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TO OVERLAP OR NOT TO OVERLAP: A NOTE ON A DOMAIN DECOMPOSITION METHOD FOR ELLIPTIC PROBLEMS*

PETTER E. BJØRSTAD† AND OLOF B. WIDLUND‡

Abstract. More than one hundred years ago, H.A. Schwarz introduced a domain decomposition method in which the original elliptic equation is solved on overlapping subregions, one after another, in an iterative process. A few years ago, Chan and Resasco, introduced a method that they classified as a domain decomposition method using nonoverlapping subdomains. In this note, it is shown that their method is an accelerated version of the classical method. It is also shown that the error propagation operator of the method can be expressed in terms of Schur complements of certain stiffness matrices and that techniques previously developed for the study of iterative substructuring algorithms can be used to derive estimates on the rate of convergence.

Key words. Schwarz's method, iterative substructuring methods, domain decomposition, elliptic equations, finite elements.

AMS(MOS) subject classification. 65F10, 65N30.

1. Introduction.

Recently there has been a strong revival of the interest in domain decomposition algorithms for elliptic problems; cf., e.g., Glowinski et al. [18] and Chan et al. [9]. This is to a large extent due to their potential on multiprocessor computing systems. It is therefore not surprising that much of the recent research effort has focused on the case of many subregions; cf., e.g., Bramble et al. [4], [5], [6], [7], [8]; Dryja [14], [15]; Dryja et al. [16], [17]; and Widlund [25]. However, in this note, we regress and consider only the case of two subregions. We show that a method recently developed by Chan and Resasco [11], [12] is identical to the classical Schwarz algorithm, cf. Schwarz [21], accelerated by the standard conjugate gradient method. We note that in a recent technical report, which we received after submitting this paper to SIAM, Chan and Goovaerts [10] give another description of our main result using a linear algebra framework similar to ours.

Domain decomposition algorithms are sometimes classified according to whether overlapping subregions are used or not. This distinction often makes sense. Thus, if overlapping is used, certain variables are associated with several subproblems, and the cost per iteration normally grows with the overlap. A decision of how to balance this effect against an increased rate of convergence has to be made, while for a method that

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† Institutt for Informatikk, University of Bergen, Thormøhlens gate 55, N-5006 Bergen, Norway. This work was supported in part by the Norwegian Research Council for Science and the Humanities under grant D.01.08.054 and by The Royal Norwegian Council for Scientific and Industrial Research under grant IT2.28.28484.

‡ Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012. This work was supported in part by the National Science Foundation under grant NSF-CCR-8703768, in part by the U.S. Department of Energy under contract DE-AC02-76ER03077-V at the Courant Mathematics and Computing Laboratory, and in part by the IBM Bergen Scientific Centre, while the author was a visiting scientist.

does not use overlapping, no such consideration arises. However, we demonstrate that the distinction is not all that clear and that the same technical tools can be used for the analysis of methods that can be regarded as members of the two different classes.

2. The Schwarz method: Several points of view. To simplify our presentation, we assume that the elliptic operator is the Laplacian and that we have a zero Dirichlet condition. Thus,

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The region Ω is bounded, two- or three-dimensional, with a Lipschitz continuous boundary. Our algorithms and results can be extended immediately to any linear, self-adjoint elliptic problem that can be formulated as a minimization problem. We use a variational formulation of the problem, which, as shown by Sobolev [22] more than fifty years ago, makes the maximum principle superfluous.

In variational form (1) is written as

$$a_\Omega(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx = f(v) \;, \qquad \forall\, v \in H_0^1(\Omega) \;,$$

where the solution $u \in H_0^1(\Omega)$, the closure in the Sobolev space $H^1(\Omega)$ of the space of smooth functions that vanish in the neighborhood of $\partial\Omega$. As always, the space $H^1(\Omega)$ is the subspace of $L_2(\Omega)$ for which $\mid u \mid^2_{H^1(\Omega)} = a(u,u)$ is finite. In our analysis we work exclusively with the inner product defined by $a(\cdot,\cdot)$. Thus in this paper orthogonality and symmetry always refer to this inner product.

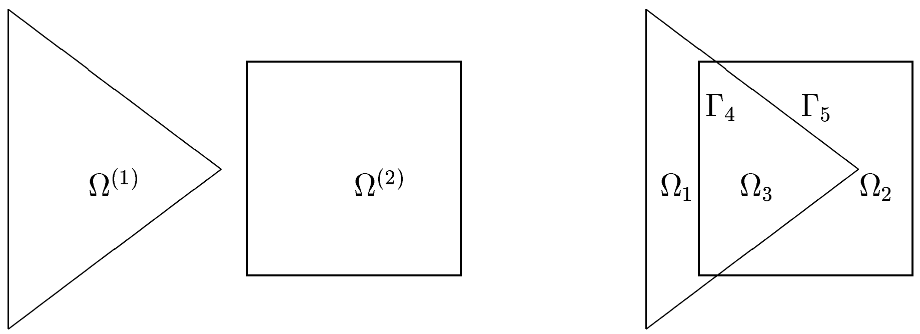

FIG. 1. *The partition of the domain.*

In this note we consider domain decomposition algorithms with two overlapping subregions $\Omega^{(1)}$ and $\Omega^{(2)}$. We also use the notations $\Omega_1 = \Omega \setminus \overline{\Omega}^{(2)}$, where $\overline{\Omega}^{(2)}$ is the closure of $\Omega^{(2)}$, $\Omega_2 = \Omega \setminus \overline{\Omega}^{(1)}$, and $\Omega_3 = \Omega^{(1)} \cap \Omega^{(2)}$. The region Ω is thus also divided into three nonoverlapping subregions Ω_1, Ω_2, and Ω_3, which are separated from each other by the curves (or surfaces) $\Gamma_4 = \overline{\Omega}_1 \cap \overline{\Omega}_3$ and $\Gamma_5 = \overline{\Omega}_2 \cap \overline{\Omega}_3$. In Fig. 1, we display separately the two subregions from which Ω is built on the left; the partitioning of Ω

into the five subsets just defined is given on the right. We assume that Γ_4 and Γ_5 follow element boundaries, that they are Lipschitz, and that they intersect only in at most a few points (or along a few curves).

The problem is discretized by finite elements in the customary fashion; cf. Ciarlet [13]. The region Ω is triangulated, and a conforming finite element space $V^h \subset H_0^1(\Omega)$ is introduced. We assume that all the elements are shape regular, i.e., there is a bound on h_K/ρ_K that is independent of the number of degrees of freedom and of K. Here h_K is the diameter of the element K and ρ_K the diameter of the largest sphere that can be inscribed in K. The approximate solution $u_h \in V^h$ is defined by

$$a_\Omega(u_h, v_h) = f(v_h)\ , \qquad \forall\, v_h \in V^h. \tag{2}$$

We return briefly to the continuous case and write down the Schwarz algorithm in its traditional form. There are two fractional steps. Let u^n be the nth iterate. Then the updated solution u^{n+1} is determined by

$$\begin{aligned} -\Delta u^{n+1/2} &= f && \text{in } \Omega^{(1)}, \\ u^{n+1/2} &= 0 && \text{on } \partial\Omega^{(1)} \cap \partial\Omega, \\ u^{n+1/2} &= u^n && \text{on } \Gamma_5, \end{aligned}$$

and

$$\begin{aligned} -\Delta u^{n+1} &= f && \text{in } \Omega^{(2)}, \\ u^{n+1} &= 0 && \text{on } \partial\Omega^{(2)} \cap \partial\Omega, \\ u^{n+1} &= u^{n+1/2} && \text{on } \Gamma_4. \end{aligned}$$

Following P. L. Lions [19], [20], who recently has obtained a number of interesting results on Schwarz's method, we transform these equations into variational form. This formulation is equally valid in the discrete case. The exact solution of (2) is denoted by u_h.

$$\begin{aligned} a_\Omega(u_h^{n+1/2} - u_h^n, v_h) &= f(v_h) - a_\Omega(u_h^n, v_h) \\ &= a_\Omega(u_h - u_h^n, v_h)\ , && \forall\, v_h \in V^h \cap H_0^1(\Omega^{(1)})\ , \\ a_\Omega(u_h^{n+1} - u_h^{n+1/2}, v_h) &= f(v_h) - a_\Omega(u_h^{n+1/2}, v_h) \\ &= a_\Omega(u_h - u_h^{n+1/2}, v_h)\ , && \forall\, v_h \in V^h \cap H_0^1(\Omega^{(2)})\ . \end{aligned}$$

Here $u_h^{n+1/2} - u_h^n \in H_0^1(\Omega^{(1)})$ and $u_h^{n+1} - u_h^{n+1/2} \in H_0^1(\Omega^{(2)})$, since the boundary values do not change from one fractional step to the next. Following Lions, these equations can conveniently be written as

$$\begin{aligned} u_h^{n+1/2} - u_h^n &= P_1(u_h - u_h^n)\ , \\ u_h^{n+1} - u_h^{n+1/2} &= P_2(u_h - u_h^{n+1/2})\ , \end{aligned}$$

where $P_i\ ,\ i = 1, 2\,$, is the orthogonal projection of V^h into $V_i = V^h \cap H_0^1(\Omega^{(i)})$. The error $e_h^n = u_h^n - u_h$ satisfies the relationship

$$e_h^{n+1} = (I - P_2)(I - P_1)e_h^n = (I - (P_1 + P_2 - P_2P_1))e_h^n.$$

In a certain sense, Schwarz's method is therefore a straightforward iterative method of solving the equation

$$(P_1 + P_2 - P_2 P_1) u_h = g_h,$$

for a certain right-hand side g_h. The operator is not symmetric, but there is an easy remedy, namely, the use of a third fractional step identical to the first. The resulting operator

$$I - (I - P_1)(I - P_2)(I - P_1) = P_1 + P_2 - P_1 P_2 - P_2 P_1 + P_1 P_2 P_1$$

is symmetric with respect to the energy inner product.

The method can be accelerated, e.g., with the conjugate gradient method. We can equally well regard the third fractional step as the first step of the next iteration, and the extra cost involved can therefore quickly be amortized. Since it can be shown that the spectra of $(I - P_2)(I - P_1)$ and $(I - P_1)(I - P_2)(I - P_1)$ are the same, it is easy to derive bounds for the rate of convergence of the different iterative methods at the same time.

The use of projections in the description of the algorithm has many merits. However, it is also useful to describe the projections and algorithms in a more concrete, traditional linear algebra framework. The symmetry required for the routine use of the conjugate gradient method can thus also be recovered by considering Schwarz's algorithm from this other point of view. From our experience with other domain decomposition methods, see, e.g., Bjørstad [1], Bjørstad and Hvidsten [2], Bjørstad and Widlund [3], and Widlund [24], [25], we have also learned that insight can be gained by expressing the algorithms in terms of stiffness matrices. With subvectors and subscripts corresponding to the degrees of freedom associated with the open sets Ω_1, Ω_2, and Ω_3 and the curves (surfaces) Γ_4 and Γ_5, the entire discrete problem can be written as

$$Kx = \begin{pmatrix} K_{11} & 0 & 0 & K_{14} & 0 \\ 0 & K_{22} & 0 & 0 & K_{25} \\ 0 & 0 & K_{33} & K_{34} & K_{35} \\ K_{14}^T & 0 & K_{34}^T & K_{44} & K_{45} \\ 0 & K_{25}^T & K_{35}^T & K_{45}^T & K_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}. \tag{3}$$

As always, the elements of K are given by $a(\phi_i, \phi_j)$, where ϕ_i and ϕ_j are finite element basis functions. The zero blocks are a consequence of the fact that, when using standard finite element basis functions, there is no direct coupling between Ω_1 and Ω_2, etc.

Stiffness matrices are often computed by the so-called subassembly method. Since the bilinear form is defined in terms of an integral,

$$a_\Omega(\phi_i, \phi_j) = a_{\widetilde{\Omega}}(\phi_i, \phi_j) + a_{\Omega \setminus \widetilde{\Omega}}(\phi_i, \phi_j),$$

for any subset $\widetilde{\Omega} \subset \Omega$. We note that for a pair of basis functions associated with $\partial \widetilde{\Omega}$, we get contributions from both terms. If we, for example, choose $\widetilde{\Omega} = \Omega_2$, then we see that the quadratic form $x^T K y$ can be assembled by adding the quadratic forms corresponding to the matrices

$$(4) \qquad \begin{pmatrix} K_{22} & K_{25} \\ K_{25}^T & K_{55}^{(2)} \end{pmatrix} \text{ and } \begin{pmatrix} K_{11} & 0 & K_{14} & 0 \\ 0 & K_{33} & K_{34} & K_{35} \\ K_{14}^T & K_{34}^T & K_{44} & K_{45} \\ 0 & K_{35}^T & K_{45}^T & K_{55}^{(3)} \end{pmatrix}.$$

Here, $K_{55}^{(2)}$ and $K_{55}^{(3)}$ represent the contributions to K_{55} attributable to the integrals over Ω_2 and Ω_3, respectively. Similarly, we can write $K_{44} = K_{44}^{(1)} + K_{44}^{(3)}$. We note that the two matrices of (4) correspond to finite element models on $\Omega^{(1)}$ and Ω_2, respectively, with natural boundary conditions on Γ_5.

In the first fractional step, described above, we solve a homogeneous Dirichlet problem on $\Omega^{(1)}$. The corresponding linear system is

$$(5) \qquad \begin{pmatrix} K_{11} & 0 & K_{14} \\ 0 & K_{33} & K_{34} \\ K_{14}^T & K_{34}^T & K_{44} \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_3 \\ c_4 \end{pmatrix}.$$

If we denote this coefficient matrix by $K^{(1)}$, and permute the unknowns appropriately, then we see that the projection P_1 corresponds to the mapping

$$y = P_1 x = \begin{pmatrix} {K^{(1)}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} K x.$$

Since it is easy to show that the residuals associated with interior nodes of the sets Ω_1, Ω_2 and Ω_3 are and remain zero after the first full step of the Schwarz algorithm, we can specialize to the case where $c_1 = 0$ and $c_3 = 0$. By block Gaussian elimination we can reduce the system (5) to

$$(6) \qquad (S^{(1)} + S^{(3)}) y_4 = c_4,$$

where the so called Schur complements $S^{(i)}$, $i = 1, 3$, are defined by

$$(7) \qquad S^{(i)} = K_{44}^{(i)} - K_{i4}^T K_{ii}^{-1} K_{i4} \ , \quad i = 1, 3.$$

The Schur complement $S^{(2)}$ is associated with the entire region $\Omega^{(2)} = \Omega_2 \cup \Omega_3 \cup \Gamma_5$. It is given by the formula

$$S^{(2)} = K_{44}^{(3)} - \begin{pmatrix} 0 & K_{34}^T & K_{45} \end{pmatrix} \begin{pmatrix} K_{22} & 0 & K_{25} \\ 0 & K_{33} & K_{35} \\ K_{25}^T & K_{35}^T & K_{55} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{34} \\ K_{45}^T \end{pmatrix}.$$

The second fractional step makes all the residuals on $\Omega^{(2)}$ equal to zero. Upon the completion of any full iterative step, the only nonzero residuals are associated with Γ_4. We can therefore reduce the original system (3) to the equation

$$(8) \qquad (S^{(1)} + S^{(2)}) x_4 = b_4.$$

By Gaussian elimination, we show straightforwardly that the residual vector associated with Γ_4 is of the form

$$(S^{(1)} + S^{(2)})e_4^{(n)}$$

This is the right-hand side of (6), and the error propagation operator is therefore

$$I - (S^{(1)} + S^{(3)})^{-1}(S^{(1)} + S^{(2)}). \tag{9}$$

Chan and Resasco [11], [12] define their algorithm in terms of the same Schur complements. They reduce the original problem to (8), and since they use the same preconditioner, $S^{(1)} + S^{(3)}$, exactly the same iterates are produced if no acceleration procedure is used. It is also easy to show that the conjugate gradient version of their algorithm is identical to an accelerated version of the classical Schwarz method.

Chan and Resasco primarily consider cases where Ω is the union of two or three rectangles. The largest possible rectangular subregions of Ω are used to enhance the rate of convergence. The subproblems can then be solved by separation of variables. The preconditioner $S^{(1)}+S^{(3)}$ is diagonalized by using the fast Fourier transform (FFT), and the work involved is, therefore, independent of the size of the overlap in these special cases. In particular, Chan and Resasco demonstrate that if $\Omega^{(1)}$ is as large as possible a subset of Ω, the condition number of $(S^{(1)} + S^{(3)})^{-1}(S^{(1)} + S^{(2)})$ remains uniformly bounded, independent of the aspect ratios of $\Omega^{(1)}$ and $\Omega^{(2)}$. Such a result is not true for certain other algorithms, such as the Neumann-Dirichlet algorithm; cf. Bjørstad and Widlund [3]. However, the aspect ratios of the rectangles have to be extreme for this to be of any real importance.

3. Remarks on theory and other related methods. The convergence analysis of Schwarz's method can be carried out in a number of ways. Traditionally, the maximum principle is used to obtain an upper bound for the spectral radius; cf. Schwarz [21] or Lions [20]. Lions [19] has also derived a different upper bound that depends on the geometry through the maximum norm of the gradient of a certain function used in a partition of unity. Our analysis above has shown that the spectrum of the error propagation operator (9) can be expressed precisely in terms of a generalized eigenvalue problem involving three Schur complements. These Schur complements, defined through block Gaussian elimination of all variables not associated with the set Γ_4, are related to the strain energies of so-called discrete harmonic functions. Thus, the strain energy $x^T Kx$ associated with the entire region Ω is equal to ${x_4}^T(S^{(1)} + S^{(2)})x_4$ if the components of Kx corresponding to the interior variables of Ω_1 and $\Omega^{(2)}$ vanish. It is known that the strain energy of a discrete harmonic function has a smaller strain energy than any other extension of the given boundary values. Similarly, since the space of extensions increases if the overlap Ω_3 increases, it can be shown that the rate of convergence of the Schwarz method increases with the overlap. In fact, every component of the expansion of the error into the eigenvectors of $(S^{(1)} + S^{(3)})^{-1}(S^{(1)} + S^{(2)})$ decreases faster.

To prove that the method is uniformly well conditioned, independent of the number of degrees of freedom, we establish uniform upper and lower bounds for the generalized Rayleigh quotient

$$x_4{}^T(S^{(1)} + S^{(2)})x_4/x_4{}^T(S^{(1)} + S^{(3)})x_4 \ ,$$

by using

$$x_4{}^T S^{(i)} x_4 \leq C x_4{}^T S^{(j)} x_4 \ , \qquad \forall x_4, \ \ i,j = 1,2,3.$$

Proofs of this so-called extension theorem are given in [3] and [23]. The constants in these inequalities depend only on Ω_1, Ω_2 and Ω_3.

It is also of interest to relate Schwarz's algorithm to the Neumann-Dirichlet algorithm, cf. Bjørstad and Widlund [3], where a region is divided into two non-overlapping subregions by an interface that we can identify with Γ_4. The Schur complement $S^{(1)}$ alone is used to precondition (8). The convergence rate, therefore, depends on the Rayleigh quotient

$$x_4{}^T(S^{(1)} + S^{(2)})x_4/x_4{}^T S^{(1)} x_4.$$

The same extension theorem establishes a uniform upper bound.

If the interface Γ_4 is straight, Ω_3 and its triangulation are mirror images of Ω_1 and its triangulation and the elements of Lagrangian type, then $S^{(3)} = S^{(1)}$. In that case, the Schwarz method and the Neumann-Dirichlet algorithm give exactly the same result if conjugate gradient acceleration is employed. In nonmodel cases, the two algorithms can differ. See Bjørstad and Hvidsten [2] for a discussion of a case where the Schwarz algorithm works much better.

Finally, we note, that there is, in fact, no need to use subregions of Ω and the given triangulation in the construction of good preconditioners. By using the extension theorem, it is easy to show that the Schur complement with respect to the variables related to Γ_4 of a finite element model on any region, for which Γ_4 is an interface or a part of boundary, provides a preconditioner for the conjugate gradient method with a rate of convergence that is independent of the size of the linear system. There is no guarantee that the unaccelerated method will converge, since the strain energy of the new model might very well be smaller than that of the original problem.

4. Numerical experiments. We give a few numerical results to confirm the theoretical results and to give the reader an idea of the performance of the algorithms. For simplicity, Ω is the union of two rectangles.

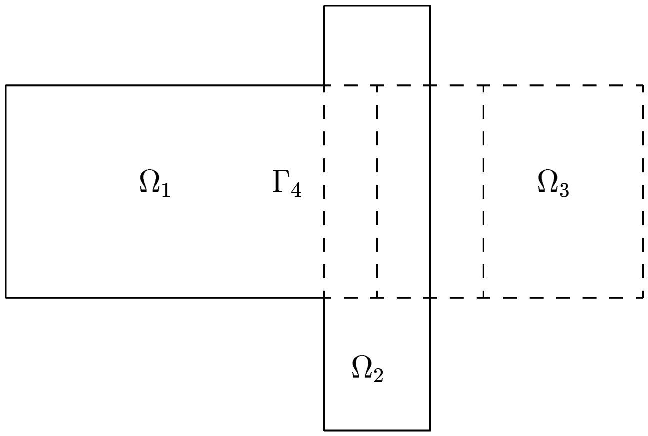

FIG. 2. *The computational domains.*

We use a mesh spacing $h = 1/256$ with the size of Ω_1 being .75 by .5, see Fig. 2. This gives a total of 40449 unknowns, with 127 along Γ_4. We consider the same problem as in [3] with the exact solution $u(x, y) = x^2 + y^2 - x \exp x \cos y$. The number of iterations required to reduce the initial error to the truncation error level (a factor of $3 \cdot 10^{-5}$) are reported both for the classical Schwarz method and for some of the other methods discussed in § 3. We create the overlap Ω_3 by extending the rectangle Ω_1 into and sometimes beyond Ω_2. Table 1 gives results for four different sizes of Ω_3: halfway into the second rectangle, all the way into the second rectangle, half the size of the mirror image of Ω_1 with respect to Γ_4, and equal to the mirror image. The second case is the Chan-Resasco algorithm. The last one, as indicated in the previous section, gives the same result as the Neumann-Dirichlet algorithm. The condition numbers relevant for the conjugate gradient method, are 1.29, 1.16, 1.177, and 1.184, respectively, in the four cases.

TABLE 1.
Number of iterations.

Schwarz algorithms	Relative Overlap			
	1/2	1	3/2	3
Unaccelerated	14	11	11	11
Accelerated	6	5	5	5

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