A Coarse Space Formulation with good parallel properties for an Additive Schwarz Domain Decomposition Algorithm

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Dedicated to Olof Widlund
on the occasion of his 60th birthday

Abstract

This paper studies a variant of the Additive Average Schwarz algorithm [1] where the coarse space consists of two subspaces. This approach results in a simpler and more parallel algorithm while we retain the essential convergence properties of the original method. Our theory is confirmed by numerical experiments showing that the new algorithm is often superior to the original variant.

1 Introduction

In [1] two additive Schwarz algorithms were described and analyzed. The first, called the Additive Diagonal Scaling method, used a diagonal scaling derived from a coarse space V_{-1} (see below), combined with the classical coarse space V_0^c. This method was not robust when the jumps in the coefficients of the PDE could be arbitrarily distributed across subdomains. The second method, called the Additive Average Schwarz method, was proven robust, but uses a potentially expensive coarse space

$$V_0^{avg} = \text{Range}(I_A)$$

with

$$I_A u = \begin{cases} u(x), & x \in \partial\Omega_{ih} \\ \bar{u}_i, & x \in \Omega_{ih}, \end{cases}$$

and

$$\bar{u}_i = \frac{1}{n_i} \sum_{x \in \partial\Omega_{ih}} u(x).$$

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Here Ω_{ih} denotes the set of all nodal points in a subdomain Ω_i and n_i is the number of nodal points $\partial\Omega_{ih}$ on its boundary. In [1], this problem was partially circumvented by considering an approximate bilinear form for the coarse space. In that way, a fairly efficient algorithm for the coarse space problem could be devised, at the expense of a higher overall iteration count for the algorithm.

In this paper we address the same problem by splitting the coarse space used in the Additive Average Method into two pieces

$$V_0^{Avg} = V_{-1} + V_0$$

that can be treated independently. The first piece is the diagonal scaling considered previously [1], that is, the restriction of our discrete space V^h to $\Gamma = \cup_i \partial\Omega_i$. Functions in V_{-1} vanish at all nodal points outside of the internal substructure boundaries Γ. The second part, V_0, is constructed from the (discontinuous) coefficients ρ_i, of the PDE in a way to be described below. The variational form of our equation is defined using the bilinear form

$$a(u,v) = \sum_{i=1}^{N} a_i(u,v), \qquad a_i(u,v) = \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \tag{1}$$

where ρ_i are positive constants. Figure 1 shows a two dimensional example with

$\mathbf{\Omega_1}$ $\rho_1 = \mathbf{1}$	$\mathbf{\Omega_2}$ $\rho_2 = \mathbf{1}$	$\mathbf{\Omega_3}$ $\rho_3 = \mathbf{4}$
$\mathbf{\Omega_4}$ $\rho_4 = \mathbf{9}$	$\mathbf{\Omega_5}$ $\rho_5 = \mathbf{1}$	$\mathbf{\Omega_6}$ $\rho_6 = \mathbf{9}$
$\mathbf{\Omega_7}$ $\rho_7 = \mathbf{4}$	$\mathbf{\Omega_8}$ $\rho_8 = \mathbf{4}$	$\mathbf{\Omega_9}$ $\rho_9 = \mathbf{16}$

Figure 1: A domain with ρ_i specified in 9 subdomains

nine subdomains. We assume that our subdomains $\Omega_i \subset R^3$ are polygonal and have diameters H_i and that our element mesh has discretization parameter h_i. We are interested in discrete solutions of our variational equation in the space $V^h \subset H_0^1(\Omega)$ of piecewise linear, continuous finite element functions where Ω is the union of Ω_i. All subspaces V_i, corresponding to the individual subdomains Ω_i, $i = 1, \ldots, N$ are defined by $V_i = H_0^1(\Omega_i) \cap V^h$ and zero outside of Ω_i.

2 The New Coarse Space

Let the set of substructures Ω_i, $i = 1, \ldots, N$ be partitioned into two sets N_I and N_B, where subdomains belonging to N_B intersect the boundary $\partial\Omega$ in at least one point, while the substructures in N_I are strictly in the interior of our domain Ω. We further partition N_B into three sets

$$N_B = N_{BF} + N_{BE} + N_{BV},$$

where N_{BF} is the set of subdomains that share at least one face with $\partial\Omega$, while N_{BE} and N_{BV} contain the domains that only share at least one edge or one vertex with $\partial\Omega$ respectively. We will also use the notation $N = N_I \cup N_B$ to denote the complete set.

We define V_0 by its basis

$$V_0 = \text{span}\,\{\phi_i\}_{i \in N}. \tag{2}$$

Here the continuous, piecewise linear (basis) function ϕ_i is associated with subdomain Ω_i, but defined at all nodal points in the entire domain as

$$\phi_i(x) = \begin{cases} \frac{\rho_i^{1/2}(x)}{\sum_j \rho_j^{1/2}(x)} & x \in \bar{\Omega}_{ih} \setminus \partial\Omega \\ 0 & x \notin \bar{\Omega}_{ih} \\ 0 & x \in \partial\Omega. \end{cases} \tag{3}$$

In (3), we have defined the function $\rho_i(x) \in V^h$ by

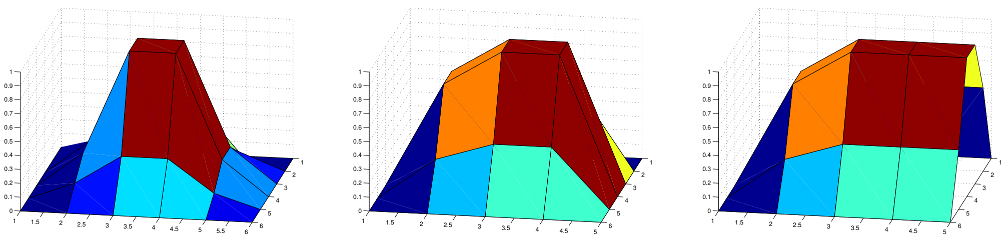

Figure 2: From left to right: the functions ϕ_5 and ϕ_6 and a modified function ϕ_6^{BF}. The functions corresponds to the subdomains and coefficients given in Figure 1. Note that $\Omega_5 \in N_I$ while $\Omega_6 \in N_B$.

$$\rho_i(x) = \begin{cases} \rho_i & x \in \bar{\Omega}_{ih} \\ 0 & x \notin \bar{\Omega}_{ih}. \end{cases} \tag{4}$$

Also, the sum $\sum_j \rho_j^{1/2}(x)$ in (3) shall extend over all $\bar{\Omega}_j$ where $x \in \bar{\Omega}_{ih}$ is a common node. We note that $\phi_i(x) = 1$ at all interior nodal points of Ω_i. In addition, $\sum_j \phi_j(x) = 1$ on all interior interface nodes $x \in \partial\bar{\Omega}_{ih} \setminus \partial\Omega$ when the sum extends over all subdomains $\bar{\Omega}_{jh}$ that share x as a common node. This property is important for our analysis.

A somewhat similar coarse space defined in terms of discrete harmonic functions v_i with data on $\partial\Omega_i$ equal to our $\phi_i(x)$ was used in [3] for an analysis of the Neumann-Neumann algorithm.

3 Analysis

With our new choice of subspaces we can take the bilinear forms $b_i(u,v) : V_i \times V_i \to R,\ i = -1, 0, 1, \ldots, N$ all equal to the form $a(u,v)$ defined by (1). Similarly, our linear projection operators $T_i : V^h \to V_i,\ i = -1, 0, 1, \ldots, N$ are defined by the corresponding bilinear forms in the standard way

$$b_i(T_i u, v) = a(u,v) \quad \forall v \in V_i,\ i = -1, 0, 1, \ldots, N.$$

Let $T = \sum_{i=-1}^{N} T_i$.

Theorem 1 The operator T is self adjoint, positive definite in the scalar product $a(\cdot,\cdot)$ and for any $u \in V^h$ we have the estimate:

$$\gamma_0 \delta^{-1} a(u,u) \le a(Tu,u) \le \gamma_1 a(u,u) \tag{5}$$

Here γ_0 and γ_1 are positive constants independent of h_i, H_i and of the jumps in ρ_i. The only mesh dependent parameter is $\delta = \max_i (H_i/h_i)$.

In a scalable parallel algorithm the ratio δ can often be kept near constant as a finer discretization will use a larger number of substructures.

In order to prove Theorem 1 we apply the general Schwarz framework as described in Chapter 5 of [5]. Three key assumptions must be verified. Referring to the notation of [5], Assumption 2 clearly has $\rho(\mathcal{E}) = 1$ when V_0 and V_{-1} are treated as two coarse spaces. Similarly, Assumption 3 has $\omega = 1$ since we use the original bilinear form $a(u,v)$ everywhere.

What remains to consider is the stability, that is,

$$\sum_{i=-1}^{N} b_i(u_i, u_i) \le c\delta a(u,u),$$

of the crucial decomposition $u = \sum_{i=-1}^{N} u_i$, with $u_i \in V_i$ for any $u \in V^h$ that is required by Assumption 1.

For any $u \in V^h$ let $u_0 \in V_0$ be defined by

$$u_0 = \sum_{i \in N} \bar{u}_i \phi_i \quad \text{where} \quad \bar{u}_i = \frac{1}{|\Omega_i|} \int_{\Omega_i} u dx. \tag{6}$$

We use the notation $|\Omega_i|$ to denote the volume of subdomain Ω_i. We next state a lemma, but for clarity of presentation defer its proof until the proof of Theorem 1 has been completed.

Lemma 1 For any $u \in V^h$ we have

$$a(u_0, u_0) \leq c\delta a(u, u) \tag{7}$$

and

$$\sum_{i=1}^{N} \rho_i ||\bar{u}_i - u_0||^2_{L^2(\Omega_i)} \leq c \sum_{i=1}^{N} \rho_i h_i H_i |u|^2_{H^1(\Omega_i)}. \tag{8}$$

Here c is independent of the mesh parameters and of the jumps of ρ_i and δ is defined above.

We now proceed with Assumption 1 in order to complete the proof of Theorem 1. Let $w = u - u_0$ and define u_{-1} by

$$u_{-1} = \begin{cases} w & \text{on } \partial\Omega_i \;\; i = 1, \ldots N \\ 0 & \text{at interior nodal points in each } \Omega_i. \end{cases} \tag{9}$$

Clearly, $u_{-1} \in V_{-1}$.

Similarly, we define the functions $u_i \in V_i$, $i = 1, \ldots, N$ on each substructure

$$u_i = \begin{cases} w & \text{at interior nodal points in } \;\; \Omega_i \\ 0 & \text{at } \;\partial\Omega_i \\ 0 & \text{outside } \;\Omega_i. \end{cases}$$

Again, we see that this construction ensures that $u_i \in V_i$ and that

$$u = \sum_{i=-1}^{N} u_i.$$

We now proceed to estimate the energy of u_{-1}, u_0 and u_i, $i \geq 1$, separately. By Lemma 1

$$b_0(u_0, u_0) \leq c\delta a(u, u), \text{ since } b_0(u, u) = a(u, u).$$

Next, consider u_{-1}. By the inverse inequality we have

$$b_{-1}(u_{-1}, u_{-1}) \leq c \sum_{i=1}^{N} \rho_i h_i^{-2} ||u_{-1}||^2_{L^2(\Omega_i)}. \tag{10}$$

Observing the definition (9) of u_{-1} and using the equivalence between the L^2 norm and the corresponding discrete norm we have

$$||u_{-1}||^2_{L^2(\Omega_i)} \leq ch_i^3 \sum_{x \in \bar{\Omega}_{ih}} u^2_{-1}(x) = ch_i^3 \sum_{x \in \partial\Omega_{ih}} w^2(x) \leq ch_i ||w||^2_{L^2(\partial\Omega_i)}. \tag{11}$$

Since $w = u - u_0$ we can write $w = u - \bar{u}_i + \bar{u}_i - u_0$ and use the triangle inequality to get

$$||w||^2_{L^2(\partial\Omega_i)} \leq 2||u - \bar{u}_i||^2_{L^2(\partial\Omega_i)} + 2||\bar{u}_i - u_0||^2_{L^2(\partial\Omega_i)}. \tag{12}$$

A standard trace inequality [4] applied to the first term gives

$$\begin{aligned}||u-\bar{u}_i||^2_{L^2(\partial\Omega_i)} &\leq c(H_i|u-\bar{u}_i|^2_{H^1(\Omega_i)} + H_i^{-1}||u-\bar{u}_i||^2_{L^2(\Omega_i)}) \\ &\leq cH_i|u|^2_{H^1(\Omega_i)},\end{aligned} \tag{13}$$

where we have used Poincare's inequality on the second term of the trace inequality.

The second term in (12) can be changed to the L^2 norm over all of Ω_i by using the discrete norm equivalence at a cost of one factor h_i^{-1}. With this observation and the results of (11), (12), and (13) inserted into (10) we obtain

$$b_{-1}(u_{-1},u_{-1}) \leq c\sum_{i=1}^N \rho_i(\frac{H_i}{h_i}|u|^2_{H^1(\Omega_i)} + h_i^{-2}||\bar{u}_i-u_0||^2_{L^2(\Omega_i)}).$$

We now apply Lemma 1 to the last term and obtain

$$b_{-1}(u_{-1},u_{-1}) \leq c\sum_{i=1}^N \rho_i\frac{H_i}{h_i}|u|^2_{H^1(\Omega_i)} = c\delta a(u,u).$$

It remains to estimate the contribution from u_i, $i=1,\ldots,N$. We can write $u_i = w - u_{-1} = (u-u_0) - u_{-1}$ everywhere in $\bar{\Omega}_i$. Therefore

$$\sum_{i=1}^N b_i(u_i,u_i) \leq 2a(w,w) + 2a(u_{-1},u_{-1}).$$

The second term has already been estimated and the first term can further be decomposed and estimated by

$$a(w,w) \leq 2a(u,u) + 2a(u_0,u_0) \leq c\delta a(u,u)$$

since we already have considered $a(u_0,u_0)$. This shows that

$$\sum_{i=1}^N b_i(u_i,u_i) \leq c\delta a(u,u).$$

Adding the three contributions u_{-1}, u_0 and u_i, $i=1,\ldots,N$, finally gives us

$$\sum_{i=-1}^N b_i(u_i,u_i) \leq c\delta a(u,u)$$

verifying Assumption 1 and therefore concluding the proof of Theorem 1.

We now proceed with the proof of Lemma 1. We first consider, separately, two different cases depending on the location of the subdomain Ω_i. In the first case Ω_i is in N_I while the second case has Ω_i belonging to N_B.

When Ω_i is in N_I, we estimate $|u_0|^2_{H^1(\Omega_i)}$ by adding a constant since this does not change the H^1 seminorm

$$\rho_i|u_0|^2_{H^1(\Omega_i)} = \rho_i|u_0 - \bar{u}_i|^2_{H^1(\Omega_i)}.$$

From the definition of u_0 in (6) it follows that $u_0 = \bar{u}_i\phi_i + \sum_j \bar{u}_j\phi_j$, on $\bar{\Omega}_i$ with j running over domains $\bar{\Omega}_j$ that share at least one node with $\bar{\Omega}_i$. Since $\phi_i + \sum_j \phi_j = 1$ on all of $\bar{\Omega}_i$ it follows that

$$u_0 - \bar{u}_i = \sum_j (\bar{u}_j - \bar{u}_i)\phi_j.$$

Using this, we readily have

$$\begin{aligned} \rho_i|u_0 - \bar{u}_i|^2_{H^1(\Omega_i)} &= \rho_i|\sum_j (\bar{u}_j - \bar{u}_i)\phi_j|^2_{H^1(\Omega_i)} \\ &\leq c\rho_i \sum_j (\bar{u}_i - \bar{u}_j)^2 |\phi_j|^2_{H^1(\Omega_i)}. \end{aligned} \tag{14}$$

Let G_{ij} be an intersection of $\bar{\Omega}_i$ and $\bar{\Omega}_j$. This can be i) a face, ii) an edge, or iii) a common vertex. Let us first consider case i), that is, $G_{ij} = F_{ij}$, a common face. We proceed to estimate the two factors $(\bar{u}_i - \bar{u}_j)^2$ and $|\phi_j|^2_{H^1(\Omega_i)}$ separately. The first term is

$$(\bar{u}_i - \bar{u}_j)^2 = ((\bar{u}_i - \bar{u}_{F_{ij}}) - (\bar{u}_j - \bar{u}_{F_{ij}}))^2, \tag{15}$$

where

$$\bar{u}_{F_{ij}} = \frac{1}{|F_{ij}|}\int_{F_{ij}} u dx.$$

We also note that

$$\bar{u}_i - \bar{u}_{F_{ij}} = -\overline{(u - \bar{u}_i)}_{F_{ij}}.$$

Therefore

$$\begin{aligned} \overline{(u - \bar{u}_i)}^2_{F_{ij}} &= (\frac{1}{|F_{ij}|}\int_{F_{ij}} (u - \bar{u}_i) dx)^2 \\ &\leq cH_i^{-4}H_i^2||u - \bar{u}_i||^2_{L^2(F_{ij})} \\ &\leq cH_i^{-2}(H_i|u - \bar{u}_i|^2_{H^1(\Omega_i)} + H_i^{-1}||u - \bar{u}_i||^2_{L^2(\Omega_i)}) \\ &\leq cH_i^{-1}|u|^2_{H^1(\Omega_i)}. \end{aligned} \tag{16}$$

Here we have used the trace inequality followed by the Poincare inequality. The second term of (15) can be estimated in exactly the same way. Using these inequalities, it is now easy to see that our first term can be estimated by

$$(\bar{u}_i - \bar{u}_j)^2 \leq cH_i^{-1}|u|^2_{H^1(\Omega_i)} + cH_j^{-1}|u|^2_{H^1(\Omega_j)}. \tag{17}$$

The second factor can be estimated as follows

$$\begin{aligned}
\rho_i|\phi_j|^2_{H^1(\Omega_i)} &\leq c\rho_i h_i^{-2}||\phi_j||^2_{L^2(\Omega_i)} \\
&\leq c\rho_i h_i^{-2} h_i^3 \textstyle\sum_{x\in\bar{\Omega}_{ih}} \phi_j^2(x) \\
&\leq c\rho_i h_i \textstyle\sum_{x\in\bar{F}_{ijh}} \phi_j^2(x) \qquad (18)\\
&\leq c\rho_i h_i (\tfrac{H_i}{h_i})^2 \\
&= c\rho_i \tfrac{H_i^2}{h_i},
\end{aligned}$$

where we have used the inverse inequality, the discrete L^2 norm (keeping in mind that $\phi_{jh}(x) = 0$ on all nodes in $\bar{\Omega}_i$ except on $\bar{F}_{ij}$), $\phi_{jh}(x) \leq 1$ on $\bar{F}_{ij}$, and that the number of nodes on $\bar{F}_{ij}$ is of order $(\frac{H_i}{h_i})^2$. Since the definition (3) of ϕ_i implies that $\rho_i\phi_j^2 = \rho_j\phi_i^2$ and since $h_i = h_j$ on the common face F_{ij}, we also have

$$\rho_i|\phi_j|^2_{H^1(\Omega_i)} \leq c\rho_j \frac{H_j^2}{h_j}. \qquad (19)$$

Combining (17) with (18) and (19)

$$\begin{aligned}
\rho_i(\bar{u}_i - \bar{u}_j)^2|\phi_j|^2_{H^1(\Omega_i)} &\leq c\rho_i \frac{H_i}{h_i}|u|^2_{H^1(\Omega_i)} + c\rho_j \frac{H_j}{h_j}|u|^2_{H^1(\Omega_j)} \\
&= c(\frac{H_i}{h_i}a_i(u,u) + \frac{H_j}{h_j}a_j(u,u)). \qquad (20)
\end{aligned}$$

Next, we must show that (20) also holds for the other cases, that is, we must derive estimates similar to (17), (18) and (19). In the case where $G_{ij} = E_{ij}$, a common edge between Ω_i and Ω_j, we proceed as in the previous case. We estimate

$$\bar{u}_i - \bar{u}_{E_{ij}} = -\overline{(u - \bar{u}_i)}_{E_{ij}},$$

where

$$\bar{u}_{E_{ij}} = \frac{1}{|E_{ij}|}\int_{E_{ij}} u dx.$$

The argument is similar to the development from (15) to (16), we note that

$$\begin{aligned}
\overline{(u - \bar{u}_i)}^2_{E_{ij}} &\leq cH_i^{-2}H_i||u - \bar{u}_i||^2_{L^2(E_{ij})} \\
&\leq cH_i^{-1}h_i^{-1}||u - \bar{u}_i||^2_{L^2(\partial\Omega_i)} \\
&\leq cH_i^{-1}h_i^{-1}(H_i|u - \bar{u}_i|^2_{H^1(\Omega_i)} + H_i^{-1}||u - \bar{u}_i||^2_{L^2(\Omega_i)}) \\
&\leq ch_i^{-1}|u|_{H^1(\Omega_i)}, \qquad (21)
\end{aligned}$$

where we have used Cauchy-Schwartz, changed from $L^2(E_{ij})$ to $L^2(\partial\Omega_i)$, at the expense of a factor h_i^{-1}. The inequality is completed by use of the trace inequality and Poincare's inequality.

The second factor is estimated as in (18) where we now have $O(H_i/h_i)$ nodes along the edge E_{ij}.

$$
\begin{aligned}
\rho_i|\phi_j|^2_{H^1(\Omega_i)} &\leq c\rho_i h_i^{-2}||\phi_j||^2_{L^2(\Omega_i)} \\
&\leq c\rho_i h_i^{-2} h_i^3 \textstyle\sum_{x\in\bar{\Omega}_{ih}} \phi^2_{jh}(x) \\
&\leq c\rho_i h_i \textstyle\sum_{x\in\bar{E}_{ijh}} \phi^2_{jh}(x) \qquad (22)\\
&\leq c\rho_i h_i (\tfrac{H_i}{h_i}) \\
&= c\rho_i H_i.
\end{aligned}
$$

Similarly, we also have

$$\rho_i|\phi_j|^2_{H^1(\Omega_i)} \leq c\rho_j H_j \qquad (23)$$

by the same argument that led to (19). Combining (21) with (22) and (23) verifies (20).

In our third case $G_{ij} = \nu_{ij}$, a common vertex of $\bar{\Omega}_i$ and $\bar{\Omega}_j$. Note that

$$
\begin{aligned}
(u(\nu_{ij}) - \bar{u}_i)^2 &\leq ch_i^{-2}||u-\bar{u}_i||^2_{L^2(\partial\Omega_i)} \\
&\leq ch_i^{-2}(H_i|u-\bar{u}_i|^2_{H^1(\Omega_i)} + H_i^{-1}||u-\bar{u}_i||^2_{L^2(\Omega_i)}) \\
&\leq cH_i h_i^{-2}|u|^2_{H^1(\Omega_i)}.
\end{aligned}
\qquad (24)
$$

For the second term we have

$$
\begin{aligned}
\rho_i|\phi_j|^2_{H^1(\Omega_i)} &\leq c\rho_i h_i^{-2}||\phi_j||^2_{L^2(\Omega_i)} \\
&\leq c\rho_i h_i^{-2} h_i^3 \textstyle\sum_{x\in\bar{\Omega}_{ih}} \phi^2_{jh}(x) \qquad (25)\\
&\leq c\rho_i h_i \phi_j^2(\nu_{ij}) \\
&= c\rho_i h_i.
\end{aligned}
$$

Similarly, we again have

$$\rho_i|\phi_j|^2_{H^1(\Omega_i)} \leq c\rho_j h_j \qquad (26)$$

by the same argument that led to (19). The estimates (24), (25), and (26) show that (20) holds in this case as well.

We can now substitute (20) into (14) and obtain

$$\rho_i|u_0|^2_{H^1(\Omega_i)} \leq c(\frac{H_i}{h_i}a_i(u,u) + \sum_j \frac{H_j}{h_j}a_j(u,u)), \qquad (27)$$

where the sum is taken over $\bar{\Omega}_j$ which intersect $\bar{\Omega}_i$. This concludes our study of the first case where $\Omega_i \in N_I$.

We next consider the case where Ω_i belongs to the set N_B. In this case on Ω_i

$$u_0 = \bar{u}_i\phi_i + \sum_j \bar{u}_j\phi_j, \qquad (28)$$

where, as usual, the sum is over $\bar{\Omega}_j$ that intersects $\bar{\Omega}_i$. We write $u_0 = u_0^{(1)} + u_0^{(2)}$ where

$$u_0^{(1)} = \bar{u}_i \tilde{\phi}_i + \sum_{j \in N_I} \bar{u}_j \phi_j + \sum_{j \in N_B} \bar{u}_j \tilde{\phi}_j \tag{29}$$

and

$$u_0^{(2)} = \bar{u}_i(\phi_i - \tilde{\phi}_i) + \sum_{j \in N_B} \bar{u}_j(\phi_j - \tilde{\phi}_j). \tag{30}$$

Here we have introduced modified functions $\tilde{\phi}_i$ in the domains Ω_i belonging to N_B. These functions are defined almost like the basis functions ϕ_i, the only difference being the removal of the third condition in (3), that is, they are nonzero on the boundary $\partial\Omega$. More precisely, we denote $\tilde{\phi}_i$ by ϕ_i^{BF}, ϕ_i^{BE}, and ϕ_i^{BV} when it is nonzero on at least one face, edge, or vertex belonging to $\partial\Omega$, respectively. In Figure 2 we show both ϕ_6 and the modified function ϕ_6^{BF} for the two dimensional example in Figure 1.

By way of this construction the first part, $u_0^{(1)}$, now has the same form as u_0 on Ω_i when $\Omega_i \in N_I$, hence our estimate (27) from the previous case is valid. We therefore proceed to estimate each term in $u_0^{(2)}$. In order to do this, we must consider three different subcases depending on whether Ω_i or Ω_j are in the sets N_{BF}, N_{BE}, or N_{BV}.

If $i \in N_{BF}$, that is, $\tilde{\phi}_i = \phi_i^{BF}$, then

$$\bar{u}_i^2 \le \frac{c}{H_i^3} ||u||^2_{L^2(\Omega_i)} \le \frac{c}{H_i} |u|^2_{H^1(\Omega_i)} \tag{31}$$

by Friedrichs' inequality, observing that u is zero on a face of Ω_i. We note that the other factor $\phi_i - \phi_i^{BF}$ is nonzero on this face and (see also (18))

$$|\phi_i - \phi_i^{BF}|^2_{H^1(\Omega_i)} \le c \frac{H_i^2}{h_i}.$$

Therefore

$$\rho_i |\bar{u}_i(\phi_i - \phi_i^{BF})|^2_{H^1(\Omega_i)} \le c \frac{H_i}{h_i} a_i(u,u). \tag{32}$$

In a similar way we estimate the terms $\bar{u}_j(\phi_j - \phi_j^{BF})$ in (30).

Next, if Ω_i belongs to N_{BE}, that is, $\tilde{\phi}_i = \phi_i^{BE}$, we have a slightly weaker estimate than (31) for the term $\bar{u}_i$

$$\bar{u}_i^2 \le \frac{c(1 + \log \frac{H_i}{h_i})}{H_i} |u|^2_{H^1(\Omega_i)} \tag{33}$$

by Lemma 6 in [3]. The difference $(\phi_i - \phi_i^{BE})$ which is nonzero only on an edge contributes a factor H_i (see (22)), so that

$$\rho_i |\bar{u}_i(\phi_i - \phi_i^{BE})|^2_{H^1(\Omega_i)} \le c \frac{H_i}{h_i} a_i(u,u),$$

where we have replaced $(1+\log(H_i/h_i)$ by its upper bound H_i/h_i. We again estimate the terms $\bar{u}_j(\phi_j - \phi_j^{BE})$ from (30) similarly.

Finally, if Ω_i belongs to N_{BV}, that is, $\tilde{\phi}_i = \phi_i^{BV}$ we have

$$\bar{u}_i^2 \le c\frac{H_i}{h_i}|u|^2_{H^1(\Omega_i)}$$

by Lemma 2.3 in [2]. The difference $(\phi_i - \phi_i^{BV})$ which is nonzero at the boundary vertex contributes a factor h_i and we obtain

$$\rho_i|\bar{u}_i(\phi_i - \phi_i^{BV})|^2_{H^1(\Omega_i)} \le cH_i a_i(u,u).$$

Again, we estimate the terms $\bar{u}_j(\phi_j - \phi_j^{BV})$ from (30) in a similar way. From this analysis it is clear that the inequality (27) also holds for $u_0^{(2)}$ in (30) and therefore that this estimate is valid for u_0 in Ω_i when $\Omega_i \in N_B$.

Now that (27) has been verified in both cases we obtain (7), the first part of Lemma 1, by summation with respect to i across the inequality. We take $\delta = \max_i(H_i/h_i)$ outside and remember that the sum with respect to j on the right side of (27) only runs over the neighbors of $\bar{\Omega}_i$. Assuming that the maximal number of neighbors of a single subdomain $\bar{\Omega}_i$ is fixed and independent of the total number of subdomains, N, completes the proof of the first part of Lemma 1.

The second part of Lemma 1 follows from the proof of the first part. We have

$$\rho_i||\bar{u}_i - u_0||^2_{L^2(\Omega_i)} \le c\rho_i \sum_j (\bar{u}_i - \bar{u}_j)^2||\phi_j||^2_{L^2(\Omega_i)}, \tag{34}$$

where we have replaced the H^1-norm with the L_2-norm in (14). As can be seen from for example (18) our estimates for $|\phi_j|^2_{L^2(\Omega_i)}$ will gain a factor h_i^2 compared with $|\phi_j|^2_{H^1(\Omega_i)}$. It can be easily verified that all cases behave similarly and that estimate (20) therefore takes the form

$$\rho_i(\bar{u}_i - \bar{u}_j)^2|\phi_j|^2_{L^2(\Omega_i)} \le c(h_iH_ia_i(u,u) + h_jH_ja_j(u,u)).$$

Substitution of this into (34) leads to

$$\rho_i||\bar{u}_i - u_0||^2_{L^2(\Omega_i)} \le c(h_iH_ia_i(u,u) + \sum_j h_jH_ja_j(u,u)).$$

A final summation of this estimate over the index i gives the second statement of Lemma 1.

Remark: If we define our coarse space V_0 to be spanned by ϕ_i for $i \in (N_I \cup N_{BE} \cup N_{BV})$ only (that is, we exclude subdomains that share a face with the external boundary $\partial\Omega$) then both Lemma 1 and Theorem 1 still holds, but we introduce a factor $2H_i$ since the effective diameter H_i in the proof now extends across two subdomains. This is clearly seen in a regular case and effectively increases the condition number by a factor two.

If we restrict the coarse space even further by only using a basis ϕ_i derived from the interior subdomains, that is, with $i \in N_I$ and there exist subdomains Ω_i with $i \in (N_{BE} \cup N_{BV})$, then the estimate (7) for $a(u_0, u_0)$ in Lemma 1 will have

$$\delta = max_i \frac{H_i}{h_i}(\frac{H_i^2}{h_i} + \log \frac{H_i}{h_i}),$$

this follows by comparing with our analysis of the case when $\Omega_i \in N_B$. The factor $\log(H_i/h_i)$ will arise when $\Omega_i \in N_I$, but a neighbor $\Omega_j \in N_{BE}$ while the factor H_i^2/h_i arises when the neighbor $\Omega_j \in N_{BV}$.

4 Numerical Results

The algorithm is currently being implemented in both two and three dimensions as part of the European ESPRIT project PARASOL, a common library of parallel algorithms for sparse linear systems. Several refinements of the algorithm will be needed in practice. For example, it is possible to use a more efficient inexact solver for the problem on the coarse space V_{-1}. This effort and a more detailed description of the parallel implementation will be reported elsewhere. Here we focus on some simple numerical tests in order to compare this algorithm with the previous variant called Additive Average Schwarz.

In our numerical experiments we have compared the method proposed in this paper with two alternative implementations described in [1]. The methods differ in the construction of the coarse space solver. In order to solve a problem corresponding to the coarse space V_0^{Avg} efficiently the bilinear form $a(u, v)$ defined in (1) was replaced by the approximation

$$b(u, v) = \sum_i \rho_i h_i^{d-2} \sum_{x \in \partial\Omega_{ih}} (u(x) - \bar{u}_i)(v(x) - \bar{v}_i). \tag{35}$$

The second alternative is to use the correct bilinear form from (1). This is not practical in actual computations due to the large coarse space problem that results, but has been included here to show the best results one can obtain with methods using this particular coarse space construction.

In Table 1, we compare three different algorithms on a simple test problem in two dimensions. Our domain is the unit square and in the case when $\rho_i = 1$ (a Poisson equation), we take the exact solution to be $u(x, y) = \sin(\omega\pi x)\sin(\omega\pi y)$ with $\omega = 1.2$. The ratio $H_i/h_i = 6$ in this example. In the first column, we show the Additive Average Schwarz method using the inexact bilinear form (35). The second column has the same results for the new method described in this paper. These results should be compared with column four where we have used the very expensive (and therefore unattractive) Additive Average Schwarz method using the exact bilinear form (1). That is, we use the proper Galerkin projection on the coarse space V_0 as a reference method. This provides a lower bound for the condition number of our iteration operator, since the method analyzed in this paper decomposes V_0 into two parts and adds the corresponding coarse space

Sub - Domains	Approximate bilinear form	Decomposed $V_{-1}+V_0$	Decomposed $V_{-1}+V_0$	Exact bilinear form
$N=$	$\rho_i=1$	$\rho_i=1$	$\rho_i=1,\dots N$	$\rho_i=1$
3×3	28 (20.2)	22 (12.7)	22 (13.1)	22 (11.0)
6×6	28 (20.9)	23 (13.7)	23 (14.2)	22 (11.5)
12×12	28 (21.2)	23 (14.1)	24 (14.8)	21 (11.6)

Table 1: Comparison of three algorithms. A square domain is decomposed into square subdomains each consisting of 72 triangular, linear finite elements. The table shows iteration counts required to reduce the residual by a factor of 10^6 and, in parentheses, an estimate for the condition number of the iteration operator.

corrections in the overall method. We observe that the method analyzed here performs better than the original Additive Average Schwarz method and that the results are indeed very close to the best we could hope for.

The third column of Table 1 verifies that the method is insensitive to jumps in the coefficient ρ_i across subdomain boundaries. We have taken $\rho_i = i$ in Ω_i in this example. We have also verified our remark at the end of the proof computationally. If we take $V_0 = \mathrm{span}\{\phi_i\}_{i\in N_I}$ then we need 37 iterations and estimate the condition number to 35.3 in the case where we have $N = 12\times 12$ above.

Sub - Domains	Approximate bilinear form	Decomposed $V_{-1}+V_0$	Decomposed $V_{-1}+V_0$	Exact bilinear form
$N=$	$\rho_i=1$	$\rho_i=1$	$\rho_i=1,\dots N$	$\rho_i=1$
9	45 (85.6)	26 (19.1)	25 (19.5)	27 (19.9)
36	74 (267.0)	36 (40.1)	35 (40.5)	36 (38.0)
144	116 (579.8)	47 (80.3)	47 (80.2)	48 (71.9)

Table 2: Comparison of three algorithms. The domain used in Table 1 has been deformed and the finite element discretization is now irregular. Iteration count and the corresponding operator condition number estimates are given.

In Table 2, we consider the same methods as in the first table, but now on a finite element problem where the elements has been significantly distorted. The resulting finite element mesh is now irregular. In this case we see an even more pronounced advantage of the new method. The condition number roughly doubles when we increase the number of unknowns in the problem by a factor of four, but it is still very close to the 'best possible' use of this coarse space (as seen in the rightmost column). Our previous method requires about twice as many iterations in this case.

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