i)}) \mid i \in J\}$. Let $L(s)$ be the maximum number of leaves of a search tree rooted at an instance of measure s. Suppose the following conditions are satisfied.*

a) There is a measure μ assigning to each instance of the algorithm an integer such that $\mu(I) \leq n$ for all instances.

b) $L(s) = 1$ *for* $s = 0$.

c) $L(s) = 0$ *for all* $s < 0$.

d) The recurrences corresponding to the branching vectors of $\mathcal{A}$ *can be written in the following form for* $k \geq 1$*:* $L(k) \leq \max\{1, \max\{L(k - c_1^{(i)}) + \ldots + L(k - c_{t_i}^{(i)}) \mid i \in J\}\}$.

Then the number of leaves of the search tree and thus the number of objects of property P *is at most* λ^n*, where* λ *is the largest branching number.*

We conclude this section by showing a lower bound for the maximum number of minimal connected dominating sets.

Proposition 1. *For every positive integer* $n = 3k + 2$ *for* $k \geq 0$*, there is an interval and distance-hereditary graph* G *with* n *vertices that has* $3^{(n-2)/3}$ *minimal connected dominating sets.*

Proof. To obtain the bound for interval and distance-hereditary graphs, consider the graph G constructed as follows for a positive integer k.

- For $i \in \{1, \ldots, k\}$, construct a triple of pairwise adjacent vertices $T_i = \{x_i, y_i, z_i\}$.
- For $i \in \{2, \ldots, k\}$, join each vertex of T_{i-1} with every vertex of T_i by an edge.
- Construct two vertices u and v and edges ux_1, uy_1, uz_1 and vx_k, vy_k, vz_k.

Clearly, G has $n = 3k + 2$ vertices. Notice that $D \subseteq V(G)$ is a minimal connected dominating set of G if and only if $u, v \notin D$ and $|D \cap T_i| = 1$ for $i \in \{1, \ldots, k\}$. Therefore, G has $3^k = 3^{(n-2)/3}$ minimal connected dominating sets.

It is straightforward to see that G is an interval graph. To show that G is distance-hereditary, let H be a connected induced subgraph of G and let $a, b \in V(H)$ be distinct vertices. If $a, b \in T_i$ for some $i \in \{1, \ldots, k\}$, then $\mathrm{dist}_H(a, b) = \mathrm{dist}_G(a, b) = 1$. If $a \in T_i$ and $b \in T_j$ for $1 \leq i < j \leq k$, then for each $i < h < j$, at least on vertex of T_h is a vertex of H, because H is connected, and $\mathrm{dist}_H(a, b) = \mathrm{dist}_G(a, b) = j - i$. By the same arguments, if $a = u$ and $b \in T_i$ or if $a \in T_i$ and $b = v$ or if $a = u$ and $b = v$, we obtain that $\mathrm{dist}_H(a, b) = \mathrm{dist}_G(a, b)$. □

3. Chordal graphs

In this section we shall heavily rely on minimal separators of graphs and minimal transversals of hypergraphs. Recall that a *hypergraph* H is a pair (V, E) where V is a nonempty set of *vertices* of H and E is a family of nonempty subsets of V called *hyperedges*. A *transversal* of a hypergraph H is a vertex set $T \subseteq V(H)$ intersecting all hyperedges of H, i.e., $T \cap A \neq \emptyset$ for all $A \in E(H)$. Furthermore, a transversal is *minimal* if no proper subset of it is a transversal. A vertex set $S \subseteq V$ is a *separator* of the graph $G = (V, E)$ if $G - S$ is disconnected. A component C of $G - S$ is *full* if every vertex of S has a neighbor in C. A separator S of G is a *minimal separator* of G if $G - S$ has at least two full components. Minimal separators of graphs have

been studied intensively in the last twenty years. They play a crucial role in minimal triangulations, and in solving problems like treewidth and minimum fill-in. For more information we refer to [26].

Let us start with a strong relationship between the minimal connected dominating sets of a graph and its minimal separators, established by Kante, Limouzy, Mary and Nourine [27]. First note that all minimal connected dominating sets of a complete graph G are singletons $\{v\}$ with $v \in V(G)$. Now we define the *minsep hypergraph* $H = (V(H), E(H))$ of a graph $G = (V, E)$. The vertex set of H consists of all vertices of G belonging to some minimal separator of G, hence $V(H) \subseteq V(G)$. The hyperedges of H are exactly the minimal separators of G. Hence $|E(H)|$ is the number of minimal separators of G.

Theorem 1 ([27])**.** *Let $G = (V, E)$ be a connected and non complete graph. Then $D \subseteq V$ is a minimal connected dominating set of G if and only if D is a minimal transversal of the minsep hypergraph H of G.*

To enumerate the minimal transversals of the minsep hypergraph of chordal graphs, we will be relying on a branching algorithm and its analysis which is due to Fomin, Grandoni, Pyatkin and Stepanov [14]. The main result of their paper is that a graph has at most 1.7159^n minimal dominating sets. The crucial result of the paper for us is the branching algorithm to enumerate all minimal set covers of a set cover instance $(\mathcal{U}, \mathcal{S})$, where $\mathcal{U}$ is a universe and $\mathcal{S}$ is a collection of subsets of $\mathcal{U}$. When studying the algorithm and its analysis one observes that it can be applied to all set cover instances $(\mathcal{U}, \mathcal{S})$. Only at the very end of the analysis the obtained general bound is applied to the particular instances obtained from graphs satisfying $|\mathcal{U}| = |\mathcal{S}|$, which includes tailoring the weights to the case $|\mathcal{U}| = |\mathcal{S}|$. The interested reader may study Sections 3 and 4 of [14] to find the following implicit result.

Theorem 2 ([14])**.** *A set cover instance $(\mathcal{U}, \mathcal{S})$ has at most $\lambda^{|\mathcal{U}|+\alpha\cdot|\mathcal{S}|}$ minimal set covers, where $\lambda = 1.156154$ and $\alpha = 2.720886$. These minimal set covers can be enumerated in time $O^*(\lambda^{|\mathcal{U}|+\alpha|\mathcal{S}|})$.*

By Theorem 1 we are interested in enumerating the minimal transversals of a hypergraph $H = (V(H), E(H))$ with $V(H) = \{v_1, v_2, \ldots v_s\}$ and $E(H) = \{E_1, E_2, \ldots, E_t\}$ where $E_j \subseteq V(H)$ for all hyperedges E_j. It is well-known that enumerating the minimal transversals of a hypergraph H is equivalent to enumerating the minimal set covers of a set cover instance (corresponding to the dual hypergraph of H) constructed as follows. First we set $\mathcal{U} = E(H)$ and then $\mathcal{S}$ is a collection of sets $S(v_1), S(v_2), \ldots S(v_s)$ such that for all $i \in \{1, 2, \ldots, s\}$ the set $S(v_i) \subseteq E(H)$ consists of all hyperedges E_j containing v_i. Consequently $|\mathcal{U}| = |E(H)|$ and $|\mathcal{S}| = |V(H)|$. By the construction enumerating the minimal set covers of the dual set cover instance $(\mathcal{U}, \mathcal{S})$ is equivalent to enumerating the minimal transversals of H. Consequently Theorem 2 implies

Corollary 1. *A hypergraph $H = (V(H), E(H))$ has at most $\lambda^{|E(H)|+\alpha\cdot|V(H)|}$ minimal transversals, where $\lambda = 1.156154$ and $\alpha = 2.720886$. These minimal transversals can be enumerated in time $O^*(\lambda^{|E(H)|+\alpha|V(H)|})$.*

Now we are ready to consider the enumeration of minimal connected dominating sets on chordal graphs.

Theorem 3. *A connected chordal graph has* 1.7159^n *minimal connected dominating sets, and these sets can be enumerated in time* $O(1.7159^n)$.

Proof. Let H be the minsep hypergraph of a chordal graph G. Clearly, $|V(H)| \leq |V(G)| = n$. Recall that a chordal graph has at most $n-1$ minimal separators (see, e.g., [5]). Then $|E(H)| \leq n$. By Corollary 1, the number of minimal transversals of H is at most $\lambda^{|E(H)|+\alpha\cdot|V(H)|}$. Hence we can upper bound the number of minimal transversals by

$$\lambda^{|E(H)|+\alpha\cdot|V(H)|} \leq \lambda^{(1+\alpha)n} < 1.7159^n.$$

Consequently the number of minimal transversals of H is at most 1.7159^n, and they can be enumerated in time $O(1.7159^n)$. Finally by Theorem 1 these minimal transversals are precisely the minimal connected dominating sets of G. □

4. Split graphs and cobipartite graphs

We denote a split graph by $G = (C, I, E)$ to indicate that its vertex set $V(G)$ can be partitioned into a clique C and an independent set I. The following simple observation made by Babel and Olariu [3] will be crucial for our branching algorithm.

Lemma 2 ([3])**.** *Let* $G = (C, I, E)$ *be a non complete connected split graph. Then* D *is a minimal connected dominating set of* G *if and only if* D *is minimal dominating set of* G *and* $D \subseteq C$.

Couturier et al. have shown that the maximum number of minimal dominating sets in a split graph is $3^{n/3}$, and that these sets can be enumerated in time $O^*(3^{n/3})$ [10]. Combined with Lemma 2, this implies that the same results hold for minimal connected dominating sets in split graphs. With a branching algorithm, we are able to establish a significant improvement.

Theorem 4. *A connected split graph has at most* $\max\{1.3803^n, n\}$ *minimal connected dominating sets, and these sets can be enumerated in time* $O(1.3803^n)$.

Proof. Let $G = (C, I, E)$ be a connected split graph. If G is a complete graph, then the set of minimal connected dominating sets is the set of single element subsets of $V(G)$, that is, G has n minimal connected dominating sets and these sets can be enumerated in time $O(n)$. From now on we assume that G is not complete. We show that G has at most 1.3803^n minimal connected dominating sets. We consider the following recursive algorithm $\textsc{EnumCDS}(K, S, X)$, where the clique $K \subseteq C$ and the independent set $S \subseteq I$ form a partition of the vertex set of the induced subgraph $H = G[K \cup S]$ of the split graph $G = (C, I, E)$. Furthermore in G, $X \subseteq C \setminus K$, X dominates $I \setminus S$, and if $X \neq \emptyset$ then X dominates C. The algorithm $\textsc{EnumCDS}(K, S, X)$ enumerates the minimal connected dominating sets D of G satisfying $X \subseteq D$, $D \cap (V(G) \setminus V(H)) = X \cap (V(G) \setminus V(H))$ and $D \cap K$ is a connected dominating set of H. This is a branching algorithm based on the property that any minimal connected dominating set of $G = (C, I, E)$ is a subset of the clique C by Lemma 2.

To describe the recursive algorithm we use the notation $d_S(x)$ and $d_K(x)$ to denote $|N_G(x) \cap S|$ and $|N_G(x) \cap K|$ respectively.

ENUMCDS(K, S, X)

1. If X is a minimal connected dominating set of G then return X and stop. If X is a connected dominating set of G but not a minimal one then stop.
2. If there is an $x \in K$ such that $d_S(x) = 0$ then *discard* x (i.e. x cannot be added to X and will be removed from H). Then call recursively ENUMCDS$(K \setminus \{x\}, S, X)$.
3. If there is a $y \in S$ such that $d_K(y) = 1$ and x is the unique neighbor of y in H then *select* x (i.e. add x to X and remove it from H) and remove y from H. Then recursively call ENUMCDS$(K \setminus \{x\}, S \setminus \{y\}, X \cup \{x\})$.
4. If there is a $x \in K$ such that $d_S(x) = 1$ and y is the unique neighbor of x in S. Let $N_H(y) = \{x, x_2, \ldots, x_t\}$ for $t \geq 2$.
   - Case 1: $t = 2$. Then branch into two branches:

     (i) select x, remove y as private neighbor of x, and discard x_2 (otherwise y would not be private). Then call recursively ENUMCDS$(K \setminus \{x, x_2\}, S \setminus \{y\}, X \cup \{x\})$.

     (ii) discard x (i.e. x cannot be added to X anymore) , select x_2, remove y (as dominated by X). Then call recursively ENUMCDS$(K \setminus \{x, x_2\}, S \setminus \{y\}, X \cup \{x_2\})$
   - Case 2: $t \geq 3$. Then branch into two branches:

     (i) select x, remove y as private neighbor of x, and discard $x_2, x_3, \ldots, x_t$ (otherwise y would not be private neighbor of x). Then call recursively ENUMCDS$(K \setminus \{x, x_2, x_3, \ldots, x_t\}, S \setminus \{y\}, X \cup \{x\})$.

     (ii) discard x and call ENUMCDS$(K \setminus \{x\}, S, X)$.
5. If there is an $x \in C$ such that $d_S(x) \geq 3$. Let $N_H(x) \cap S = \{y_1, y_2, \ldots, y_t\}$ and $t \geq 3$. Then branch into two branches:
   (i) select x, remove $y_1, y_2, \ldots, y_t$ (as dominated by x) and call recursively ENUMCDS$(K \setminus \{x\}, S \setminus \{y_1, y_2, \ldots y_t\}, X \cup \{x\})$.
   (ii) discard x and call recursively ENUMCDS$(K \setminus \{x\}, S, X)$.
6. If there is a $y \in S$ such that $d_K(y) = 2$ then let $N_H(y) = \{x_1, x_2\}$ and for all $i = 1, 2$ let w_i be the unique neighbor of x_i in S different from y. Then branch into two branches:
   (i) select x_1, remove y and w_1. Call recursively ENUMCDS$(K \setminus \{x_1\}, S \setminus \{y, w_1\}, X \cup \{x_1\})$.
   (ii) discard x_1, select x_2, remove y and w_2. Call recursively ENUMCDS$(K \setminus \{x_1, x_2\}, S \setminus \{y, w_2\}, X \cup \{x_2\})$.
7. If there is a $y \in S$ such that $d_K(y) = 3$ then let $N_H(y) = \{x_1, x_2, x_3\}$ and for all $i = 1, 2, 3$ let w_i be the unique neighbor of x_i in S different from y. Then branch into three branches:
   (i) select x_1, remove y and w_1. Call recursively ENUMCDS$(K \setminus \{x_1\}, S \setminus \{y, w_1\}, X \cup \{x_1\})$.
   (ii) discard x_1, select x_2, remove y and w_2 and call recursively ENUMCDS$(K \setminus \{x_1, x_2\}, S \setminus \{y, w_2\}, X \cup \{x_2\})$.
   (iii) discard x_1, discard x_2, select x_3, remove y and w_3. Call recursively ENUMCDS$(K \setminus \{x_1, x_2, x_3\}, S \setminus \{y, w_3\}, X \cup \{x_3\})$.

8. If there is a $y \in S$ such that $d_K(y) = 4$ then let $N_H(y) = \{x_1, x_2, x_3, x_4\}$ and for all $i = 1, 2, 3, 4$ let w_i be the unique neighbor of x_i in S different from y. Then branch into four branches:
   (i) select x_1, remove y and w_1. Call recursively ENUMCDS$(K \setminus \{x_1\}, S \setminus \{y, w_1\}, X \cup \{x_1\})$.
   (ii) discard x_1, select x_2, remove y and w_2 and call recursively ENUMCDS$(K \setminus \{x_1, x_2\}, S \setminus \{y, w_2\}, X \cup \{x_2\})$.
   (iii) discard x_1, discard x_2, select x_3, remove y and w_3. Call recursively ENUMCDS$(K \setminus \{x_1, x_2, x_3\}, S \setminus \{y, w_3\}, X \cup \{x_3\})$.
   (iv) discard x_1, discard x_2, discard x_3, select x_4, remove y and w_4. Call recursively ENUMCDS$(K \setminus \{x_1, x_2, x_3, x_4\}, S \setminus \{y, w_4\}, X \cup \{x_4\})$.
9. If there is an $x \in K$ with neighbors y and y' in S then $d_K(y) \geq 5$ and $d_K(y') \geq 5$ then branch into three branches:
   (i) select x, remove y as private neighbor of x, remove y' since dominated by X, discard all vertices of $N_H(y) \setminus \{x\}$ (since none of them can be added to X, otherwise y is not private of x). Then call recursively ENUMCDS$(K \setminus N_H(y), S \setminus \{y, y'\}, X \cup \{x\})$.
   (ii) select x, remove y' as private neighbor of x, remove y, remove $N_H(y') \setminus \{x\}$ (which cannot be added to X since y' private of x) Then call recursively ENUMCDS$(K \setminus N_H(y'), S \setminus \{y, y'\}, X \cup \{x\})$.
   (iii) discard x and call recursively ENUMCDS$(K \setminus \{x\}, S, X)$.

We call ENUMCDS$(C, I, \emptyset)$ to enumerate the minimal connected dominating sets of $G = (C, I, \emptyset)$.

The construction of the branching algorithm implies its correctness; a careful case analysis with respect to the degrees d_S and d_K of vertices in K and S in the graph H is done. Since this is not hard to verify, we only mention a few of the cornerstones. If step 4 is applied then steps 1 – 3 cannot be applied, hence $d_K(y) \geq 2$ for all $y \in S$. If step 5 is applied then steps 1 – 4 cannot be applied, hence $d_S(x) \geq 2$ for all $x \in K$. If none of steps 1 – 8 can be applied then $d_K(y) \geq 5$ for all $y \in S$ and thus step 9 will be applied. Consequently any recursive call of ENUMCDS(K, S, X) either leads to new recursive calls or stops in a leaf of the search tree by applying step 1. Summarizing if $G = (C, I, E)$ is non complete then ENUMCDS$(C, I, \emptyset)$ enumerates the minimal connected dominating sets of G.

To analyze the running time of such a branching algorithm one has to determine all branching vectors for all branching rules, i.e. steps 4 – 9 for our algorithm. This requires to define a measure for all instances (K, S, X) of all the (recursive) calls of the algorithm. We simply define the measure of (K, S, X) as $|K| + |S|$, i.e. the number of vertices of H. Notice that for the the original instance $(C, I, \emptyset)$, the measure $|C|+|I| = n$. It remains to analyze the decreases of the measure of an instance to the measures of the instances of its branches, when a certain branching rule is applied to this instance. We list the least branching vectors for all branching rules and provide their branching numbers The branching vector of step 4 for $t = 2$ is $(3, 3)$ with branching number : < 1.26. The branching vector of step 4 for $t \geq 3$ is $(t + 1, 1)$ and we have the same branching vector in step 5. Hence, the maximum value of the branching number is < 1.3803 (for $t = 3$) for these cases. The branching vectors of steps 6, 7 and 8 are

$(3,4)$, $(3,4,5)$ and $(3,4,5,6)$, and their branching numbers are < 1.2208, 1.3247 and 1.3803 respectively. Finally in step 9 all vertices of S have at least 5 neighbors in K, hence the branching vector is (at least) $(7,7,1)$ and its branching number is 1.3422. Consequently the largest overall branching number is less than 1.3803 (steps 3, 4 and 8). Hence, the algorithm has running time $O(1.3803^n)$. By Lemma 1, the maximal number of minimal connected dominating sets of a non complete n-vertex split graph, is at most 1.3803^n. □

Due to the structure of minimal connected dominating sets in split graphs, it can be observed that the bound holds for the number of minimal total dominating sets (see [25] for the definition). Theorem 4 also implies an improvement on the number of minimal (connected) dominating sets for n-vertex cobipartite graphs. The previous best known bound is $O(1.4511^n)$ [10].

Corollary 2. *A connected cobipartite graph has at most* $2 \cdot 1.3803^n + n^2$ *minimal (connected) dominating sets, and these sets can be enumerated in time* $O(1.3803^n)$.

Proof. Let $G = (C_1, C_2, E)$ be a cobipartite graph where its vertex set can be partitioned into cliques C_1 and C_2. Let D be a minimal dominating set of $G = (C_1, C_2, E)$. Then $D = \{x, y\}$ with $x \in C_1$ and $y \in C_2$, $D \subseteq C_1$ or $D \subseteq C_2$. Hence with the exception of the $O(n^2)$ minimal dominating sets $D = \{x, y\}$, all other minimal dominating sets are connected. There is a one-to-one relation of the minimal (connected) dominating sets of $G = (C_1, C_2, E)$ being a subset of C_1 and the minimal connected dominating sets of the split graph $G = (C_1, C_2, E)$ obtained by transforming C_2 into an independent set. Similarly, there is a one-to-one relation of the minimal (connected) dominating sets of $G = (C_1, C_2, E)$ being a subset of C_2 and the minimal connected dominating sets of the split graph $G = (C_2, C_1, E)$ obtained by transforming C_1 into an independent set. Hence the maximum number of minimal (connected) dominating sets of an n-vertex cobipartite graphs is at most the sum of twice the maximum number of minimal connected dominating sets in an n-vertex split graph and n^2. This together with Theorem 4 implies the corollary. □

The above-mentioned one-to-one correspondence can also be used to obtain the best known lower bound for the maximum number of minimal connected dominating sets in an n-vertex split graph which is 1.3195^n, based on a lower bound construction for cobipartite graphs given in [8]. The following corollary will be useful in the next section.

Corollary 3. *A red-blue graph G has at most* 1.3803^n *minimal red dominating sets, and these sets can be enumerated in time* $O(1.3803^n)$.

Proof. Let $G = (R, B, E)$ be a red-blue graph. If G has a blue vertex without red neighbors, then G has no red dominating set. If G has no blue vertex, then $\emptyset$ is the unique minimal connected dominating set. Assume that we are not in one of those cases. We construct a split graph $G' = (R, B, E)$ with clique R and independent set B. Clearly, G' is connected. Then there is a one-to-one relation between the minimal red dominating sets of the red-blue graph G and the minimal connected dominating sets of the split graph G. Using this and Theorem 4 we get the result. □

5. AT-free graphs

We need the following folklore observation about the number of induced paths (see, e.g., [29] for the proof).

Lemma 3. *For every pair of vertices u and v of a graph G, G has at most $3^{(n-2)/3}$ induced (u,v)-paths, and these paths can be enumerated in time $O^*(3^{n/3})$.*

Using Lemma 3, we can obtain the tight upper bound for the number of minimal connected dominating sets for interval graphs. Let G be an interval graph with at least two non-adjacent vertices. Consider an interval model of G, i.e., a collection of closed intervals on the real line corresponding to the vertices of G such that two vertices are adjacent in G if and only if their intervals intersect. Let u be the vertex of G corresponding to an interval with the leftmost right end-point and let v be the vertex corresponding to an interval with the rightmost left end-point. Notice that $u \neq v$ and u and v are not adjacent, because G is not a complete graph. It can be shown that $D \subseteq V(G)$ is a minimal connected dominating set of G if and only if D is the set of inner vertices of an induced (u,v)-path. This observation together with Lemma 3 immediately implies the following proposition.

Proposition 2. *A connected interval graph has at most $\max\{3^{(n-2)/3}, n\}$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proposition 1 shows that the bound is tight. To extend it to AT-free graphs we need some additional terminology and auxiliary results. A path P in a graph G is a *dominating path* if $V(P)$ is a dominating set of G. A pair of vertices $\{u,v\}$ of G is a *dominating pair* if any (u,v)-path in G is a dominating path.

Lemma 4 ([7])**.** *Every connected AT-free graph has a dominating pair.*

We show the following properties of minimal connected dominating sets of AT-free graphs. Notice that if D is a connected dominating set of an AT-free graph G, then $G[D]$ is a connected AT-free graph and, therefore, $G[D]$ has a dominating pair by Lemma 4.

Lemma 5. *Let D be a minimal connected dominating set of an AT-free graph G. Let $\{u,v\}$ be a dominating pair of $H = G[D]$ and suppose that $P = v_1 \dots v_k$, where $u = v_1$ and $v = v_k$, is a shortest (u,v)-path in H. Let $X_1 = N_G(\{v_1, v_2\}) \setminus N_G[v_4]$, $X_2 = N_G(\{v_{k-1}, v_k\}) \setminus N_G[v_{k-3}]$, $Y_1 = N_G(X_1) \setminus N_G[\{v_1, \dots, v_4\}]$ and $Y_2 = N_G(X_2) \setminus N_G[\{v_{k-3}, \dots, v_k\}]$. Then $D \subseteq N_G[V(P)]$ and if $k \geq 6$, the following holds:*

i) $D \subseteq V(P) \cup X_1 \cup X_2$,

ii) for the (v_6, v_k)-subpath P_1 of P, $V(P_1) \cap N_G[\{v_1, \dots, v_4\} \cup Y_1] = \emptyset$, and for the (v_1, v_{k-5})-subpath P_2 of P, $V(P_2) \cap N_G[\{v_{k-3}, \dots, v_k\} \cup Y_2]) = \emptyset$.

Proof. Since $\{u,v\}$ is a dominating pair of H, $D \subseteq N_H[V(P)] \subseteq N_G[V(P)]$. Let $k \geq 6$.

To prove i), suppose for the sake of contradiction that $w \in D \setminus (V(P) \cup X_1 \cup X_2) \neq \emptyset$. Because P is a dominating path in H, w is not a cut vertex of H. Hence, there is a private x of w with respect to D, because D is a minimal connected dominating set. We show that $\{x, u, v\}$ is an asteroidal triple in G. Clearly, u and v are not adjacent. Because x is a private of $w \neq u, v$, $xu, xv \notin E(G)$. Because $w \notin V(P)$ and x is a private of w, $V(P) \cap N_G[x] = \emptyset$ and, therefore, P is a (u, v)-path that avoids the neighborhood of x. Since P is a dominating path in H, there is a vertex $v_i \in V(P)$ such that $v_i w \in E(G)$. Assume that $i \in \{1, \ldots, k\}$ is the minimum index such that $v_i w \in E(G)$. Suppose that w is adjacent to v_k. Because $w \notin X_2$, $w \in N_G[v_{k-3}]$. Then the (u, v)-path $v_1 \ldots v_{k-3} w v_k$ has length $k - 2$ contradicting the choice of P. Hence, $wv_k \notin E(G)$. In particular, it means that $i \leq k - 1$. If $i = k - 1$, then since $w \notin X_2$, $w \in N_G[v_{k-3}]$, i.e., w is adjacent to v_{k-3} and $i \leq k - 3$; a contradiction. Therefore, $i \leq k - 2$. Consider the path $P' = v_1 \ldots v_i w x$. Because w is not adjacent to v_k and $v_1, \ldots, v_i$ are not adjacent to v_k, P' is a (u, x)-path that avoids the neighborhood of v. By symmetry, there is a (v, x)-path that avoids the neighborhood of u. We conclude that $\{x, u, v\}$ is an asteroidal triple, but this contradicts the AT-freeness of G.

By symmetry, to show ii), it is sufficient to prove that $V(P_1) \cap N_G[\{v_1, \ldots, v_4\} \cup Y_1] = \emptyset$. Because P is a shortest (u, v)-path in H, P is an induced (u, v)-path and, therefore, $V(P_1) \cap N_G[\{v_1, \ldots, v_4\}] = \emptyset$. Assume that there is $v_i \in V(P_1) \cap N_G[Y_1]$. Clearly, $i \geq 6$. We prove that $\{v_1, v_4, v_i\}$ is an asteroidal triple in G. Because P is an induced path, v_1, v_4, v_i are pairwise non-adjacent. Also we have that the (v_1, v_4)-subpath of P avoids the neighborhood of v_i, and the (v_4, v_i)-subpath of P avoids the neighborhood v_1. It remains to prove that there is a (v_1, v_i)-path that avoids the neighborhood of v_4.

Suppose that $v_i \in Y_1$. Then there is $w \in X_1$ such that $wv_i \in E(G)$. Notice that w is not adjacent to v_4 and is adjacent to v_1 or v_2. We obtain that the path $v_1 w v_i$ or $v_1 v_2 w v_i$ avoids the neighborhood of v_4. Therefore, $\{v_1, v_4, v_i\}$ is an asteroidal triple.

Assume now that $v_i \in N_G(Y_1)$. We have that there is $w' \in Y_1$ adjacent to v_i. By the definition of Y_1, there is $w \in X_1$ adjacent to w'. Notice that w is not adjacent to v_4 and is adjacent to v_1 or v_2. Observe also that w' is not adjacent to v_4. We obtain that the path $v_1 w w' v_i$ or $v_1 v_2 w w' v_i$ avoids the neighborhood of v_4. Therefore, $\{v_1, v_4, v_i\}$ is an asteroidal triple.

The obtained contradiction with the AT-freeness of G proves ii). □

Now we are ready to enumerate the minimal connected dominating sets of AT-free graphs.

Theorem 5. *A connected AT-free graph has at most $3^{n/3} \cdot (n^{10} + n^9)$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proof. First, we show that there are at most $3^{n/3} \cdot n^9$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u, v\}$ with $\mathrm{dist}_G(u, v) \leq 8$ and enumerate these sets. Consider all the at most $\binom{n}{1} + \ldots + \binom{n}{9} \leq n^9$ possible choices of a pair of vertices $\{u, v\}$ and an induced path $P = v_1 \ldots v_k$ with $u = v_1$ and $v = v_k$ such that $k \leq 9$. For each $\{u, v\}$ and P we enumerate the minimal connected dominating

sets D such that $\{u,v\}$ is a dominating pair in $H = G[D]$ and P is a shortest (u,v)-path in H.

Let P be any induced path $P = v_1 \dots v_k$ with $u = v_1$ and $v = v_k$ such that $k \leq 9$. Consider the red-blue graph $G' = G - V(P)$, where the set of red vertices is $R = N_G(V(P))$ and the set of blue vertices is $B = V(G') \setminus R$. Let D be a minimal connected dominating set of G such that $\{u,v\}$ is a dominating pair of $H = G[D]$ and P is a shortest (u,v)-path in H. By Lemma 5, $D \subseteq N_G[V(P)]$. It is straightforward to see that $D \setminus V(P)$ is a red dominating set of G' that dominates all blue vertices and by minimality, $D \setminus V(P)$ is a minimal red dominating set. By Corollary 3, there are at most $1.3803^{|V(G')|} \leq 3^{n/3}$ such sets D and they can be enumerated in time $O(3^{n/3})$. We obtain that there are at most $3^{n/3} \cdot n^9$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u,v\}$ with $\mathrm{dist}_G(u,v) \leq 8$, and these sets can be enumerated in time $O(3^{n/3} \cdot n^9)$.

Now we enumerate minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u,v\}$ with $\mathrm{dist}_G(u,v) \geq 9$. Consider all the at most $\binom{n}{1} + \dots + \binom{n}{10} \leq n^{10}$ possible choices of a pair of vertices $\{u,v\}$ and 2 disjoint induced paths $P_1 = x_1 \dots x_5$ and $P_2 = y_1 \dots y_5$ with $u = x_1$ and $v = y_5$. For each $\{u,v\}$, P_1 and P_2 we enumerate the minimal connected dominating sets D such that $\{u,v\}$ is a dominating pair in $H = G[D]$ and H has a shortest (u,v)-path $P = v_1 \dots v_k$ such that $v_i = x_i$ for $i \in \{1,\dots,5\}$ and $v_i = y_{i+5-k}$ for $i \in \{k-4,\dots,k\}$.

Denote by $X_1 = N_G(\{x_1,x_2\}) \setminus N_G[x_4]$, $X_2 = N_G(\{y_4,y_5\}) \setminus N_G[y_2]$, $Y_1 = N_G(X_1) \setminus N_G[\{x_1,\dots,x_4\}]$ and $Y_2 = N_G(X_2) \setminus N_G[\{y_2,\dots,y_5\}]$. Consider the red-blue graph $G_1 = G[X_1 \cup X_2 \cup Y_1 \cup Y_2]$, where the set of red vertices $R = X_1 \cup X_2$ and the set of blue vertices $B = Y_1 \cup Y_2$. Let $n_1 = |V(G_1)|$. Let D be a minimal connected dominating set of G such that $\{u,v\}$ is a dominating pair of $H = G[D]$ and P is a shortest (u,v)-path in H. By Lemma 5, $D \subseteq N_G[V(P)]$ and $D' = D \setminus V(P) \subseteq X_1 \cup X_2$ is a red dominating set of G_1 that dominates all blue vertices and by minimality, D' is a minimal red dominating set. By Corollary 3, there are at most $1.3803^{n_1} \leq 3^{n_1/3}$ minimal red dominating sets in G_1, and they can be enumerated in time $O(3^{n_1/3})$.

Let $G_2 = G - (V(G_1) \cup \{x_1,\dots,x_4\} \cup \{y_2,\dots,y_5\})$ and let $n_2 = |V(G_2)|$. By Lemma 5, the (v_5, v_{k-4})-subpath of P is an induced (x_5,y_1)-path in G_2. By Lemma 3, there are at most $3^{(n_2-2)/3}$ such paths, and they can be enumerated in time $O^*(3^{n_2/3})$.

Since $D = D' \cup V(P)$, we obtain that there are at most $3^{n_1/3} \cdot 3^{(n_2-1)/3} \leq 3^{(n_1+n_2)/3} \leq 3^{n/3}$ minimal connected dominating sets D with the dominating pair $\{u,v\}$ in $H = G[D]$ and such that H has a shortest (u,v)-path $P = v_1 \dots v_k$ such that $v_i = x_i$ for $i \in \{1,\dots,5\}$ and $v_i = y_{i+5-k}$ for $i \in \{k-4,\dots,k\}$. Moreover, these sets can be enumerated in time $O(3^{n/3})$. It follows that there are at most $3^{n/3} \cdot n^{10}$ minimal connected dominating sets D such that $H = G[D]$ has a dominating pair $\{u,v\}$ with $\mathrm{dist}_G(u,v) \geq 9$, and these sets can be enumerated in time $O(3^{n/3} \cdot n^{10})$.

We conclude that G has at most $3^{n/3} \cdot (n^{10} + n^9)$ minimal connected dominating sets that can be enumerated in time $O^*(3^{n/3})$. □

Proposition 1 implies that the bound for AT-free graphs is tight up to a polynomial factor.

6. Strongly chordal graphs

A vertex u of a graph G is *simple* if for any two neighbors x and y, $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. In other words, the closed neighborhoods of the neighbors of u are linearly ordered by inclusion. An ordering $v_1, \ldots, v_n$ of $V(G)$ is a *simple elimination ordering* if for each $i \in \{1, \ldots, n\}$, v_i is a simple vertex of $G[\{x_i, \ldots, x_n\}]$.

Lemma 6 ([12])**.** *A graph is strongly chordal if and only if it has a simple elimination ordering.*

Theorem 6. *A connected strongly chordal graph has at most $3^{n/3}$ minimal connected dominating sets, and these set can be enumerated in time $O^*(3^{n/3})$.*

Proof. We consider the following ENUMCDS(H, X) algorithm that for a connected induced subgraph H of a connected strongly chordal graph G and a set of vertices $X \subseteq V(G)$ enumerates the minimal connected dominating sets D of G such that $X \subseteq D$, $D \cap (V(G) \setminus V(H)) = X \cap (V(G) \setminus V(H))$ and $D \cap V(H)$ is a connected dominating set of H. This is a branching algorithm based on the property that any strongly chordal graph has a simple vertex by Lemma 6.

ENUMCDS(H, X)

1. If $X \cap V(H)$ is a connected dominating set of H, then return X if X is a minimal connected dominating set of G and stop.
2. If H is a complete graph, then for each $v \in V(H)$, return $X \cup \{v\}$, and stop.
3. Consider a simple vertex $u \in V(H)$, and for each $v \in N_H(u)$, let $H' = H - (N_H[u] \setminus \{v\})$, $X' = X \cup \{v\}$ and call ENUMCDS(H', X').

We call ENUMCDS$(G, \emptyset)$ to enumerate minimal connected dominating sets of G.

The correctness of the algorithm is proved via the following claim.

Claim ($*$). *Suppose that D is a minimal connected dominating set of G, $X \subseteq D$ and H is a connected induced subgraph G such that*

i) $D \cap (V(G) \setminus V(H)) = X \cap (V(G) \setminus V(H))$ and X dominates $V(G) \setminus V(H)$,

ii) if a vertex $w \in V(H)$ is dominated by X in G, then for each component F of $G[X]$, if $N_G[w] \cap V(F) \neq \emptyset$, then $N_G[w] \cap (V(F) \cap V(H)) \neq \emptyset$,

iii) for any component F of $G[X]$, $G[V(F) \cap V(H)]$ is a non-empty component of $G[X \cap V(H)]$.

Then

a) if $X \cap V(H)$ is a connected dominating set of H, then $X = D$,

b) otherwise, if H is a clique, then $D = X \cup \{v\}$ for some $v \in V(H)$,

c) otherwise, if u is a simple vertex of H, then there is $v \in N_H(u)$ such that $X' = X \cup \{v\} \subseteq D$ and i)–iii) are fulfilled for $H' = H - (N_H[u] \setminus \{v\})$ and X'.

Proof of Claim (∗). To show a), it is sufficient to observe that if $X \cap V(H)$ is a connected dominating set of H, then by i), X is a dominating set of G, and by iii), $G[X]$ is connected, i.e., X is a connected dominating set of G. Hence, $D = X$ by the minimality of D.

To see b), notice that if H is a clique and $V(H) \cap X \neq \emptyset$, then we have the case a) and $X = D$. Assume that $X \cap V(H) = \emptyset$. By iii), $X = \emptyset$. Then by i), $V(G) \setminus V(H) = \emptyset$, i.e., $H = G$. Clearly, $D = \{v\}$ for $v \in V(G)$ in this case.

It remains to show c). In this case H is not a complete graph and, therefore, $|V(H)| \geq 3$. Let u be a simple vertex of H.

First, we prove that $D \cap N_H(u) \neq \emptyset$. Suppose that u is dominated by X. Then by ii), u is dominated by $X \cap V(H)$. If u is dominated by a vertex $v \in X \cap V(H)$ such that $v \neq u$, then $v \in D \cap N_H(u) \neq \emptyset$. Assume that u is dominated only by itself and $u \in X \cap V(H)$. As $X \cap V(H)$ is not a connected dominating set of H, at least one vertex of $N_H(u)$ is in D by i) and iii). Suppose now that u is not dominated by X. Clearly, u is dominated by some $v \in D$. If $v \neq u$, then $v \in D \cap N_H(u) \neq \emptyset$. Suppose that u is a private for itself with respect to D. It means that $D = \{u\}$. Because u is a simple vertex of H, we obtain that H is a complete graph and we have b) instead of c); a contradiction.

We have that $D \cap N_H(u) \neq \emptyset$. Let $v \in D \cap N_H(u)$ be such that $N_H[v]$ is maximal by inclusion. We prove that i)–iii) hold for this choice of v.

To show i), we first prove that $D \cap N_H[u] = (X \cap N_H[u]) \cup \{v\}$. Clearly, $(X \cap N_H[u]) \cup \{v\} \subseteq D \cap N_H[u]$. To obtain a contradiction, suppose that there is $w \in D \cap N_H[u]$ such that $w \notin X$ and $w \neq v$. Since u is a simple vertex of H, $N_H[w] \subseteq N_H[v]$ by the selection of v. Hence, every vertex of H dominated by w is dominated by v as well. As $w \notin X$, w is not a cut vertex of $G[X]$, and because $N_H[w] \subseteq N_H[v]$, we obtain that $D \setminus \{w\}$ is connected. This contradicts the minimality of D.

As $D \cap N_H[u] = X \cap N_H[u] \cup \{v\}$, $D \cap (V(G) \setminus V(H')) = X' \cap (V(G) \setminus V(H'))$. Recall that X dominates $V(G) \setminus V(H)$. Because v dominates $N_H[u]$, we have that X' dominates $V(G) \setminus V(H')$.

To prove ii), let w be a vertex of H' dominated by $X' \setminus V(H')$ in G and let F be a component of $G[X']$ that dominates w. If $v \in V(F)$ and v dominates w then, trivially, w is dominated by $V(F) \cap V(H')$. Suppose that v does not dominate w. Then w is dominated by X and there is a component F' of $G[X]$ such that $V(F') \subseteq V(F)$ and F' dominates w. By ii) for H and X, w is dominated by a vertex $z \in V(F') \cap V(H)$. If $z \in V(H')$, w is dominated by $V(F) \cap V(H')$ and ii) is fulfilled. Assume that $z \notin V(H')$. Then $z \in N_H[u] \setminus \{v\}$. But v is adjacent to z and w by the choice of v. We have that $v \in V(F)$ and v dominates w, i.e., we have the previous case; a contradiction.

To show iii), let F be a component of $G[X']$.

Assume first that $v \in V(F)$. Then, trivially, $V(F) \cap V(H') \neq \emptyset$. If F has the unique vertex v, then $G[V(F) \cap V(H')]$ is a component of $G[X \cap V(H')]$. Assume that $|V(F)| \geq 2$.

Suppose that $v \in X$. Then F is a component of $G[X]$. By iii) for H and X, $V(F) \cap V(H) \neq \emptyset$ and $G[V(F) \cap V(H)]$ is a component of $G[X \cap V(H)]$. If $V(F) \cap (N_H[u] \setminus \{v\}) = \emptyset$, then $G[V(F) \cap V(H)] = G[V(F) \cap V(H')]$ is a component of $G[X \cap V(H')]$. If $V(F) \cap (N_H[u] \setminus \{v\}) \neq \emptyset$, then because $N_H[v]$ contains all the

vertices of $G[V(F)\cap V(H')]$ that are adjacent to the vertices of $V(F)\cap(N_H[u]\setminus\{v\})$, $G[V(F)\cap V(H')]$ is connected.

Let $v\notin X$. Denote by $F_1,\dots,F_r$ the components of $F-v$. Notice that $F_1,\dots,F_r$ are components of $G[X]$. By iii) for H and X, $G[V(F_i)\cap V(H)]\neq\emptyset$ and $G[V(F_i)\cap V(H)]$ is a component of $G[X\cap V(H)]$ for $i\in\{1,\dots,r\}$. Moreover, by ii) for H and X, v is adjacent to a vertex of $G[V(F_i)\cap V(H)]$. If $V(F_i)\cap(N_H[u]\setminus\{v\})=\emptyset$, then $G[V(F_i)\cap V(H')]$ is a component of $G[X\cap V(H')]$ and, consequently, $G[(V(F_i)\cup\{v\})\cap V(H')]$ is connected. If $V(F_i)\cap(N_H[u]\setminus\{v\})\neq\emptyset$, then recall that $N_H[v]$ contains all the vertices of $G[V(F_i)\cap V(H')]$ that are adjacent to the vertices of $V(F_i)\cap(N_H[u]\setminus\{v\})$. Hence, $G[(V(F_i)\cup\{v\})\cap V(H')]$ is connected. Since $G[(V(F_i)\cup\{v\})\cap V(H')]$ is connected for every $i\in\{1,\dots,r\}$, $G[V(F)\cap V(H')]$ is a component of $G[X\cap V(H')]$.

Suppose now that $v\notin V(F)$. We have that F is a component of $G[X]$. By iii) for H and X, $G[V(F)\cap V(H)]\neq\emptyset$ and $G[V(F)\cap V(H)]$ is a component of $G[X\cap V(H)]$. Because $v\notin V(F)$, $N_H[v]\cap(V(F)\cap V(H))=\emptyset$ and, therefore, $V(F)\cap(N_H[u]\setminus\{v\})=\emptyset$. We have that $G[V(F)\cap V(H)]=G[V(F)\cap V(H')]$ is a non-empty component of $G[X\cap V(H')]$.

This completes the proof of the claim. □

Observe that the conditions i)–iii) of Claim $(*)$ are fulfilled for $H=G$ and $X=\emptyset$. Applying Claim $(*)$ recursively, we obtain that for any minimal connected dominating set D of G, ENUMCDS$(G,\emptyset)$ outputs D at least once, i.e., ENUMCDS$(G,\emptyset)$ enumerates the minimal connected dominating sets.

To obtain an upper bound for the number of minimal connected dominating sets, it is sufficient to upper bound the number of leaves in the search tree produced by ENUMCDS. Notice that when we perform branching on Step 3 of ENUMCDS for a simple vertex u with $t=d_H(u)$, we get d branches and in each branch we call ENUMCDS for a graph with $|V(H)|-t$ vertices, that is, we have the branching vector $(\underbrace{t,\dots,t}_{t})$ for $t\geq 2$. Since the maximum value of the branching number is achieved for $t=3$ and is $3^{1/3}$, we obtain that G has at most $3^{n/3}$ minimal connected dominating sets by Lemma 1.

To complete the proof, notice that it is known that a simple elimination ordering of a strongly chordal graph can be found in polynomial time (see, e.g., [2, 5]). Observe also that for any induced subgraph H of G, the ordering of its vertices induced by the simple elimination ordering for G is a simple elimination ordering. As each step of ENUMCDS can be done in linear time, we conclude that ENUMCDS runs in time $O^*(3^{n/3})$. □

As the class of interval graphs is a subclass of the strongly chordal graphs, the bound $3^{n/3}$ is tight by Proposition 1.

7. Distance-hereditary graphs

Note that all cographs are distance-hereditary. First, we observe that the number of minimal connected dominating sets of a cograph is polynomial.

Proposition 3. *A connected cograph G with at least 3 vertices has at most $m = |E(G)|$ minimal connected dominating sets, and these can be enumerated in time $O(m)$.*

Proof. Let G be a connected cograph. We claim that each minimal connected dominating set either is a singleton or a set of two adjacent vertices.

The proof is by induction on the number of vertices. If G has one vertex, then the claim is trivial. Assume that G has at least two vertices. Recall (see, e.g., [5, 24]) that any connected cograph is the *join* of two cographs G_1 and G_2, i..e, $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$). Let D be a minimal connected dominating set of G. If $D \cap V(G_1) \neq \emptyset$ and $D \cap V(G_2) \neq \emptyset$, then by minimality, $D = \{u, v\}$, where $u \in V(G_1)$ and $v \in V(G_2)$ and $|D| = 2$. If $D \subseteq V(G_1)$ or $D \subseteq V(G_2)$, then D is a minimal connected dominating set of G_1 or G_2 respectively and, by induction, $|D| \leq 2$. Hence, the claim follows.

Let G be a cograph with at least 3 vertices. Let

$$X = \{v \mid \{v\} \text{ is a minimal connected dominating set of } G\}$$

and

$$Y = \{uv \mid \{u, v\} \text{ is a minimal connected dominating set of } G\}.$$

By minimality, $Y \subseteq E(G - X)$. Notice that for each $v \in X$, $d_G(v) = n - 1$. We have that $m \geq (n-1)|X| - |X|(|X| - 1)/2 + |E(G - X)| \geq |X| + |Y|$.

Using the fact that every cograph can be constructed recursively from isolated vertices using disjoint union and join operations and this decomposition can be obtained in linear time [5, 24], it is straightforward to check that all minimal connected dominating sets can be enumerated in time $O(m)$. □

Notice that this bound is tight, e.g., for complete bipartite graphs. Now we consider distance-hereditary graphs. First, we need some additional terminology.

Let G be a connected graph and $u \in V(G)$. Denote by $L_0(u), \ldots, L_{s(u)}(u)$ the levels in the breadth-first search (BFS) of G starting at u. Hence for all $i \in \{0, \ldots, s(u)\}$, $L_i(u) = \{v \in V(G) \mid \mathrm{dist}_G(u, v) = i\}$. Clearly, the number of levels in this decomposition is $s(u) + 1$. For $i \in \{1, \ldots, s(u)\}$, we denote by $\mathcal{G}_i(u)$ the set of components of $G[L_i(u) \cup \ldots \cup L_{s(u)}(u)]$, and $\mathcal{G}(u) = \cup_{i=1}^{s(u)} \mathcal{G}_i(u)$. Let $H \in \mathcal{G}_i(u)$ and $B = N_G(V(H))$. Clearly, $B \subseteq L_{i-1}(u)$. We say that B is the *boundary of H in* $L_{i-1}(u)$. For $i \in \{0, \ldots, s(u) - 1\}$, $\mathcal{B}_i(u)$ is the set of boundaries in $L_i(u)$ of the graphs of $\mathcal{G}_{i+1}(u)$ and $\mathcal{B}(u) = \cup_{i=0}^{s(u)-1} \mathcal{B}_i(u)$.

Lemma 7 ([4, 11])**.** *A connected graph G is distance-hereditary if and only if for any vertex $u \in V(G)$, any $H \in \mathcal{G}(u)$ with the boundary B, the following holds: $N_G(v) \cap V(H) = N_G(w) \cap V(H)$ for $v, w \in B$.*

We also need the next observation that is implicit in [4, 11] but also can be easily proved directly and we provide the proof for completeness.

Lemma 8 ([4, 11])**.** *Let G be a connected distance-hereditary graph and $u \in V(G)$. Then for any $B_1, B_2 \in \mathcal{B}(u)$, either $B_1 \cap B_2 = \emptyset$ or $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.*

Proof. Assume for the sake of contradiction that there are $B_1, B_2 \in \mathcal{B}(u)$ such that $B_1 \cap B_2 \neq \emptyset$ but neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$. Let B_1 and B_2 be the boundaries of H_1 and H_2 respectively. Let $x \in N_G(B_1) \cap V(H_1)$ and $y \in N_G(B_2) \cap V(H_2)$. By Lemma 7, $H_1 \neq H_2$ and $\mathrm{dist}_G(x, y) = 2$. Consider $G' = G - B_1 \cap B_2$. Because $B_1 \setminus B_2 \neq \emptyset$ and $B_2 \setminus B_1 = \emptyset$, x and y are in the same component of G', because G' has paths that connect u with x and y respectively, but $\mathrm{dist}_{G'}(x, y) \geq 3$; a contradiction. □

For a graph G and $u \in V(G)$, denote by $\mathcal{B}'(u)$ the set of inclusion minimal sets of $\mathcal{B}(u)$. Notice that by Lemma 8, the sets of $\mathcal{B}'(u)$ are pairwise disjoint if G is distance-hereditary. The enumeration of minimal connected dominating sets for distance-hereditary graphs is based on the following lemma.

Lemma 9. *Let G be a connected distance-hereditary graph with at least two vertices and $u \in V(G)$. Then for any minimal connected dominating set D with $u \in D$, $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ and $|B \cap D| = 1$ for $B \in \mathcal{B}'(u)$.*

Proof. Let $H \in \mathcal{G}(u)$ and let $B \in \mathcal{B}(u)$ be its boundary. If $v \in V(H) \cap D \neq \emptyset$, then $B \cap D \neq \emptyset$, because $G[D]$ has an (u, v)-path and this path has a vertex of B. If $V(H) \cap D = \emptyset$, then D has a vertex v that dominates a vertex of H. Clearly, $v \in B$. We conclude that for each $B \in \mathcal{B}'(u)$, $|B \cap D| \geq 1$.

Since $n \geq 2$, $s(u) \geq 1$ and, therefore, $\mathcal{G}(u) \neq \emptyset$ and $\mathcal{B}'(u) \neq \emptyset$. In particular $\{u\} \in \mathcal{B}'(u)$. Recall that the sets of $\mathcal{B}'(u)$ are pairwise disjoint. Hence, there is a $D' \subseteq D$ such that $D' \subseteq \cup_{B \in \mathcal{B}'(u)} B$ and $|B \cap D'| = 1$ for $B \in \mathcal{B}'(u)$. If $H \in \mathcal{G}_i(u)$ for $i \in \{1, \ldots, s(u)\}$, then any vertex of its boundary dominates $V(H) \cap L_i(u)$ by Lemma 7. Therefore, for each $H \in \mathcal{G}_i(u)$, the vertices of $V(H) \cap L_i(u)$ are dominated. Hence, D' is a dominating set. Because for each $v \in D'$, $G[D']$ has a (u, v)-path by Lemma 7, D' is a connected dominating set. By minimality, we obtain that $D = D'$. □

Theorem 7. *A connected distance-hereditary graph has at most $3^{n/3} \cdot n$ minimal connected dominating sets, and these sets can be enumerated in time $O^*(3^{n/3})$.*

Proof. If $n = 1$, then G has one connected dominating set and $1 \leq 3^{n/3} \cdot n$. Suppose that G is a connected distance-hereditary graph and $n \geq 2$. For each $u \in V(G)$, G has at most $\prod_{B \in \mathcal{B}'(u)} |B|$ sets $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ such that $|D \cap B| = 1$ for $B \in \mathcal{B}'(u)$. By Lemma 9, G has at most $\prod_{B \in \mathcal{B}'(u)} |B|$ minimal connected dominating sets containing u. Because the sets of $\mathcal{B}'(u)$ are pairwise disjoint, $\sum_{B \in \mathcal{B}'(u)} |B| \leq n$. It is well known (see, e.g., [17]) that $\prod_{B \in \mathcal{B}'(u)} |B| \leq 3^{n/3}$ in this case. We obtain that the total number of minimal connected dominating sets is at most

$$\sum_{u \in V(G)} \prod_{B \in \mathcal{B}'(u)} |B| \leq 3^{n/3} \cdot n.$$

It is trivial to enumerate all minimal connected dominating sets if $n = 1$. To enumerate minimal connected dominating sets of a connected graph G with $n \geq 2$, we consider all possible choices of a vertex u. For each u, we perform the breadth-first search from u that can be done in linear time, and construct in time $O(nm)$ $\mathcal{B}'(u)$. Then

the sets $D \subseteq \cup_{B \in \mathcal{B}'(u)} B$ such that $|D \cap B| = 1$ for $B \in \mathcal{B}'(u)$ can be enumerated in a straightforward way in time $O^*(3^{n/3})$ by the recursive branching algorithm with the branching vectors $(\underbrace{t, \ldots, t}_{\text{t}})$ for $t \geq 2$. Hence, the total running time is $O^*(3^{n/3})$. $\square$

By Proposition 1 the obtained upper bound for distance-hereditary graphs is tight up to factor n.

8. Open questions

The most challenging question concerns the maximum number of minimal connected dominating sets in an n-vertex graph. No upper bound c^n with $c < 2$ is known so far, leaving a large gap between the known lower bound $3^{n/3}$ and the trivial upper bound 2^n. It seems unlikely that the maximum number of minimal connected dominating sets is upper bounded by $3^{n/3}$. If this were the case and the upper bound could be obtained by an enumeration algorithm, then this would drastically improve the running time of the best algorithm solving the minimum connected dominating set problem from $O(1.8619^n)$ to $O(1.4423^n)$.

Acknowledgments

We are grateful to the anonymous referees for constructive comments which helped us to improve the presentation of our results.

References

[1] F. N. Abu-Khzam, A. E. Mouawad, M. Liedloff, An exact algorithm for connected red-blue dominating set, J. Discrete Algorithms 9 (3) (2011) 252–262.

[2] R. P. Anstee, M. Farber, Characterizations of totally balanced matrices, J. Algorithms 5 (2) (1984) 215–230.

[3] L. Babel, S. Olariu, Domination and steiner tree problems on graphs with few P4's, in: WG '98, vol. 1517, Springer, 1998, pp. 337–350.

[4] H. Bandelt, H. M. Mulder, Distance-hereditary graphs, J. Comb. Theory, Ser. B 41 (2) (1986) 182–208.

[5] A. Brandstädt, V. B. Le, J. P. Spinrad, Graph classes: a survey, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[6] J. M. Byskov, Enumerating maximal independent sets with applications to graph colouring, Operations Research Letters 32 (6) (2004) 547 – 556.

[7] D. G. Corneil, S. Olariu, L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (3) (1997) 399–430.

[8] J. Couturier, P. Heggernes, P. van 't Hof, D. Kratsch, Minimal dominating sets in graph classes: Combinatorial bounds and enumeration, Theor. Comput. Sci. 487 (2013) 82–94.

[9] J. Couturier, P. Heggernes, P. van 't Hof, Y. Villanger, Maximum number of minimal feedback vertex sets in chordal graphs and cographs, in: COCOON 2012, vol. 7434 of Lecture Notes in Computer Science, Springer, 2012, pp. 133–144.

[10] J. Couturier, R. Letourneur, M. Liedloff, On the number of minimal dominating sets on some graph classes, Theor. Comput. Sci. 562 (2015) 634–642.

[11] A. D'Atri, M. Moscarini, Distance-hereditary graphs, steiner trees, and connected domination, SIAM J. Comput. 17 (3) (1988) 521–538.

[12] M. Farber, Characterizations of strongly chordal graphs, Discrete Mathematics 43 (2-3) (1983) 173–189.

[13] F. V. Fomin, S. Gaspers, A. V. Pyatkin, I. Razgon, On the minimum feedback vertex set problem: Exact and enumeration algorithms, Algorithmica 52 (2) (2008) 293–307.

[14] F. V. Fomin, F. Grandoni, A. V. Pyatkin, A. A. Stepanov, Combinatorial bounds via measure and conquer: Bounding minimal dominating sets and applications, ACM Transactions on Algorithms 5 (1).

[15] F. V. Fomin, P. Heggernes, D. Kratsch, Exact algorithms for graph homomorphisms, Theor. Comp. Sys. 41 (2) (2007) 381–393.

[16] F. V. Fomin, P. Heggernes, D. Kratsch, C. Papadopoulos, Y. Villanger, Enumerating minimal subset feedback vertex sets, Algorithmica 69 (1) (2014) 216–231.

[17] F. V. Fomin, D. Kratsch, Exact exponential algorithms, Texts in Theoretical Computer Science. An EATCS Series, Springer, Heidelberg, 2010.

[18] F. V. Fomin, Y. Villanger, Finding induced subgraphs via minimal triangulations, in: STACS 2010, vol. 5 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010, pp. 383–394.

[19] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Co., New York, NY, USA, 1979.

[20] S. Gaspers, S. Mackenzie, On the number of minimal separators in graphs, CoRR abs/1503.01203.

[21] S. Gaspers, M. Mnich, Feedback vertex sets in tournaments, Journal of Graph Theory 72 (1) (2013) 72–89.

[22] P. A. Golovach, P. Heggernes, D. Kratsch, Enumeration and maximum number of minimal connected vertex covers in graphs, CoRR abs/1602.07504.
URL `http://arxiv.org/abs/1602.07504`

[23] P. A. Golovach, P. Heggernes, D. Kratsch, R. Saei, Subset feedback vertex sets in chordal graphs, J. Discrete Algorithms 26 (2014) 7–15.

[24] M. C. Golumbic, Algorithmic graph theory and perfect graphs, vol. 57 of Annals of Discrete Mathematics, 2nd ed., Elsevier Science B.V., Amsterdam, 2004.

[25] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of domination in graphs, vol. 208 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1998.

[26] P. Heggernes, Minimal triangulations of graphs: A survey, Discrete Mathematics 306 (3) (2006) 297–317.

[27] M. M. Kanté, V. Limouzy, A. Mary, L. Nourine, On the enumeration of minimal dominating sets and related notions, SIAM J. Discrete Math. 28 (4) (2014) 1916–1929.

[28] J. W. Moon, L. Moser, On cliques in graphs, Israel J. Math. 3 (1965) 23–28.

[29] J. A. Telle, Y. Villanger, Connecting terminals and 2-disjoint connected subgraphs, in: WG 2013, vol. 8165 of Lecture Notes in Computer Science, Springer, 2013, pp. 418–428.