MULTIPLICATIVE AND ADDITIVE SCHWARZ' METHODS: CONVERGENCE IN THE 2-DOMAIN CASE. *

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Abstract. We consider the classical Schwarz alternating algorithm and an additive version more suitable for parallel processing. The two methods are compared and analyzed in the case of two domains. We show that the rate of convergence for both methods, can be directly related to a generalized eigenvalue problem, derived from subdomain contributions to the global stiffness matrix. Analytical expressions are given for a model case.

Key Words. Schwarz' Method, Orthogonal Projections, Elliptic Equations, Domain Decomposition, Finite Elements, Parallel Computation.

1. Introduction. Recently there has been a strong revival of the interest in domain decomposition algorithms for the solution of elliptic problems; cf. e.g. Glowinski et al [12], and Chan et al [7]. This is to a large extent due to their potential in parallel computing environments.

Substructuring methods, with a long history from the structural analysis community [20, 1], are methods where the global domain is partitioned into disjoint (non-overlapping) pieces. More recently, the use of iterative methods for this class of problems, has been investigated, see [3] and [6] and the references given in those papers.

In a famous paper more than hundred years ago, Schwarz [21] developed his domain decomposition algorithm, based on computing the solution of two overlapping subproblems in an alternating fashion. This algorithm is therefore quite sequential in its original form, and not necessarily the best candidate for a parallel implementation. More recently P. L. Lions [14, 15] has obtained a number of interesting results on Schwarz' method and inspired new work on this type of algorithms. His reformulation of the classical method in terms of orthogonal projections has motivated the development of an alternative additive algorithm, see [11]. This particular formulation was apparently first described by Matsokin and Nepomnyashchikh [17].

Grid refinement algorithms can also be formulated and analyzed using orthogonal projections see the papers by Dryja and Widlund [24] and Mandel and McCormick [16]. The close relationship between Schwarz type methods, iterative substructuring methods [3] and promising iterative grid refinement algorithms; cf. e.g. McCormick et al [18, 19, 13] and Bramble et al [5], has only recently been realized.

As a beginning contribution to the analysis of these algorithms, we will derive precise relations between the classical (multiplicative) and the newly proposed additive Schwarz' methods in the case of two domains. We will do this by expressing the rele-

* To appear in the *Proceedings of the Second International Symposium on Domain Decomposition Methods, January 14-16, 1988, University of California, Los Angeles,* to be published by *SIAM.*

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vant orthogonal projection operators in terms of elementary contributions to the global stiffness matrix from the subdomains.

A model problem and the necessary notation is described in section 2. We review the classical Schwarz algorithm, including its variational form in section 3. The related additive form, is introduced in section 4. Since this algorithm is relatively new, we also describe some implementation details. In section 5, we explicitly compute the projection operators introduced in section 2, in terms of stiffness contributions from the individual subdomains. It is important to note that this derivation is quite general, and valid for any geometrical shape of the domains. This leads to a decomposition of the spectrum giving a precise relationship between the multiplicative and the additive algorithm. We also note that the classical Schwarz' algorithm is identical to a method recently proposed by Chan and Resasco [8, 9]. A closed form expression for the eigenvalues as a function of the geometry of the subdomains can only be given for special cases. We give the eigenvalues as a function of the aspect ratio, for the case where Ω is rectangular in section 6. Section 7 contains a few numerical experiments, confirming the theoretical results.

2. Notations and Preliminaries. To simplify our presentation, we assume that the elliptic operator is the Laplacian and that we have a zero Dirichlet condition. Thus,

$$\begin{aligned} -\Delta u &= f \qquad && \textit{in } \Omega, \\ u &= 0 && \textit{on } \partial\Omega. \end{aligned} \tag{1}$$

The region Ω is bounded, two- or three-dimensional, with a Lipschitz continuous boundary. Our algorithms and results can be extended immediately to linear, self-adjoint elliptic problems. We use a variational formulation of the problem, which, as shown by Sobolev [22] makes the maximum principle superfluous.

In variational form equation 1 is written as

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx = f(v) \ , \qquad \forall\, v \in H_0^1(\Omega) \ ,$$

where the solution $u \in H_0^1(\Omega)$, the closure in the Sobolev space $H^1(\Omega)$ of the space of smooth functions which vanish in a neighborhood of $\partial\Omega$. As always, the space $H^1(\Omega)$ is the subspace of $L_2(\Omega)$ for which $\mid u \mid_{H^1(\Omega)}^2 = a(u,u)$ is finite. In our analysis we work exclusively with the inner product defined by $a(\cdot,\cdot)$. Thus in this paper orthogonality and symmetry always refer to this inner product.

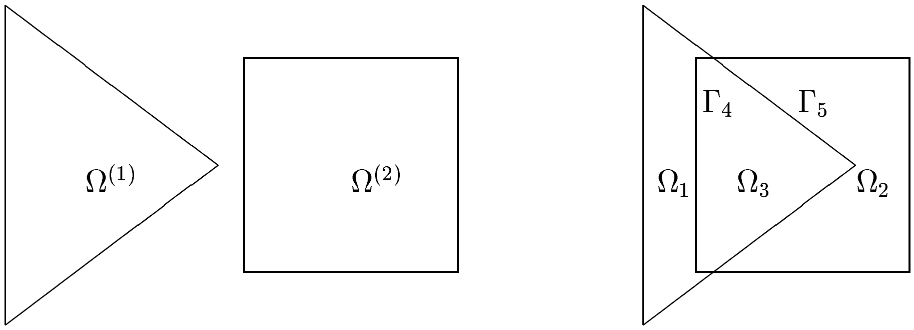

Fig. 1. *The partition of the domain.*

In this paper we consider domain decomposition algorithms with two overlapping subregions $\Omega^{(1)}$ and $\Omega^{(2)}$. We also use the notations $\Omega_1 = \Omega \setminus \overline{\Omega}^{(2)}$, where $\overline{\Omega}^{(2)}$ is the closure of $\Omega^{(2)}$, $\Omega_2 = \Omega \setminus \overline{\Omega}^{(1)}$ and $\Omega_3 = \Omega^{(1)} \cap \Omega^{(2)}$. The region Ω is thus also divided into three nonoverlapping subregions Ω_1, Ω_2, and Ω_3 which are separated from each other by the curves (or surfaces) $\Gamma_4 = \overline{\Omega}_1 \cap \overline{\Omega}_3$ and $\Gamma_5 = \overline{\Omega}_2 \cap \overline{\Omega}_3$. In figure 1 on the left, we display separately, the two subregions from which Ω is built and on the right the partitioning of Ω into the five subsets just defined. We assume that Γ_4 and Γ_5 follow element boundaries, that they are Lipschitz, and that they intersect only in at most a few points (or along a few curves).

The problem is discretized by finite elements in the customary fashion; cf. Ciarlet [10]. The region Ω is triangulated and a conforming finite element space $V^h \subset H_0^1(\Omega)$ is introduced. We assume that all the elements are shape regular, i.e. there is a bound on h_K/ρ_K which is independent of the number of degrees of freedom and of K. Here h_K is the diameter of the element K and ρ_K the diameter of the largest sphere that can be inscribed in K. The approximate solution $u_h \in V^h$ is defined by

$$a_\Omega(u_h, v_h) = f(v_h)\ , \qquad \forall\, v_h \in V^h. \tag{2}$$

We note that the exact solution to the discrete problem is given as $u_h = P_{V^h} u$, the orthogonal projection of the continuous solution $u \in H_0^1(\Omega)$, from $H_0^1(\Omega)$ into V^h.

3. The multiplicative Schwarz' algorithm. We return to the continuous case and write down the Schwarz algorithm in its traditional form. There are two fractional steps. Let u^n be the n-th iterate. Then the updated solution u^{n+1} is determined by,

$$\begin{aligned} -\Delta u^{n+1/2} &= f && \textit{in } \Omega^{(1)},\\ u^{n+1/2} &= 0 && \textit{on } \partial\Omega^{(1)} \cap \partial\Omega,\\ u^{n+1/2} &= u^n && \textit{on } \Gamma_5, \end{aligned}$$

and,

$$\begin{aligned} -\Delta u^{n+1} &= f && \textit{in } \Omega^{(2)},\\ u^{n+1} &= 0 && \textit{on } \partial\Omega^{(2)} \cap \partial\Omega,\\ u^{n+1} &= u^{n+1/2} && \textit{on } \Gamma_4. \end{aligned}$$

Following P. L. Lions [14, 15], we write these equations in variational form. This formulation is also valid in the discrete case.

$$
\begin{aligned}
a_\Omega(u_h^{n+1/2} - u_h^n, v_h) &= f(v_h) - a_\Omega(u_h^n, v_h) \\
&= a_\Omega(u_h - u_h^n, v_h) \;, && \forall\, v_h \in V^h \cap H_0^1(\Omega^{(1)}) \;, \\
a_\Omega(u_h^{n+1} - u_h^{n+1/2}, v_h) &= f(v_h) - a_\Omega(u_h^{n+1/2}, v_h) \\
&= a_\Omega(u_h - u_h^{n+1/2}, v_h) \;, && \forall\, v_h \in V^h \cap H_0^1(\Omega^{(2)}) \;.
\end{aligned}
$$

Here $u_h^{n+1/2} - u_h^n \in H_0^1(\Omega^{(1)})$ and $u_h^{n+1} - u_h^{n+1/2} \in H_0^1(\Omega^{(2)})$, since the boundary values do not change from one fractional step to the next. The equations can now be restated using orthogonal projections,

$$
\begin{aligned}
u_h^{n+1/2} - u_h^n &= P_1(u_h - u_h^n) \;, \\
u_h^{n+1} - u_h^{n+1/2} &= P_2(u_h - u_h^{n+1/2}) \;,
\end{aligned}
$$

where P_i , $i = 1, 2$, is the orthogonal projection of V^h into $V_i = V^h \cap H_0^1(\Omega^{(i)})$. The error $e_h^n = u_h^n - u_h$ satisfies the relationship

$$
e_h^{n+1} = (I - P_2)(I - P_1)e_h^n = (I - (P_1 + P_2 - P_2P_1))e_h^n.
$$

In a certain sense Schwarz' method is therefore a straightforward iterative method of solving the equation

$$
(P_1 + P_2 - P_2P_1)u_h = g_m, \tag{3}
$$

for a certain right hand side g_m.

4. The Additive Schwarz' Algorithm. The product term P_2P_1 in 3 prevents a straight forward parallel implementation, although many subdomain computations can be carried out simultaneously in the case where the classical algorithm is extended to many subdomains. Assuming that we can solve linear systems involving polynomials in P_1 and P_2, it is natural to consider alternatives to equation 3.

The simplest, additive form of Schwarz more suitable for parallel processing, appears by simply removing the product term in 3. We are then faced with solving the system

$$
(P_1 + P_2)u_h = g_a. \tag{4}
$$

In order to retain the same solution u_h, the right hand side g_m in 3 has changed to g_a. In order to use 4 for the computation of u_h, we must first find this new right hand side g_a. Fortunately it is computable from solving subproblems on the individual subdomains. We compute g_a by writing

$$
g_a = \sum_i g_i
$$

where $g_i \in V_i$ and each g_i solves the subproblem

$$a(g_i, v_h) = a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_i.$$

Once the appropriate right hand side g_a has been computed, we can apply a conjugate gradient iteration (working with the K-inner product) to solve 4 with no further preconditioning. This strategy requires a procedure for computing the products $v_h = P_i w_h$ for a given function w_h. Let us consider this problem in some detail for the subdomain $\Omega^{(1)}$ corresponding to $i = 1$. The orthogonal projection $P_1 w_h$ of w_h on V_1, is defined by

$$a_\Omega(P_1 w_h, \hat{v}_h) = a_\Omega(w_h, \hat{v}_h)\ , \quad \forall\ \hat{v}_h \in V_1,$$

and $P_1 w_h \in H_0^1(\Omega^{(1)})$. If we write $w_h = P_1 w_h + \hat{w}_h$, we conclude that $\hat{w}_h$ must be harmonic in $\Omega^{(1)}$ and have the boundary value w_h on Γ_5. We must take $\hat{w}_h = 0$ on the rest of the boundary $\partial\Omega \cap \partial\Omega^{(1)}$. Clearly, $\hat{w}_h = w_h$ in $\Omega \setminus \Omega_1$, making $v_h = 0$ outside Ω_1. The conjugate gradient iteration for solving 4 therefore requires the computation of a harmonic function on each subdomain in every iteration. All the subdomains can be processed in parallel. A representation of the projection operator in terms of the discrete matrix operator of the appropriate subdomain can also be given. Let K be the global stiffness matrix for the problem on Ω and let $K^{(i)}$ be the corresponding matrix for a problem on $\Omega^{(i)}$. We define the correspondence between the discrete finite element functions v_h and w_h and the vectors of nodal values y and x in the usual fashion. If the nodal unknowns are numbered appropriately, we can write:

$$y = P_i x = \begin{pmatrix} K^{(i)^{-1}} & 0 \\ 0 & 0 \end{pmatrix} Kx.$$

5. Representation of the Projection Operators in terms of Block Stiffness Matrices. The computer program implementation of the finite element method most often uses the so called subassembly method when building the matrix corresponding to the discrete, global problem. This principle is based on the observation that the bilinear form in 2 can be written as a sum of bilinear forms over disjoint subregions as expressed in equation 6 below. In this way contributions from the elementary elements can be added to form substructures of the full model, the substructures can again be added to form the full matrix.

The same concept is very useful in the study of certain domain decomposition methods. Many properties of these algorithms can be understood and interpreted in terms of the stiffness submatrices corresponding to the subdomains into which the domain has been decomposed [3].

With subvectors and subscripts corresponding to the degrees of freedom associated with the open sets Ω_1, Ω_2 and Ω_3 and the curves (surfaces) Γ_4 and Γ_5, the entire discrete problem can be written as

$$Kx = \begin{pmatrix} K_{11} & 0 & 0 & K_{14} & 0 \\ 0 & K_{22} & 0 & 0 & K_{25} \\ 0 & 0 & K_{33} & K_{34} & K_{35} \\ K_{14}^T & 0 & K_{34}^T & K_{44} & K_{45} \\ 0 & K_{25}^T & K_{35}^T & K_{45}^T & K_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}. \tag{5}$$

As always the elements of K are given by $a(\phi_i, \phi_j)$ where ϕ_i and ϕ_j are finite element basis functions. The zero blocks are a consequence of the fact that, when using standard finite element basis functions, there is no direct coupling between Ω_1 and Ω_2 etc..

Since the bilinear form is defined in terms of an integral,

$$a_\Omega(\phi_i, \phi_j) = a_{\widetilde{\Omega}}(\phi_i, \phi_j) + a_{\Omega \setminus \widetilde{\Omega}}(\phi_i, \phi_j), \tag{6}$$

for any subset $\widetilde{\Omega} \subset \Omega$. We note that for a pair of basis functions associated with $\partial\widetilde{\Omega}$, we get contributions from both terms. Thus, for example the submatrix K_{44} in 5 can be written $K_{44} = K_{44}^{(1)} + K_{44}^{(3)}$ where $K_{44}^{(i)}$ is the contribution from triangles in Ω_i $, i = 1, 3$.

Using the procedure to compute $v_h = P_i w_h$, described in the previous section, we can easily derive explicit expressions for the operator P_i itself. We first introduce some notations for Schur complements that frequently occur in the calculations to follow. Schur complements arise quite naturally in block Gaussian elimination and play an important role in the analysis of iterative substructuring methods as well [3]. We define the Schur complement $S_j^{(i)}$ as the Schur complement corresponding to the domain Ω_i with respect to the interior subdomain boundary Γ_j, as follows:

$$S_j^{(i)} = K_{jj}^{(i)} - K_{ij}^T K_{ii}^{-1} K_{i,j}, \;\; j = 4, \; i = 1, 3 \;\; \textit{and} \;\; j = 5, \; i = 2, 3.$$

In a similar way, we define $S_4^{(2)}$ to be the Schur complement corresponding to the entire region $\Omega^{(2)} = \Omega_2 \cup \Omega_3 \cup \Gamma_5$ with respect to Γ_4. It is of the form

$$S_4^{(2)} = K_{44}^{(3)} - \begin{pmatrix} 0 & K_{34}^T & K_{45} \end{pmatrix} \begin{pmatrix} K_{22} & 0 & K_{25} \\ 0 & K_{33} & K_{35} \\ K_{25}^T & K_{35}^T & K_{55} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{34} \\ K_{45}^T \end{pmatrix}.$$

We also define

$$S_{45} = K_{45} - K_{34}^T K_{33}^{-1} K_{35}$$

and write $S_4 = S_4^{(1)} + S_4^{(3)}$ and $S_5 = S_5^{(2)} + S_5^{(3)}$. We can now write down an explicit, block matrix representation for the projection operators.

$$P_1 x = \begin{pmatrix} I & 0 & 0 & 0 & C_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & B_4 \\ 0 & 0 & 0 & I & A_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

and

$$P_2 x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & C_5 & 0 \\ 0 & 0 & I & B_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_5 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

We have introduced the quantities A_i , B_i and C_i , $i = 4, 5$ in order to simplify the notation. These quantities are defined as:

$$\begin{aligned} A_4 &= S_4^{-1} S_{45} \\ A_5 &= S_5^{-1} S_{45}^T \\ B_4 &= K_{33}{}^{-1}(K_{35} - K_{34} A_4) \\ B_5 &= K_{33}{}^{-1}(K_{34} - K_{35} A_5) \\ C_4 &= -K_{11}{}^{-1} K_{14} A_4 \\ C_5 &= -K_{22}{}^{-1} K_{25} A_5. \end{aligned}$$

We can now find the matrix representation of the operators defined in 3 and 4,

$$(7) \qquad P_1 + P_2 = \begin{pmatrix} I & 0 & 0 & 0 & C_4 \\ 0 & I & 0 & C_5 & 0 \\ 0 & 0 & 2I & B_5 & B_4 \\ 0 & 0 & 0 & I & A_4 \\ 0 & 0 & 0 & A_5 & I \end{pmatrix}$$

and

$$(8) \qquad P_1 + P_2 - P_2 P_1 = \begin{pmatrix} I & 0 & 0 & 0 & C_4 \\ 0 & I & 0 & 0 & -A_4 C_5 \\ 0 & 0 & I & 0 & -A_4 B_5 \\ 0 & 0 & 0 & I & A_4 \\ 0 & 0 & 0 & 0 & I - A_5 A_4 \end{pmatrix}.$$

We observe that 7 is 2 by 2 block upper triangular with an invariant subspace corresponding to the unknowns on the two interior interfaces $\Gamma_i \quad i = 4, 5$, while 8 is block upper triangular. The spectra of the operators are readily available and we observe a very simple relationship between the two. Let the number of unknowns on Ω_i or Γ_i be $N_i \quad i = 1, 2, 3, 4, 5$.

In the additive case, we have $N_1 + N_2$ eigenvalues equal to 1 and N_3 eigenvalues equal to 2. The remaining eigenvalues have the form $1 \pm \mu_i$ where μ^2 is an eigenvalue of

$$A_5 A_4 y_5 = \mu^2 y_5$$

or in terms of the Schur complements

$$S_{45}^T S_4^{-1} S_{45} y_5 = \mu^2 S_5 y_5. \tag{9}$$

One should note that if $N_4 \neq N_5$, then there will be $|N_5 - N_4|$ additional eigenvalues equal to one. This observation may be important in cases where there is a significant difference between the number of points on Γ_4 and Γ_5. Only the smallest number of unknowns (shortest boundary) on the two interfaces will contribute nontrivial eigenvalues to the operator.

The algorithms presented in this paper and the finite element framework provided the motivation for this work. We have obtained a complete characterization of the spectra of the two operators from 3 and 4. It is possible to obtain a similar characterization for projection operators in general, we refer to a joint paper with J. Mandel [2] on this topic.

The matrix 7 is only symmetric in the K-inner product, for completeness we therefore include the explicit representations of 7 and 8 multiplied by the global stiffness matrix K.

$$K(P_1 + P_2) = \begin{pmatrix} K_{11} & 0 & 0 & K_{14} & 0 \\ 0 & K_{22} & 0 & 0 & K_{25} \\ 0 & 0 & 2K_{33} & 2K_{34} & 2K_{35} \\ K_{14}^T & 0 & 2K_{34}^T & K_{44} + D_4 & 2K_{45} \\ 0 & K_{25}^T & 2K_{35}^T & 2K_{45}^T & K_{55} + D_5 \end{pmatrix} \tag{10}$$

where

$$D_4 = K_{34}^T K_{33}^{-1} K_{34} + S_{45} S_5^{-1} S_{45}^T$$

and

$$D_5 = K_{35}^T K_{33}^{-1} K_{35} + S_{45}^T S_4^{-1} S_{45}.$$

For the multiplicative case,

$$K(P_1 + P_2 - P_2 P_1) = \begin{pmatrix} K_{11} & 0 & 0 & K_{14} & 0 \\ 0 & K_{22} & 0 & 0 & K_{25} \\ 0 & 0 & K_{33} & K_{34} & K_{35} \\ K_{14}^T & 0 & K_{34}^T & K_{44} & K_{45} + E_{45} \\ 0 & K_{25}^T & K_{35}^T & K_{45}^T & K_{55} \end{pmatrix} \tag{11}$$

with

$$E_{45} = S_{45}(I - A_5 A_4).$$

We observe that the matrix 10 is symmetric and that the entries corresponding to Ω_3 have been multiplied by two as should be expected. Except for this, the matrix is different from K only by the addition of a symmetric, positive definite term to each of the block diagonal positions representing the two interior interfaces. The matrix 11 differs from K only in the (4,5) block position. The matrix is therefore unsymmetric. Notice that the correction term is closely related to 9.

It remains to show that μ in 9 is bounded away from one, independent of the discretization. Consider the generalized eigenvalue problem

$$(S_4^{(1)} + S_4^{(2)})x_4 = \lambda(S_4^{(1)} + S_4^{(3)})x_4. \tag{12}$$

This problem arises if we consider the (non-overlapping) domain decomposition of Ω into $\Omega^{(2)}$ and Ω_1 with the common interface Γ_4. If 5 is reduced by block Gaussian elimination, to a system on Γ_4 only, and this system is preconditioned by the Schur complement resulting from a similar reduction of the equations on $\Omega^{(1)}$ to Γ_4, then $\lambda_{max}/\lambda_{min}$ is the relevant condition number of the iteration operator. We can establish uniform lower and upper bounds on the corresponding generalized Rayleigh quotient by using:

$$x_4{}^T S_4^{(i)} x_4 \leq C x_4{}^T S_4^{(j)} x_4 \;, \qquad \forall x_4, \;\; i,j = 1,2,3.$$

Proofs of this so called extension theorem are given in [3] and [23]. The constants in these inequalities depend only on Ω_1, Ω_2 and Ω_3.

It should be noted that this iterative method has recently been proposed by Chan and Resasco [8, 9]. Their results show that this method is quite robust, when applied to certain model problems, having subdomains with large aspect ratios, in the sense that the rate of convergence seems to be independent of the aspect ratios. As opposed to this, it is shown in [3] that the so called Neumann-Dirichlet algorithm do have a weak aspect ratio dependent rate of convergence.

We remark that the Neumann-Dirichlet algorithm [3], results when the preconditioning in 12 contains the term $S_4^{(1)}$ only. A simple calculation will show that

$$S_4^{(2)} = S_4^{(3)} - S_{45} S_5^{-1} S_{45}^T.$$

Substituting this into 12 gives

$$\lambda = 1 - \mu^2$$

where μ^2 solves the eigenvalue problem 9.

We have therefore shown that $1 - \mu^2$ is uniformly bounded from below and above independent of the discretization. This observation shows that the classical Schwarz' algorithm can be reduced to a symmetric form on the subspace corresponding to the interface Γ_4, and accelerated by conjugate gradients. At the same time this argument shows that the method proposed by Chan and Resasco is nothing but the classical Schwarz' method accelerated by conjugate gradients. We refer to a joint paper with Widlund [4] for a more detailed discussion of this point.

The condition number of the iteration operator providing an upper bound on the number of conjugate gradient iterations in order to achieve a given tolerance, is $(1+\mu_1)/(1-\mu_1)$ for the additive method. From 8, it follows that the appropriate eigenvalues are $1-\mu_i^2$ in the multiplicative case. The condition number of the iteration operator is therefore $1/(1-\mu_1^2)$ in this case. It follows that the condition number of the additive method is bounded by 4 times the condition number of the multiplicative method. We conclude that the number of conjugate gradient iterations needed in the additive method in order to achieve a given tolerance, cannot be more than twice the number of iterations when using the multiplicative algorithm.

This result is somewhat negative, showing that in the worst case we would solve a problem using the additive algorithm and two processors in the same time as one processor would run the multiplicative algorithm. Domain decomposition clearly depends on having (many) more than two subdomains in order to be efficient on parallel processor systems. Unfortunately, the analysis presented in this paper does not extend easily to the case of many subdomains.

6. Exact Results for a Model Problem. The results of the previous section are quite general, they do not depend on the shape of the various domains, the particular discretization, or the specific form of the elliptic equation. The actual numerical values of the spectra and in particular, the condition number of the iteration operators can, if necessary, be computed from equation 9.

More details can be given in model cases only. We will in this section derive the exact values for the rectangular model problem. Study of model problems can lead to a better understanding of the influence of the relative sizes of the subdomains on the rate of convergence. We consider the rectangle shown in figure 2. The domain has q interior nodes in the vertical direction, while there are r , $n-l-1$, and l interior nodes in the horizontal direction of Ω_1, Ω_2 and Ω_3 respectively. We use linear triangular elements, this discretization is equivalent with the well known 5-point finite difference stencil. The notation in this section will be consistent with the paper [3]. Let

$$\lambda_{iq} = 4sin(\frac{i\pi}{2(q+1)})^2 , \qquad i = 1, 2, .. \ q ,$$

and

$$a_{iq} = 1 + \lambda_{iq}/2 - (\lambda_{iq} + (\lambda_{iq}/2)^2)^{1/2}.$$

The quantity a_{iq} appears frequently in the analysis of model problems using Fourier analysis. It is bounded by

$$3 - 2\sqrt{2} < a_{iq} < 1.$$

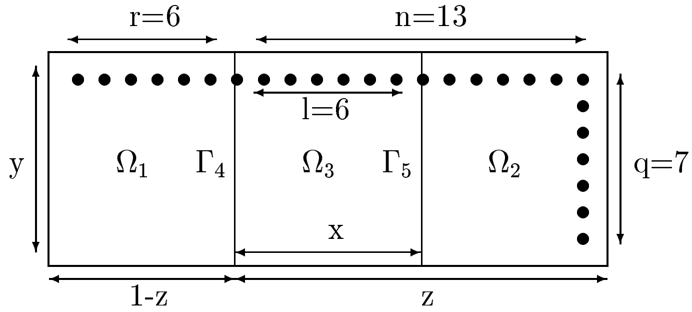

Fig. 2. *The model problem geometry.*

Using a derivation very similar to the one in section 4 of [3], we proceed to diagonalize the eigenvalue problem 9. After a rather tedious calculation one obtains the following exact formula for the eigenvalues μ_i:

$$\mu_i^2 = \frac{a_{iq}^{2r}(a_{iq}^{2n+4} - a_{iq}^{2l+4}) - a_{iq}^{2n+2} + a_{iq}^{2l+2}}{a_{iq}^{2r}(a_{iq}^{2n+2l+6} - a_{iq}^{2l+4}) - a_{iq}^{2n+2} + 1}. \tag{13}$$

We observe that $l = n$ ie. full overlap gives $\mu_i = 0$ and one can derive the upper bound

$$\mu_i \leq \frac{2a_{iq}^2}{(l+1)(1+a_{iq})(1-a_{iq})}.$$

The bound therefore decreases with the size of the overlap as $1/(l+1)$, where l is the number of interior nodes in the x-direction, inside the overlap subdomain Ω_3.

We can also compute the eigenvalues μ_i^2 in the continuous limit as the discretization parameter $h = 0$. We scale the rectangle to unit length and introduce the relative sizes x, y, z as indicated in figure 2. The eigenvalues are

$$\mu_i^2 = \frac{(e^{\alpha_i z} - e^{\alpha_i})(e^{\alpha_i z} + e^{\alpha_i})(e^{\alpha_i z} - e^{\alpha_i x})(e^{\alpha_i z} + e^{\alpha_i x})}{(e^{\alpha_i z} - 1)(e^{\alpha_i z} + 1)(e^{\alpha_i z} - e^{\alpha_i (x+1)})(e^{\alpha_i z} + e^{\alpha_i (x+1)})} \tag{14}$$

where we write $\alpha_i = i\pi/y$. It turns out that 14 is a very good approximation to 13 for almost all interesting cases. The example in figure 2 has $\mu_1 = 0.06$, while an overlap corresponding to only one grid line in the figure, increases μ_1 to 0.67 .

7. Numerical Examples. We give a few numerical results to confirm the theoretical results and to give the reader an idea of the performance of the algorithms. For simplicity, Ω is the union of two rectangles.

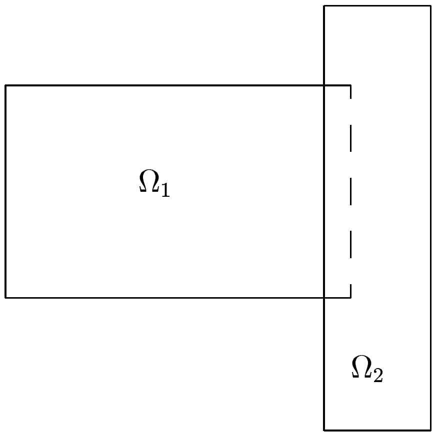

Fig. 3. *The computational domain.*

We use a uniform mesh spacing $h = 1/256$ with the size of Ω_1 being .75 by .5, and $\Omega^{(2)}$ extending 3/16 above and 5/16 below Ω_1, see figure 2. This gives a total of 40449 unknowns, with 127 along Γ_4. We consider the same problem as in [3] with the exact solution $u(x, y) = x^2 + y^2 - xe^x \cos y$. The number of iterations required to reduce the initial error to the truncation error level (a factor of $3 \cdot 10^{-5}$) are reported both for the classical Schwarz' method and for the additive version. The conjugate gradient method is used in both cases. We create the overlap Ω_3 by extending the rectangle Ω_1 a distance x into Ω_2. Table 1 gives results for six different sizes of Ω_3. The first column is the overlap length x. Column 2 shows the largest eigenvalue μ_1, the next two columns give the condition number and the number of iterations N_m, for the multiplicative method. Column 5 and 6 have the same information for the additive method. The last column lists μ_1^R in the continuous limit, for the rectangular model case just studied, with $y = .5$ and $\Omega^{(2)}$ being an extension of Ω_1 of length .25.

x	μ_1	$\frac{1}{1-\mu_1^2}$	N_m	$\frac{1+\mu_1}{1-\mu_1}$	N_a	μ_1^R
1/128	0.95	10.70	17	40.75	33	0.95
1/64	0.91	5.61	13	20.41	24	0.90
1/32	0.82	3.08	9	10.24	17	0.81
1/16	0.68	1.85	7	5.19	11	0.66
1/8	0.47	1.29	6	2.80	9	0.41
1/4	0.37	1.16	5	2.19	7	0.00

Table 1. *Multiplicative and additive Schwarz.*

We note that the estimated eigenvalues from our conjugate gradient iteration, behaves exactly as predicted by the theory. The numerical results demonstrate that the condition number of the additive method approaches four times the size of the multiplicative condition number, as the overlap tends to zero. The actual number of iterations required by the additive method, is also two times what is needed for the multiplicative method when the overlap is very small. The iteration counts of the two methods are closer in the case of a little larger, perhaps more realistic overlap. Finally, observe that the exact eigenvalues for the rectangular model case in the continuous limit, are quite close to the actual eigenvalues for this T-shaped region, except for the special case when the model problem has complete overlap.

8. Acknowledgments. The author thanks professor Olof Widlund for many valuable discussions related to the present work.

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