Finding Clubs in Graph Classes☆

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Abstract

For a positive integer s, an s-club in a graph G is a set of vertices that induces a subgraph of G of diameter at most s. We study a relation of clubs and cliques. For a positive integer s, we say that a graph class $\mathcal{G}$ has the s-clique-power property if for every graph $G \in \mathcal{G}$, every maximal clique in G^s is an s-club in G. Our main combinatorial results show that both 4-chordal graphs and AT-free graphs have the s-clique-power property for all $s \geq 2$. This has various algorithmic consequences. In particular we show that a maximum s-club in G can be computed in polynomial time when G is a chordal bipartite or a strongly chordal or a distance hereditary graph. On weakly chordal graphs, we obtain a polynomial-time algorithm when s is an odd integer, which is best possible as the problem is NP-hard for even values of s. We complement these results by proving the NP-hardness of the problem for every fixed s on 4-chordal graphs. Finally, if G is an AT-free graph, we prove that the problem can be solved in polynomial time when $s \geq 2$, which gives an interesting contrast to the fact that the problem is NP-hard for $s = 1$ on this graph class.

Keywords: s-club, k-chordal graphs, AT-free graphs

1. Introduction

Max Clique is one of the most fundamental problems in graph algorithms. Cliques model highly connected or correlated parts of networks and data sets, and consequently they find applications in numerous diverse fields. For many real problems, however, cliques present a too restrictive measure of connectivity (see e.g., [1, 15, 27, 33]), and the notion of clubs were proposed to give more

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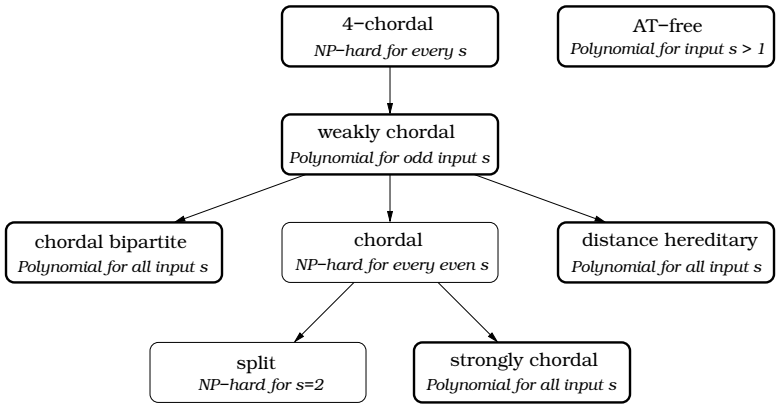

Figure 1: The inclusion relationship among the mentioned graph classes, where each child is a subset of its parent. AT-free graphs are not related to the rest. Boxes with thicker frames contain the results proved in this paper.

realistic models [3, 28]. Given a graph $G = (V, E)$ on n vertices and an integer s between 1 and n, a vertex subset $S \subseteq V$ is an s-club if the subgraph of G induced by S has diameter at most s. Hence 1-clubs are exactly cliques, and every s-club is also an $(s + 1)$-club by definition. Notice the non-hereditary nature of s-clubs, which makes their behavior different from that of cliques for $s \geq 2$: although every subset of a clique is a clique, the same is not true for an s-club. In fact, deciding whether a given s-club is maximal, in the sense that no superset of it is an s-club, is NP-complete for every fixed $s \geq 2$ [29].

Given a graph G and an integer s, the objective of the Max s-Club problem is to compute an s-club of maximum cardinality. We are interested in the exact solution of this problem. Note that the problem becomes trivial if G has diameter at most s.

Max s-Club is NP-hard for every fixed s, even on graphs of diameter $s+1$ [7]. It remains NP-hard on bipartite graphs for every fixed $s \geq 3$, and on chordal graphs for every even fixed $s \geq 2$ [4]. Max 2-Club is NP-hard on graphs that become bipartite by deleting one vertex, on graphs that can be covered by three cliques, and on graphs with domination number two and diameter three [22]. On split graphs Max 2-Club is NP-hard [4], whereas Max s-Club has a trivial solution for all input $s \geq 3$. On general graphs, the problem is fixed-parameter tractable when parameterized by the solution size [13] for every fixed $s \geq 2$, or by the dual of the solution size [32] for every fixed s. Fixed-parameter tractability of Max 2-Club has been studied also with respect to various other parameters [21, 22]. Furthermore, Max s-Club can be solved by an $O(1.62^n)$-time algorithm [13]. The problem can be solved in polynomial time on trees and interval graphs for all input values of s, and on graphs of bounded treewidth and graphs of bounded clique-width for every fixed s that is not a part of the input [31].

In this paper we show that Max s-Club can be solved in polynomial time

for all odd input values of s on weakly chordal graphs. For subclasses of weakly chordal graphs, we show that the problem can be solved in polynomial time for all input values of s on chordal bipartite graphs, strongly chordal graphs, and distance hereditary graphs. To complement these positive results, we show that on 4-chordal graphs, which form a superclass of weakly chordal graphs, the problem is NP-hard for every fixed s. In addition to these results, we show that the problem is solvable in polynomial time for all input $s \geq 2$ on AT-free graphs. Interestingly, Max Clique is NP-hard on this graph class. The inclusion relationship among the graph classes mentioned above is illustrated in Fig. 1, which also summarizes our results.

2. Definitions and first observations

We refer to the textbook by Diestel [17] for any undefined graph terminology. We consider finite undirected graphs without loops or multiple edges. Such a graph $G = (V, E)$ is identified by its vertex set V and its edge set E. Throughout the paper, we let $n = |V|$ and $m = |E|$. The subgraph of G induced by $U \subseteq V$ is denoted by $G[U]$. For a vertex v, we denote by $N_G(v)$ the set of vertices that are adjacent to v in G. The *distance* $\mathrm{dist}_G(u, v)$ between vertices u and v of G is the number of edges on a shortest path between them. The *diameter* $\mathrm{diam}(G)$ of G is $\max\{\mathrm{dist}_G(u, v) \mid u, v \in V\}$. The *complement* of G is the graph $\overline{G}$ with vertex set V, such that any two distinct vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. For a positive integer k, the *k-th power* G^k of G is the graph with vertex set V, such that any two distinct vertices u, v are adjacent in G^k if and only if $\mathrm{dist}_G(u, v) \leq k$. We say that P is a (u, v)-path if P is a path that joins u and v. The vertices of P different from u and v are the *inner* vertices of P. The *chordality* $ch(G)$ of a graph G is the length of the longest induced cycle in G; if G has no cycles, then $ch(G) = 0$. A set of pairwise adjacent vertices is a *clique*. A clique is *maximal* if no proper superset of it is a clique, and *maximum* if it has maximum size.

For a non-negative integer k, a graph G is *k-chordal* if $ch(G) \leq k$. A graph G is *weakly chordal* if both G and $\overline{G}$ are 4-chordal. A graph is *chordal bipartite* if it is both 4-chordal and bipartite. A graph is *chordal* if it is 3-chordal. A graph is a *split* graph if its vertex set can be partitioned in an independent set and a clique. A chord xy in a cycle C of even length is said to be *odd* if the distance in C between x and y is odd. A graph is *strongly chordal* if it is chordal and every cycle of even length at least 6 has an odd chord. A graph G is a *distance hereditary* if for any connected induced subgraph H of G, if u and v are in H, then $\mathrm{dist}_G(u, v) = \mathrm{dist}_H(u, v)$. An *asteroidal triple (AT)* is a set of three non-adjacent vertices such that between each pair of them there is a path that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. Each of these graph classes can be recognized in polynomial (in most cases linear) time and they are closed under taking induced subgraphs [10, 19]. See the monographs by Brandstädt et al. [10] and Golumbic [19] for more properties and characterizations of these classes and their inclusion relationships.

Let s be a positive integer. A set of vertices S in G is an *s-club* if $\mathrm{diam}(G[S]) \leq s$. An s-club of maximum size is a *maximum* s-club. Given a graph G and a positive integer s, the Max s-Club problem is to compute a maximum s-club in G. Cliques are exactly 1-clubs, and hence Max 1-Club is equivalent to Max Clique.

Observation 1. *Let G be a graph and let s be a positive integer. If S is an s-club in G then S is a clique in G^s.*

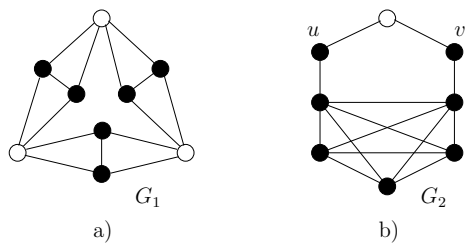

Figure 2: Cliques in G^2 and 2-clubs.

Although Observation 1 is easy to see, it is important to note that the backward implication does not hold in general: a (maximal) clique in G^s is not necessarily an s-club. To see this, let $s = 2$ and consider the graphs shown in Fig. 2 a) and b). The set of vertices S shown in black in Fig. 2 a) is the unique maximum clique of G_1^2, but clearly S is not a 2-club in G_1, as $G_1[S]$ is not even connected. The example in Fig. 2 b) shows that it does not help to require that $G[S]$ is connected: the set of black vertices S is a maximal clique in G_2^2, $G_2[S]$ is connected, but S is not a 2-club in G_2, because $\mathrm{dist}_{G_2[S]}(u, v) = 3$. Observe also that a maximum s-club in G is not necessarily a maximal clique in G^s. Furthermore, the maximum size of a clique in G_1^2 is strictly greater than the maximum size of a 2-club in G_1. For the set of black vertices S in Fig. 2 b), $S \setminus \{u\}$ and $S \setminus \{v\}$ are maximum 2-clubs, whereas S is a maximal clique in G_2^2.

As we will show in Section 3, for some graph classes, maximal cliques in s-th powers are in fact s-clubs. For a positive integer s, we say that a graph class $\mathcal{G}$ has the *s-clique-power* property if for every graph $G \in \mathcal{G}$, every maximal clique in G^s is an s-club in G. Furthermore, we say that $\mathcal{G}$ has the *clique-power* property if every maximal clique in G^s is an s-club in G, for every positive integer s and every graph $G \in \mathcal{G}$. Due to Observation 1, we see that if G belongs to a graph class that has the clique-power property, then a vertex set S in G is a maximal s-club if and only if S is a maximal clique in G^s. As G^s can be computed in time $O(n^3)$ for any positive s, the following is immediate, and it will be the framework in which we obtain our results.

Proposition 1. *Let $\mathcal{G}$ be a graph class that has the clique-power property and let s be a positive integer.*

- *If* Max Clique *can be solved in time $O(f(n))$ on $\{G^s \mid G \in \mathcal{G}\}$, then* Max s-Club *can be solved in time $O(f(n) + n^3)$ on $\mathcal{G}$.*

- *If* Max Clique *is* NP*-hard on* $\{G^s \mid G \in \mathcal{G}\}$*, then* Max s-Club *is* NP*-hard on* $\mathcal{G}$*.*

3. Graph classes that have the clique-power property

In this section we show that 4-chordal graphs and AT-free graphs have the clique-power property. We start with 4-chordal graphs, and we consider the cases $s = 2$ and $s \geq 3$ separately in the next two lemmas.

Lemma 1. *4-Chordal graphs have the* 2*-clique-power property.*

Proof. To obtain a contradiction, assume that there is a 4-chordal graph $G = (V, E)$ such that G^2 has a maximal clique S, but S is not a 2-club in G. Let $u, v \in S$ be vertices at distance at least 3 in $G[S]$. This means in particular that u, v are not adjacent in G.

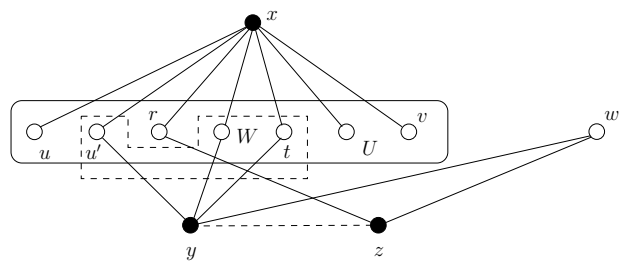

Figure 3: Construction of P, Q, R.

Since $u, v \in S$, there is a (u, v)-path of length at most 2 in G. As u and v are not adjacent, every such a path has length 2. Let P be a (u, v)-path of length 2 in G such that the middle vertex x of P is adjacent to the maximum number of pairwise non-adjacent vertices of S. Let $U \subseteq S$ be a maximum size set of pairwise non-adjacent vertices of S that are adjacent to x and contains u and v. Vertex x is at distance at least 3 from a vertex $w \in S$ in G. Otherwise x would belong to maximal clique S in G^2, and u, v would be at distance at most 2 in $G[S]$. It follows that w is not adjacent to the vertices of U, because G has no (x, w)-path of length 2. Clearly, $xw \notin E$. Let u' be any vertex of U. Since $u', w \in S$, there is a (u', w)-path of length at most 2. Since $u'w \notin E$, any such a path has length 2. Let Q be a (u', w)-path of length 2 such that $u' \in U$ and the middle vertex y of Q is adjacent to the maximum number of vertices of U. By the choice of P, vertex y is not adjacent to some vertices of U, say $yr \notin E$ for some $r \in U$. Let $W \subseteq U$ be the set of vertices of U adjacent to y. Now there is a (r, w)-path R of length at most 2 in G. We observe that R has length 2 and denote by z the middle vertex of R. By the choice of Q, z is not adjacent to at least one vertex $t \in W$, and $y \neq z$. The construction of the paths P, Q and R is shown in Fig. 3.

Observe that $xy, xz \notin E$, because $\mathrm{dist}_G(x, w) \geq 3$. Recall also that r, t, w are pairwise non-adjacent and $ry, tz \notin E$. If $yz \notin E$, then $\{r, x, t, y, w, z\}$ induces a

chordless cycle of length 6 in G; contradicting that G is 4-chordal. If $yz \in E$, then $\{r, x, t, y, z\}$ induces a chordless cycle of length 5 in G, and we again get a contradiction. $\square$

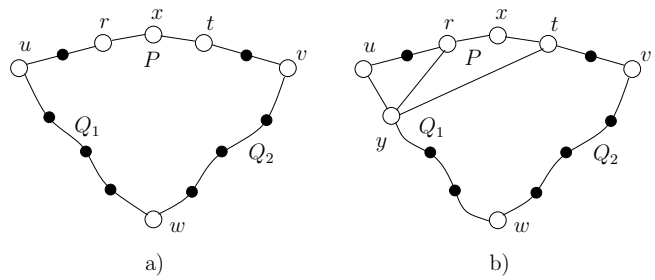

Figure 4: Construction of P, Q_1, Q_2.

Lemma 2. 4-*Chordal graphs have the* s-*clique-power property for every* $s \geq 3$.

Proof. To obtain a contradiction, assume that there is an integer $s \geq 3$ and a 4-chordal graph G such that G^s has a maximal clique S, but S is not an s-club in G. Let $u, v \in S$ be vertices at distance at least $s+1$ in $G[S]$. In particular, u, v are not adjacent in G. Since $u, v \in S$, $\mathrm{dist}_G(u, v) \leq s$. Any shortest (u, v)-path in G has at least one vertex that is at distance at least $s+1$ from some vertex of S; otherwise all inner vertices of some (u, v)-path of length at most s would belong to maximal clique S in G^s, and u, v would be at distance at most s in $G[S]$. Consider a shortest (u, v)-path P in G that has the minimum number of vertices at distance at least $s+1$ from some vertex of S. Let x be an inner vertex of P at distance at least $s+1$ from a vertex $w \in S$ in G. Denote by r and t the vertices adjacent to x in P. Since P is a shortest path, r and t are not adjacent. For every vertex $h \in N_G(r) \cap N_G(t)$, let P_h be the path obtained from P by replacing subpath rxt with rht. Observe that by the choice of P, for every $h \in N_G(r) \cap N_G(t)$, P_h is a shortest (u, v)-path, and h is at distance at least $s+1$ from some vertex of S. For every vertex $h \in N_G(r) \cap N_G(t)$, let

$$U_h = \{g \in S \setminus \{u, v\} \mid \mathrm{dist}_G(g, h) = s-1 \text{ and } \mathrm{dist}_G(g, r) = \mathrm{dist}_G(g, t) = s\}.$$

We may assume that $|U_x| = \max\{|U_h| \mid h \in N_G(r) \cap N_G(t)\}$, because otherwise we can replace rxt with rht in P. Notice that the set U_x might be empty.

Let Q_1 be a shortest (u, w)-path in G, and let Q_2 be a shortest (v, w)-path in G. Note that the length of each of these paths is at most s. Since $\mathrm{dist}_G(x, w) > s$, x is not in Q_1 or Q_2, and x is not adjacent to w or any inner vertex of Q_1 or Q_2. The construction of P, Q_1 and Q_2 is shown in Fig. 4 a). Let X be the union of the vertices belonging to P, Q_1, and Q_2. Observe that $G[X]$ contains an induced cycle C that includes vertices x, r, t and edges xr, xt, because $G[X \setminus \{x\}]$ is connected by our construction. Since r, t are not adjacent and x is not adjacent to any vertex of $X \setminus \{r, t\}$, it follows that C has at least four

vertices. Since G is 4-chordal, C has exactly four vertices. Let y be a vertex of C different from r, x, t (see Fig. 4 b). Clearly, y belongs to Q_1 or Q_2. It follows that $\mathrm{dist}_G(w, y) \leq s - 1$, $\mathrm{dist}_G(w, r) \leq s$, and $\mathrm{dist}_G(w, t) \leq s$. As $\mathrm{dist}_G(w, x) > s$, we conclude that $\mathrm{dist}_G(w, y) = s - 1$, and $\mathrm{dist}_G(w, r) = \mathrm{dist}_G(w, t) = s$.

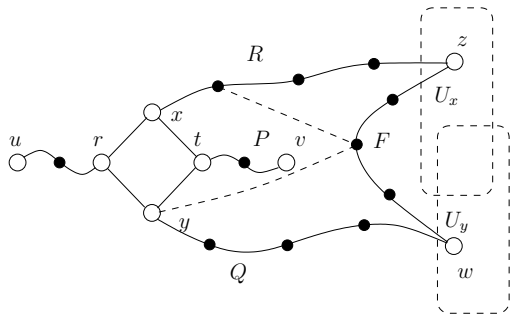

Figure 5: Construction of Q, R, F.

Denote by Q a shortest (w, y)-path in G, and observe that Q has length $s - 1$. Vertex y belongs to $N_G(r) \cap N_G(t)$. Notice that $w \in U_y$, and thus $U_y \neq \emptyset$. Since $|U_x| \geq |U_y|$ by our construction, and $w \notin U_x$, there is a vertex $z \in U_x \setminus U_y$. By the definition of U_x, $\mathrm{dist}_G(z, r) = \mathrm{dist}_G(z, t) = s$. Since $z \notin U_y$, $\mathrm{dist}_G(z, y) \neq s-1$. The assumption that $\mathrm{dist}_G(z, y) \leq s-2$ immediately implies that $\mathrm{dist}_G(z, r) \leq s - 1$, which gives a contradiction. Hence, $\mathrm{dist}_G(z, y) \geq s$. Denote by R a shortest (z, x)-path in G, and note that R has length $s - 1$ by the definition of U_x. The construction of Q and R is shown in Fig. 5. Notice that r or t does not belong to Q or R. Furthermore, r or t is not adjacent to any vertex of Q or R, except y and x.

We claim that Q and R have no common vertex and there is no edge between a vertex of Q and a vertex of R. For contradiction, assume first that Q and R have a common vertex h. Let R' be the (h, x)-subpath of R, and let Q' be the (h, y)-subpath of Q. Denote by ℓ_1 the length of R' and by ℓ_2 the length of Q', and assume that $\ell_1 \leq \ell_2$. Then consider the following path from w to x: first take the (w, h)-subpath of Q and then from h to x the (h, x)-subpath of R. It follows that the length of this path is at most the length of Q, i.e., $s-1$, which contradicts that $\mathrm{dist}_G(w, x) > s$. Hence if Q and R have a common vertex h then $\ell_1 > \ell_2$. Now consider the following path from z to y: first take the (z, h)-subpath of R and then the (h, y)-subpath of Q, which has length at most $s - 1$. This implies $\mathrm{dist}_G(z, y) \leq s - 1$, which contradicts our previous conclusion that $\mathrm{dist}_G(z, y) \geq s$. Consequently P and Q cannot have a common vertex. Suppose now that G has an edge $z'w'$ where z' is in R and w' is in Q. We choose $z'w'$ in such a way that the distance between x and z' in R is minimum. Recall that $xy \notin E$. If $z'y \in E$, then $\mathrm{dist}_G(z, y) \leq s - 1$. Hence, $z'y \notin E$. By the same arguments, $xw' \notin E$. Then the concatenation of xry, the (y, w')-subpath of Q, $w'z'$, and the (z', x)-subpath of R is an induced cycle on at least 5 vertices, contradicting that G is 4-chordal. We conclude that Q and

R have neither common vertices nor adjacent vertices.

Since $z, w \in S$, we know that $\mathrm{dist}_G(z, w) \leq s$. Let F be a shortest (z, w)-path in G (see Fig. 5). We claim that vertices x, y, r do not belong to F, and neither x nor r is adjacent to any vertex of F. If x is in F, then the (z, x)-subpath of F has length at least the length of R, namely $s - 1$. But since $xw \notin E$, this contradicts that $\mathrm{dist}_G(z, w) \leq s$. Symmetrically, we observe that y is not in F either. If r is in F, since $\mathrm{dist}_G(z, r) = s$, then the (z, r)-subpath of F has length at least s, which contradicts either $\mathrm{dist}_G(z, w) \leq s$ or F is a shortest (z, w)-path. Now assume that x is adjacent to some vertex h of F. Since $\mathrm{dist}_G(w, x) \geq s + 1 \geq 4$, the (z, h)-subpath of F has length at most $s - 3$, but then $\mathrm{dist}_G(z, x) \leq s - 2$; again a contradiction. Let r be adjacent to a vertex h of F. Then because $\mathrm{dist}_G(r, w) = s$, the (w, h)-subpath of F has length at least $s - 1$, but then $\mathrm{dist}_G(r, z) \leq 2 < s$, and we again obtain a contradiction. To complete the proof, it remains to notice that the union of the vertices of Q, R, and F, together with r, induces a subgraph of G with an induced cycle on at least 5 vertices, but this contradicts our assumption that G is 4-chordal. $\square$

Theorem 1. 4-*Chordal graphs have the clique-power property.*

Theorem 1 immediately follows from Lemmas 1 and 2. The example shown in Fig. 2 b) shows that this result is tight in the sense that 5-chordal graphs do not have the clique-power property.

Now we turn to AT-free graphs, and we show that they also have the clique-power property. In the following proof, we use additional terminology: Let u, v be vertices of G, and let P be a (u, v)-path in G. We say that P *sees* a vertex x of G if x belongs to P or x is adjacent to an inner vertex of P.

Theorem 2. *AT-free graphs have the clique-power property.*

Proof. To obtain a contradiction, assume that there is an integer $s \geq 2$ and an AT-free graph $G = (V, E)$, such that G^s has a maximal clique S, but S is not an s-club in G. Let $u, v \in S$ be vertices at distance at least $s + 1$ in $G[S]$. In particular, u and v are not adjacent in G.

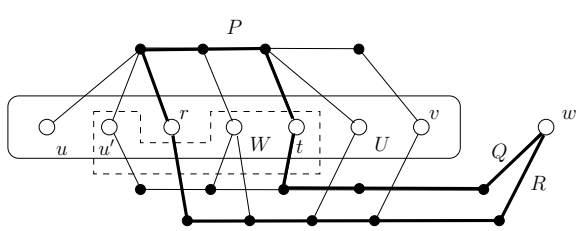

Figure 6: Construction of P, Q, R.

Since $u, v \in S$, G has a (u, v)-path of length at most s. Let P be a (u, v)-path of length at most s in G that sees the maximum number of pairwise non-adjacent vertices of S. Let $U \subseteq S$ be a maximum size set of pairwise non-adjacent vertices

of S containing u and v that are seen by P. Clearly, $u, v \in U$. The path P has an inner vertex x at distance at least $s+1$ from a vertex $w \in S$ in G. Otherwise, all inner vertices of P would be included in the maximal clique S in G^s, and u, v would be at distance at most s in $G[S]$. It follows that w is not adjacent to any vertex of U, because the distance between x and z in the subgraph of G induced by the vertices of P and U is at most $s-1$ for any $z \in U$. Furthermore, by construction, P does not see w. Let u' be any vertex of U. Since $u, w \in S$, G has a (u', w)-path of length at most s. Let Q be a (u', w)-path of length at most s for any $u' \in U$ that sees the maximum number of vertices of U. By the choice of P, there is at least one vertex $r \in U$ such that Q does not see r. Let $W \subseteq U$ be the set of vertices of U seen by Q; hence $r \notin W$. Now there is a (r, w)-path R of length at most s in G, and by the choice of Q, there is at least one vertex $t \in W$ such that R does not see t. The construction of the paths P, Q and R is shown in Fig. 6.

We claim that $\{r, t, w\}$ is an AT, contradicting that G is AT-free. Observe that $r, t \in U$, the vertices of P together with U induce a connected subgraph of G, and w is not adjacent to any vertex of P or U. Thus G has an (r, t)-path that does not contain a neighbor of w. Similarly, G has a (t, w)-path each of whose vertices belongs to Q or W, that does not contain a neighbor of r. Finally, R is an (r, w)-path that contains no neighbor of t. □

4. Algorithmic consequences

In this section we obtain tractability and intractability results by combining the results of Section 3 with Proposition 1.

4.1. Polynomial cases

Max s-Club has been studied on chordal graphs by Asahiro et al. [4]. Their results assume that chordal graphs have the clique-power property, but the property is neither stated nor proved in [4]. Balakrishnan and Paulraja [5, 6] (see also [2]) showed that odd powers of chordal graphs are chordal. Consequently, Proposition 1 and Theorem 1 immediately imply that Max s-Club can be solved in polynomial time on chordal graphs. We can now generalize this result to weakly chordal graphs using Theorem 1, since weakly chordal graphs are 4-chordal. Brandstädt et al. [9] proved that odd powers of weakly chordal graphs are weakly chordal. Hayward et al. [23, 24] showed that Max Clique can be solved in time $O(nm)$ on weakly chordal graphs. As a consequence of these results, Proposition 1, and Theorem 1, we obtain the following result.

Theorem 3. Max s-Club *can be solved in time $O(n^3)$ on weakly chordal graphs, for all positive odd input integers s.*

Recall that for every even integer s, Max s-Club is NP-hard on chordal graphs [4], and thus also on weakly chordal graphs and on 4-chordal graphs. For the strongly chordal subclass of chordal graphs we are able to show polynomial-time solvability for all values of s. Lubiw [26] showed that any power of a

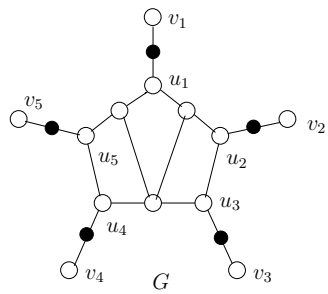

Figure 7: A chordal bipartite graph G is shown. For a positive even integer s and $i = 1, \ldots, 5$, the vertices u_i and v_i are joined by a path of length $(s-2)/2$ ($u_i = v_i$ for $s = 2$) in G. Then the vertices $v_1, \ldots, v_5$ induce a cycle of length 5 in G^s.

strongly chordal graph is strongly chordal. With this result, Proposition 1 and Theorem 1 immediately give the following.

Theorem 4. Max s-Club *can be solved in time $O(n^3)$ on strongly chordal graphs, for all positive input integers s.*

We move to distance hereditary graphs. Recall that they are 4-chordal, and consequently we can apply Theorem 1. Bandelt et al. [8] proved that even powers of distance hereditary graphs are chordal, and thus weakly chordal. It also follows from their results that odd powers of distance hereditary graphs are weakly chordal. Combining this with Proposition 1 and Theorem 1, we obtain the following result.

Theorem 5. Max s-Club *can be solved in time $O(n^3)$ on distance hereditary graphs, for all positive input integers s.*

Notice that if s is a fixed integer and not a part of the input, then the problem can be solved in linear time on distance hereditary graphs [31] because these graphs have clique-width at most 3 [20].

Chordal bipartite graphs form another subclass of 4-chordal graphs and of weakly chordal graphs. By Theorem 3, Max s-Club can be solved in polynomial time on chordal bipartite graphs, for odd values of s. For even values of s, s-th powers of chordal bipartite graphs are not necessary weakly chordal. In fact they are not even perfect, as shown in Fig. 7. A graph is *perfect* if neither the graph nor its complement contains an induced cycle of odd length [14]. Perfect graphs form a superclass of weakly chordal graphs, and Max Clique is solvable in polynomial time on them. Unfortunately, we cannot use this due to Fig. 7. Still, we are able to solve Max s-Club on chordal bipartite graphs in polynomial time for even s using the following structural result that we find interesting also on its own.

Lemma 3. *Let $G = (X, Y, E)$ be a chordal bipartite graph and let s be any positive integer. Then $G^s[X]$ and $G^s[Y]$ are chordal graphs.*

Proof. By symmetry, it is sufficient to prove the lemma for $G^s[X]$. Since the lemma is trivially true for $s = 1$, let us assume that $s \geq 2$ for the rest of the proof. For contradiction suppose there is an induced cycle $C = x_0, x_1, \ldots, x_{k-1}, x_0$ of length at least 4 in $G^s[X]$. For ease of notation, let $x_k = x_0$, and read all indices modulo k throughout the proof. It follows that for every i between 0 and $k-1$, there is a shortest (x_i, x_{i+1})-path P_i of length at most s in G.

For every i, we first show that there is no edge $xy \in E$ with $x \in P_i$ and $y \in P_j$, for $j \notin \{i-1, i, i+1\}$. For contradiction suppose there is such an edge xy for some i. Without loss of generality we may assume that $k > j > i+1$. Since C is an induced cycle, $x_i x_j$ is not an edge of G^s and $x_{i+1}x_{j+1}$ is not an edge of G^s. Let P be the path obtained by the concatenation of the (x_i, x)-subpath of P_i and edge xy and the (y, x_j)-subpath of P_j (see Fig. 8 a). Similarly, let Q be the path obtained by the concatenation of the (x_{i+1}, x)-subpath P_i and edge xy and the (y, x_{j+1})-subpath of P_j. Observe that both P and Q have length at least $s+1$, because otherwise $x_i x_j$ or $x_{i+1}x_{j+1}$ would be an edge of G^s. Let ℓ_1 be the length of the (x_i, x)-subpath of P, and let ℓ_2 be the length of the (y, x_j)-subpath of P. Thus the length of P is $\ell_1 + \ell_2 + 1$, and consequently $\ell_1 + \ell_2 \geq s$. Observe that the length of Q is at most $2s - \ell_1 - \ell_2 + 1$, and since the length of Q is at least $s+1$, we have $2s - \ell_1 - \ell_2 \geq s$. Combining this inequality with $\ell_1 + \ell_2 \geq s$, we conclude that $\ell_1 + \ell_2 = s$. This implies that both P and Q have length exactly $s+1$. We can further conclude that both P_i and P_j have length exactly s. However, since $x_i, x_{i+1}, x_j, x_{j+1}$ are all in X, and G is a bipartite graph, there cannot be paths of length both s and $s+1$ between pairs of these vertices in G, which gives the desired contradiction.

The above also implies that P_i and P_j do not have common vertices for $j \notin \{i-1, i, i+1\}$, since this would imply an edge between a vertex of P_i and a vertex of P_j, under the above assumptions.

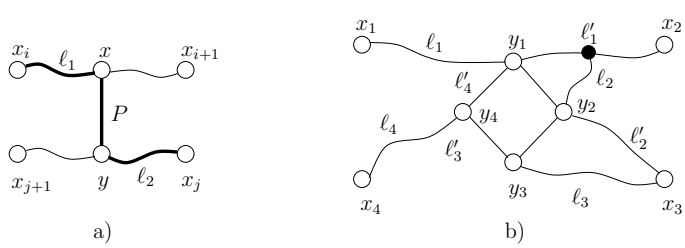

Figure 8: The paths described in the proof of Lemma 3.

As a consequence of the above, if there is an edge in G between a vertex of P_i and a vertex of P_j, then we can assume that $j = i+1$. Observe that there is always an edge between every pair of consecutive paths P_i and P_{i+1}, and they might also share some vertices. For every i, we will call an edge $xy \in E$ with $x \in P_i$ and $y \in P_{i+1}$ a *long chord with respect to* x if there is no other vertex y' in the (y, x_{i+2})-subpath of P_{i+1} such that $xy' \in E$. Observe that x_i is not adjacent to any vertex of P_{i+1}, since C is an induced cycle in G^s and thus $x_i x_{i+2}$ is not an edge in G^s. We will now follow the long chords between consecutive

paths, and construct an induced cycle C' in G as follows. Start with any vertex $x \in P_1$. Pick a vertex y in P_0 such that $yx \in E$ and the (y, x_1)-subpath of P_0 is longest. We traverse P_1 from x to x_2, and as soon as we come to a vertex that is adjacent to a vertex in P_2, we take the first long chord and go to P_2. For each i from 2 to $k-1$, we continue in this manner from P_i to P_{i+1}: once we are on P_i we continue on P_i towards x_{i+1} and we take the first long chord to P_{i+1}. At the end once we are in P_{k-1}, we take the first edge $y'y''$ such that $y' \in P_{k-1}$ and $y'' \in P_0$ and y'' is not an inner vertex of the (y, x_1)-subpath of P_0. If y' has other neighbors that are on the (y'', y)-subpath of P_0 then we take as y'' such a neighbor that is closest to y. We continue from y'' to y on P_0 and use the edge yx to close the cycle. Observe that, by the choices we made, the (y'', y)-subpath of P_0 is the only portion on P_0 that contributes to C', and no vertex on this subpath is adjacent to y' or x, except y'' that is adjacent to y', and y that is adjacent to x. All other edges of C' are long chords or portions of P_i that do not contain any neighbor of P_{i+1}, and hence C' is an induced cycle. Since there is no induced cycle of length more than 4 in G, and C' contains distinct vertices from each P_i for $0 \leq i \leq k-1$, we conclude that $k = 4$. Consequently, C' consists of y_0, y_1, y_2, y_3, such that $y_i \in P_i$ and $y_i y_{i+1} \in E$, for $i = 0, 1, 2, 3$. A possible way these paths can interact is depicted in Fig. 8 b). Let ℓ_i be the length of the (x_i, y_i)-subpath of P_i, and let ℓ'_i be the length of the (y_i, x_{i+1})-subpath of P_i, for $i = 0, 1, 2, 3$. Since C is an induced cycle in G^s, $x_i x_{i+2}$ is not an edge of G^s, and we can conclude that $\ell_i + \ell'_i \leq s$ and $\ell_i + \ell'_{i+1} \geq s$, for $i = 0, 1, 2, 3$. Adding up all four pairs of inequalities, we obtain that $\ell_i + \ell'_i = s$ and $\ell_i + \ell'_{i+1} = s$, for $i = 0, 1, 2, 3$. Consequently, we have a path between x_i and x_{i+2} of length $s+1$ using the (x_i, y_i)-subpath of P_i, the edge $y_i y_{i+1}$, and the (y_{i+1}, x_{i+2})-subpath of P_{i+1}, whereas the length of P_i is s. Since $x_0, \ldots x_{k-1} \in X$ and G is bipartite, we cannot have paths of length both s and $s+1$ between pairs of them. Therefore our initial assumption that $G^s[X]$ contained an induced cycle of length at least 4 is wrong, and $G^s[X]$ is chordal. □

Theorem 6. Max s-Club *can be solved in time $O(n^4)$ on chordal bipartite graphs, for all positive input integers s.*

Proof. For a chordal bipartite graph $G = (X, Y, E)$, denote by $G^s_{\text{bip}}[X, Y]$ the bipartite graph obtained from G^s by removing the edges of G^s whose both endpoints are either in X or in Y. Any maximal clique S in G^s can be partitioned into S_1 and S_2 such that S_1 is a clique in $G^s[X]$, S_2 is a clique in $G^s[Y]$, and $G^s_{\text{bip}}[S_1, S_2]$ is a complete bipartite graph (biclique). We can thus generate all maximal cliques of G^s and pick a maximum one as follows. For every maximal clique X' of $G^s[X]$ and every maximal clique Y' of $G^s[Y]$, find a maximum biclique of $G^s_{\text{bip}}[X', Y']$. By Lemma 3, the disjoint union of $G^s[X]$ and $G^s[Y]$ is a chordal graph on n vertices. Every such chordal graph has at most n maximal cliques, and these can be generated in $O(n^2)$ time [34]. After this, in at most n^2 bipartite graphs we need to find a maximum biclique. Maximum bicliques in bipartite graphs can be found in polynomial time [16], and hence we can find a maximum clique in G^s in polynomial time. From Proposition 1 and Theorem 1, it then follows that Max s-Club can be solved in polynomial time on

chordal bipartite graphs for all s. Using arguments similar to those in the proof of Lemma 3, it can be shown that $G^s_{\text{bip}}[X, Y]$ is a chordal bipartite graph, for every positive s; this also follows from the results of Chandran and Mathew [11]. A maximum biclique can be computed in time $O(n^2)$ on chordal bipartite graphs [25], and hence the claimed running time follows. □

Finally we move to AT-free graphs. Max Clique is NP-hard on AT-free graphs [30], and hence Max 1-Club is NP-hard on AT-free graphs. Chang et al. [12] showed that for every $s \geq 2$ and every AT-free graph G, G^s is a cocomparability graph. Cocomparability graphs form a subclass of AT-free graphs. Fortunately Max Clique can be solved in polynomial time on cocomparability graphs [19]. This, combined with Proposition 1 and Theorem 2, gives the next result.

Theorem 7. Max s-Club *can be solved in time $O(n^3)$ on AT-free graphs, for all positive input integers $s \geq 2$.*

4.2. Hardness on 4-chordal graphs

In Section 4.1 we proved that Max s-Club can be solved in polynomial time on several subclasses of 4-chordal graphs. Here we complement these results by showing that the problem is NP-hard on 4-chordal graphs. By Proposition 1, it is sufficient to show that Max Clique is NP-hard on powers of 4-chordal graphs.

A *2-subdivision* of a graph is obtained by replacing every edge with a path of length three. A graph is a *2-subdivision* if it is a 2-subdivision of some graph. Given a graph G and an integer k, the decision problem Clique asks whether G has a clique of size at least k, and the decision problem Independent Set whether G has an independent set of size at least k. Clearly, X is a clique in G if and only if X is an independent set in $\overline{G}$. We use this duality to obtain the following lemma.

Lemma 4. Clique *is* NP*-complete on 4-chordal graphs of diameter at most 2.*

Proof. Let $G = (V, E)$ be a 2-subdivision on at least 5 vertices. We show that $\overline{G}$ is a 4-chordal graph of diameter at most 2. First, $\overline{G}$ cannot have an induced cycle on 5 vertices, as it would imply that G has an induced cycle on 5 vertices, but every induced cycle in G has length at least 9. If C is an induced cycle on at least 6 vertices in $\overline{G}$, then the vertices of C induce a subgraph with a triangle in G, but G has no triangles. Hence, $\overline{G}$ is 4-chordal. Let x and y be any pair of distinct vertices of G. If they are not adjacent in $\overline{G}$, then $xy \in E$. Because G is a 2-subdivision with at least 5 vertices, there is a vertex z in G such that $z \neq x, y$, and z is not adjacent to x or y. It follows that z is adjacent to both x and y in $\overline{G}$, and hence $\text{dist}_{\overline{G}}(x, y) \leq 2$. Therefore, $\text{diam}(\overline{G}) \leq 2$.

To conclude the proof of the lemma, it is sufficient to observe that Independent Set is known to be NP-complete for 2-subdivisions [19, 30]. □

Theorem 8. Clique *is* NP*-complete on $\{G^s \mid G$ is 4-chordal$\}$, for every positive integer s.*

Proof. Asahiro et al. [4] proved that Clique is NP-complete on even powers of chordal graphs, and consequently on even powers of 4-chordal graphs. For $s = 1$, the statement of Theorem 8 follows immediately from Lemma 4. Hence, it is sufficient to prove the theorem for odd $s > 1$. Let $s = 2r + 1$ for $r \geq 1$. We give a reduction from Clique on 4-chordal graphs of diameter at most 2, which is NP-complete by Lemma 4.

Let $G = (V, E)$ be a 4-chordal graph of diameter at most 2, which is input to Clique together with an integer k. Let $V = \{u_1, \ldots, u_n\}$, and let H be the graph obtained from G as follows: for each $i \in \{1, \ldots, n\}$, we add r new vertices $v_i^1, \ldots, v_i^r$ and r new edges $u_i v_i^1$, $v_i^1 v_i^2$, ..., $v_i^{r-1} v_i^r$ to G. In other words, we attach a path $v_i^1, v_i^2; \ldots, v_i^r$ to every vertex u_i of G, via edge $u_i v_i^1$. Let us denote by U the set of vertices $\{u_1, \ldots, u_n\} \cup (\cup_{i=1}^{r-1} \{v_1^i, \ldots v_n^i\})$. Since $\mathrm{diam}(G) \leq 2$ and $s = 2r + 1$, we can observe the following:

- U is a clique in H^s,
- for every $i \in \{1, \ldots n\}$, v_i^r is adjacent to every vertex of U in H^s,
- for every pair $i, j \in \{1, \ldots, n\}$, v_i^r and v_j^r are adjacent in H^s is and only if u_i and u_j are adjacent in G.

Consequently, every maximal clique in H^s contains U as a subset. Furthermore, any set $\{u_{i_1}, \ldots, u_{i_k}\}$ of k vertices in G is a clique of G if and only if $\{v_{i_1}^r, \ldots, v_{i_k}^r\} \cup U$ is a clique in H^s. Since $|U| = rn$, we conclude that G has a clique of size at least k if and only if H^s has a clique of size at least $k + rn$, which completes the reduction. □

Theorem 8 and Proposition 1 immediately give the following result.

Theorem 9. Max s-Club *is* NP*-hard on 4-chordal graphs, for every positive integer* s.

5. Conclusions

Structural and algorithmic properties of clubs in 4-chordal graphs, AT-free graphs and their subclasses have been studied. Our main combinatorial results state that all these graph classes have the s-clique-power property for every $s \geq 2$. Among the algorithmic consequences are polynomial time algorithms to compute a maximum s-club in chordal bipartite, strongly chordal, distance hereditary and AT-free graphs for any fixed s, and in weakly chordal graphs for fixed odd s.

It is natural to ask whether other important graph classes have the s-clique-power property for all fixed $s \geq 2$. Also it is interesting to consider Max s-Club for other graphs classes. To mention one candidate, we believe that it is interesting to know whether this problem is tractable for circular-arc graphs. Notice that the Maximum Clique can be solved in polynomial time on this graph class [19], but circular-arc graphs have no clique-power property. To see it, it is sufficient to observe that the graphs shown in Fig. 2 are circular-arc.

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