Upper Bounds on Error-Correcting Runlength-Limited Block Codes

Øyvind Ytrehus, Member IEEE

Dept. of Informatics, University of Bergen,
Thormøhlensgt. 55, 5020 Bergen, Norway.
E-mail: oyvind@ii.uib.no

IEEE Trans. Inf. Th. May 1991, pp. 941–945

Abstract — Upper bounds are derived on the number of codewords in error correcting (d, k)-constrained block codes. The results suggest that in such codes, fairly long block lengths are necessary to achieve code rates close to capacity.

Index terms — Runlength-limited codes, error-correction.

This work was supported by the Norwegian Research Council for Science and the Humanities (NAVF).

1. Introduction

Runlength-limited or (d, k)-constrained codes are frequently employed as modulation codes, particularly in magnetic and optical recording systems.

A binary sequence is (d, k)-*constrained* (where it is always assumed that $k > d$) if

- every pair of binary "1"s in the sequence are separated by at least d "0"s, and
- any $k + 1$ consecutive symbols contain at least one "1".

Traditional (d, k)-constrained codes are designed for a noiseless channel. For an introduction to such codes, see , for example, [1, 2, 3, 4, 5]. Of course the channels are not really noiseless after all. Necessary error correction is achieved by concatenating the inner (d, k)-constrained code with an outer error correcting code, usually a Reed-Solomon code.

In recent years, some attention has been given (examples include [6, 7, 8, 9, 10]) to the problem of designing (d, k)-constrained codes with error-correction capabilities. This research has been motivated by the idea that some of the inherent redundancy of (d, k)-constrained codes can be exploited towards error correction.

In this correspondence we derive some upper bounds on the size of (d, k)-constrained, simple-error correcting block codes. There are two directions in which one might like to extend the results of this correspondence: To (d, k)-constrained, error correcting *trellis* codes, and to more complex error correcting schemes.

Concerning trellis codes (i. e., "nonlinear convolutional codes", or codes generated by encoders with more than one state), we note that the set of codewords obtained by a fixed length input sequence, is a block code. Thus the bounds

obtained here also imply limitations on what is achievable by a trellis code of a given complexity.

The correspondence focuses on simple error correcting schemes, such as single error correction, since the recording channels on which these codes are used usually have a very low bit error rate. Multi-error correcting codes with a relatively short block length are not practical since a high code rate is a major concern.

In the next section, the notion of a (d, k)-constrained block code is discussed. Section 3 introduces the concept of combined codes, and several bounds are derived. Finally, in section 4 several examples are given.

2. (d, k)-constrained block codes

Let $\mathcal{C}$ be a (d, k)-constrained block code, i. e. a set of binary (d, k)-constrained sequences (denoted *codewords*) of fixed length. Consider a sequence $S' = (\ldots, c_{i-1}, c_i, c_{i+1}, \ldots)$, where each $c_j \in \mathcal{C}$. Then S is itself (d, k)-constrained only if the (d, k)-constraints are preserved across all the c_j, c_{j+1} boundaries. In order to ensure this, it is necessary to impose boundary constraints (l, L, t, T) on the codewords in $\mathcal{C}$. Thus,

- every codeword in $\mathcal{C}$ starts with at least l and at most L "0"s, and
- every codeword in $\mathcal{C}$ ends with at least t and at most T "0"s.

The actually transmitted or recorded sequence is then

$$S = (\ldots, c_{i-1}, b_i, c_i, b_{i+1}, c_{i+1}, b_{i+2}, \ldots)$$

where each b_j is a *bridge* of $\beta(\geq 0)$ bits that is inserted between each pair of adjacent codewords. The value of the bridge bits may or may not depend on the

preceding and the succeeding codeword. Tang and Bahl [1] investigated (d, k)-constrained block codes with boundary constraints $(0, k, 0, k)$ and bridgelength β=$d + 2$. More efficient are Constructions 1 and 2 of [5]; both have β=d. Construction 1 corresponds to $(l, L, t, T) = (0, L, 0, k - d - L)$, where the bridges all consist of d "0"s. For Construction 1, it has been shown [8] that, in the noiseless case, the optimal choice for L is $\lfloor \frac{k-d}{2} \rfloor$. Construction 2 corresponds to $(l, L, t, T) = (0, k - d, 0, k - d)$. In this case, a bridge can contain a single "1" if the k constraint would otherwise be violated by an all-zero bridge.

Since leading or trailing fixed zeros carry no information, they can be thought of as belonging to the bridges. Hence we can assume without loss of generality that $l = t = 0$.

In the following, assume that $n > min\{L, T\}$.

Definition 2.1. Let $\mathcal{M}(n, d, k, L, T)$ be the set of (d, k)-constrained vectors of length n that conform to the boundary constraints $(0, L, 0, T)$. Further, let $\mathcal{M}_w(n, d, k, L, T)$ be the subset of $\mathcal{M}(n, d, k, L, T)$ consisting of vectors of weight w.

Definition 2.2. For $n > i + j, w \geq 1$, let $\gamma_w(n, d, k, i, j)$ be the number of (d, k)-constrained vectors of length n, of Hamming weight w, that start with exactly i "0"s and end with exactly j "0"s.

We state without proof the following Lemmas.

Lemma 2.3 *([8, 9]).*

$$\gamma_w(n, d, k, i, j) =$$

$$\begin{cases} \sum_{p=0}^{w-1} (-1)^p \binom{w-1}{p} \binom{n-(i+j+2+d(w-1)+p(k-d+1))}{n-(i+j+1+(d+1)(w-1)+p(k-d+1))} & \text{if } w > 1 \\ \delta_{i+j+1,n} & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\delta_{i,j}$ *is* 1 *if* $i = j$, 0 *otherwise.*

Lemma 2.4. $|\mathcal{M}(n,d,k,L,T)| = \sum_{w=w_{min}}^{w_{max}} \sum_{i=0}^{L} \sum_{j=0}^{T} \gamma_w(n,d,k,i,j)$, *where*

$$w_{min} = \left\lceil \frac{n-L-T-1}{k+1} \right\rceil + 1$$

and

$$w_{max} = \left\lfloor \frac{n-1}{d+1} \right\rfloor + 1$$

are the minimum and maximum weight in $\mathcal{M}(n,d,k,L,T)$, *respectively.*

Lemma 2.4. *([11]). The code rate of any* (d,k)*-constrained code is bounded by the* capacity $C(d,k) = \log_2\lambda$, *where* λ *is the largest solution to*

$$x^{k+1} - \sum_{i=0}^{k-d} x^i = 0.$$

3. Bounds on Error-Correcting Codes

There is no universally agreed upon model of how noise affects a magnetic recording channel. Depending on the physical characteristics of the recording system, the channel can be modelled as one of the following:

- The *Binary Symmetric Channel* (BSC) has been considered by many researchers. In this model, "1" is changed into "0" and "0" is changed into "1" with equal probabilities. However,
- the *Asymmetric Channel* (ASC) is more suited to the asymmetric structure of recording systems employing NRZI modulation, where "1"s ("0"s) are represented by presence (absence) of magnetic transitions. In the ASC model, "1" is changed into "0" with some positive probability but "0" is never changed into "1".

- On the *Shift Channel* (SC), a binary "1" can be shifted (at most) one position in either direction. This is a frequent type of error when the recording density is high and d is large.
- (*All of the above* is probably the most realistic channel model. [12])

Let $N_{L,T;d,k,\Delta}(n)$ be the number of codewords in the largest code $\mathcal{C} \subset \mathcal{M}(n,d,k,L,T)$, such that the code has error-correcting capability specified by Δ.

Bounds from Computer Search

The exact determination of $N_{L,T;d,k,\Delta}(n)$ is in general a very difficult problem. For small values of n (i. e., $|\mathcal{M}(n,d,k,L,T)|$ not much larger than approximately 150), the exact values can be found by computer search [9]. For slightly larger values of n, decreasingly sharp bounds can still be found by this approach.

Sphere-Packing Bounds

Sphere-packing is a well-known technique for obtaining bounds on the size of an error-correcting code. As examples, we shall derive upper bounds on the size of a single asymmetric (AS) error correcting code, i. e., Δ="1–AS error correction", and on the size of a single shift (S) error correcting code, i. e., Δ="1–S error correction" .

Definition 3.1. For $\mathbf{x} \in \{0,1\}^n$, define

$$\mathcal{E}_{1-\mathrm{AS}}(\mathbf{x}) = \left\{\mathbf{y} \in \{0,1\}^n | \text{at most one AS error transforms } \mathbf{x} \text{ into } \mathbf{y}\right\}.$$

Definition 3.2. Let $\mathcal{M}_w = \{\mathbf{x} \in \mathcal{M} \mid w_H(\mathbf{x}) = w\}$ and define

$$\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M},w) \stackrel{\triangle}{=}$$
$$\left\{\mathbf{x} \in \{0,1\}^n \;\middle|\; w_H(\mathbf{x}) = w, \exists \mathbf{y} \in \left(\mathcal{M}_w \bigcup \mathcal{M}_{w+1}\right) : \mathbf{x} \in \mathcal{E}_{1-\mathrm{AS}}(\mathbf{y})\right\}.$$

Lemma 3.3.

$$
\begin{aligned}
|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}(n,d,k,L,T),w)| &= |\mathcal{M}_w(n,d,k,L,T)| \\
&+ \sum_{i=max\{d+1,L+1\}}^{L+k+1} \sum_{j=0}^{T} \gamma_w(n,d,k,i,j) \\
&+ \sum_{i=0}^{L} \sum_{j=0}^{T} \sum_{p=k+2}^{2k+2} (w-1)\gamma_{w-1}(n-p,d,k,i,j) \\
&+ \sum_{i=0}^{L} \sum_{j=max\{d+1,T+1\}}^{T+k+1} \gamma_w(n,d,k,i,j).
\end{aligned}
$$

Proof. A binary vector **x** of length n and Hamming weight w is in $\mathcal{A}_{1-AS}(\mathcal{M}(n,d,k,L,T),w)$ iff

a) $\mathbf{x} \in \mathcal{M}_w(n,d,k,L,T)$, or

b) $\mathbf{x} \notin \mathcal{M}_w(n,d,k,L,T)$, but **x** is the result of some vector in $\mathcal{M}_{w+1}(n,d,k,L,T)$ being corrupted by a single AS error, and hence contains exactly one "too long" run of "0"s, either in the beginning, the middle or the end of the vector.

□

Lemma 3.4. *Let $\mathcal{M}$ be a set of binary vectors of length n, let $\mathcal{C}$ ($\subset \mathcal{M}$) be a 1–AS-error correcting code, and let $\{z_w\}$ be the weight distribution of $\mathcal{C}$, i. e., $z_w = |\{\mathbf{x} \in \mathcal{C} | w_H(\mathbf{x}) = w\}|$. (Clearly, $|\mathcal{C}| = \sum_{i=w_{min}}^{w_{max}} z_i$, where w_{min} and w_{max} are the smallest and largest weights in $\mathcal{M}$, respectively.) Then*

a)

$$0 \le z_w \le |\mathcal{M}_w|$$

b)

$$z_{w-1} + w z_w \le |\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, w-1)|. \qquad (1)$$

Proof. a) is obvious. To show b), note that for each $\mathbf{c} \in \mathcal{C}$, it holds that $\mathcal{E}_{1-AS}(\mathbf{c})$ contains exactly $1 + w_H(\mathbf{c})$ vectors. Since the code is single AS error correcting, the sets $\mathcal{E}_{1-AS}(\mathbf{c})$ must be non-overlapping, and b) follows.

□

Theorem 3.5. *Let $\mathcal{M}, w_{min}, w_{max}$ be defined as in Lemma 3.4. Define $\{y_i\}$ by*

- $y_{w_{min}-1} = 0$, *and*
- *for* $w_{min} \leq i \leq w_{max}$:

$$y_i = min\left\{ \left\lfloor \frac{|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, i-1)| - y_{i-1}}{i} \right\rfloor, |\mathcal{M}_i| \right\} \qquad (2)$$

It holds that

$$N_{L,T;d,k,1-\mathrm{AS\ correction}} \leq \sum_{i=w_{min}}^{w_{max}} y_i.$$

Proof. Assume that $\mathcal{C} \subset \mathcal{M}$, is a 1–AS correcting code with weight distribution $\{z_i\}$ such that

$$\sum_{i=w_{min}}^{w_{max}} z_i > \sum_{i=w_{min}}^{w_{max}} y_i. \qquad (3)$$

Let $\{i_1, \ldots, i_t\}, i_1 < \cdots < i_t$, be the set of indices i for which $z_i > y_i$. Then, for $1 < j \leq t$, it holds that $i_j - i_{j-1} \geq 2$. For if $z_i > y_i$, then by Lemma 3.4

$$z_{i+1} \leq min\left\{ \left\lfloor \frac{|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, i)| - z_i}{i+1} \right\rfloor, |\mathcal{M}_{i+1}| \right\} \leq$$
$$min\left\{ \left\lfloor \frac{|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, i)| - y_i}{i+1} \right\rfloor, |\mathcal{M}_{i+1}| \right\} = y_{i+1}.$$

Also, from (1) and (2), $i_1 > w_{min} (> 0)$. Now, if $z_i - y_i = m > 0$, then $i \geq 2$ and

$$\left\lfloor \frac{|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, i-1)| - z_{i-1}}{i} \right\rfloor - \left\lfloor \frac{|\mathcal{A}_{1-\mathrm{AS}}(\mathcal{M}, i-1)| - y_{i-1}}{i} \right\rfloor =$$
$$\frac{y_{i-1} - z_{i-1} + r}{i} \geq m, \qquad (\text{where } |r| < i)$$
$$\Rightarrow y_{i-1} - z_{i-1} \geq m.$$

But then

$$\sum_{j=1}^{t} \left(z_{i_j} - z_{i_j-1}\right) \leq \sum_{j=1}^{t} \left(y_{i_j} - y_{i_j-1}\right),$$

contradicting (3).

□

For the asymmetric channel, it is appropriate to derive bounds on codes based on Construction 2, since this construction appears to be the most efficient for (d, k)-constrained block codes. Let us now turn the attention towards deriving a bound for single *shift* correcting codes. In this case we shall base the code construction and the bounds on Construction 1. This is because the bridges of Construction 2 may contain nonzero symbols, which by a shift error may cause non-shift errors in the leading or trailing bits of a codeword. In contrast, the bridge bits of Construction 1 are fixed "0"s. If the bridge length $\beta = d \geq 2$, one can assume that the first and last bridge bits belong to the preceding (coordinate position $n + 1$) and succeeding (coordinate position 0) codeword, respectively. Thus, any single-position shift will only affect a single codeword. This also means that a shift error will not change the weight of a codeword. Consequently, it is sufficient to consider constant weight subcodes.

Definition 3.6. Let $\mathcal{M}_w$ be a set of binary, fixed length vectors of constant weight w. Then

$$\mathcal{A}_{1-\mathrm{S}}(\mathcal{M}_w) \stackrel{\triangle}{=} \left\{\mathbf{x} \in \{0,1\}^n | \exists \mathbf{y} \in \mathcal{M}_w : \text{at most one S error transforms } \mathbf{x} \text{ into } \mathbf{y}\right\}.$$

Lemma 3.7

$$\begin{aligned} |\mathcal{A}_{1-\mathrm{S}}(\mathcal{M}_w(n,d,k,L,T))| =& |\mathcal{M}_w(n,d,k,L,T)| \\ &+ \sum_{i=1}^{7} S_i(n,d,k,w,L,T) - \sum_{i=8}^{13} S_i(n,d,k,w,L,T), \end{aligned}$$

where

$$S_1(n,d,k,L,T) = \sum_{j=0}^{T} \gamma_w(n,d,k,0,j),$$

$$S_2(n,d,k,L,T) = \sum_{i=0}^{L} \gamma_w(n,d,k,i,0),$$

$$S_3(n,d,k,L,T) = \sum_{j=0}^{T} \gamma_w(n,d,k,L,j),$$

$$S_4(n,d,k,L,T) = \sum_{i=0}^{L} \gamma_w(n,d,k,i,T),$$

$$S_5(n,d,k,L,T) = \sum_{i=0}^{L}\sum_{j=0}^{T} (w-1)\gamma_{w-1}(n-d,d,k,i,j),$$

$$S_6(n,d,k,L,T) = \sum_{i=0}^{L}\sum_{j=0}^{T} (w-1)\gamma_{w-1}(n-(k+2),d,k,i,j),$$

$$S_7(n,d,k,L,T) = \sum_{i=0}^{T}\sum_{j=0}^{T} 2(w-2)\gamma_{w-2}(n-(d+k+2),d,k,i,j),$$

$$S_8(n,d,k,L,T) = \sum_{j=0}^{T} \gamma_{w-2}(n-(2d+1),d,k,0,j),$$

$$S_9(n,d,k,L,T) = \sum_{i=0}^{L} \gamma_{w-2}(n-(2d+1),d,k,i,0),$$

$$S_{10}(n,d,k,L,T) = \sum_{i=0}^{L}\sum_{j=0}^{T} (w-3)\gamma_{w-3}(n-(3d+2),d,k,i,j),$$

$$S_{11}(n,d,k,L,T) = \sum_{j=0}^{T} \gamma_{w-2}(n-(2k+3),d,k,L,j),$$

$$S_{12}(n,d,k,L,T) = \sum_{i=0}^{L} \gamma_{w-2}(n-(2k+3),d,k,i,T),$$

$$S_{13}(n,d,k,L,T) = \sum_{i=0}^{L}\sum_{j=0}^{T} (w-3)\gamma_{w-3}(n-(3k+4),d,k,i,j).$$

Proof. Let $\mathbf{x}$ be a binary vector in $\mathcal{A}_{1-\mathrm{S}}(\mathcal{M}(n,d,k,L,T)) \setminus \mathcal{M}(n,d,k,L,T)$.

- If the leading (resp. trailing) "1" in $\mathbf{x}$ occurs in position 0 or $L+2$ (resp. $n-T-1$ or $n+1$), then $\mathbf{x}$ is the single shift corrupted version of some vector

in $\mathcal{M}(n, d, k, L, T)$ with the leading (resp. trailing) "1" in position 1 or $L+1$ (resp. $n-T$ or n). The number of vectors with "1" in these positions is given by $S_1, \ldots, S_4$.

- Assume that the first and last "1" in **x** do *not* violate the $(0, L, 0, T)$ constraints on the number of leading and trailing zeros. Then

  i. there is, somewhere in **x**, a *short* run of "0"s of length $d-1$, and the rest of the "0"-runs satisfy the (d, k) constraints. The number of such vectors is given by S_5. However, we need not count those vectors where, (relative to the short "0"-run), both the previous and the next "0"-run already is as short as possible. S_8, S_{10}, S_9 give the number of such vectors, where the short "0"-run follows the first "1", precedes the last "1" or is somewhere in the middle, *OR*

  ii. there is, somewhere in **x**, a *long* run of "0"s of length $k+1$, and the rest of the "0"-runs satisfy the (d, k) constraints. The number of such vectors is given by S_6. However, we need not count those vectors where, (relative to the long "0"-run), both the previous and the next "0"-run already is as long as possible. S_{11}, S_{13}, S_{12} give the number of such vectors, where the long "0"-run follows the first "1", precedes the last "1" or is somewhere in the middle, *OR*

  iii. there are, somewhere in **x**, two adjacent runs of "0", of lengths $d-1$ and $k+1$, respectively, and the rest of the "0"-runs satisfy the (d, k) constraints. The number of such vectors is given by S_7.

□

Theorem 3.8 *For a single shift correcting code $\mathcal{C} \subset \mathcal{M}$, let $\{z_w\}$ be the weight distribution of $\mathcal{C}$. Then*

$$z_w \leq \left\lfloor \frac{|\mathcal{A}_{1-\mathrm{S}}(\mathcal{M}_w)|}{2w+1} \right\rfloor.$$

Proof. For $\mathbf{x} \in \mathcal{C}$, define

$$\mathcal{E}_{1-\mathrm{S}}(\mathbf{x}) = \left\{\mathbf{y} \in \{0,1\}^n | \text{at most one S error transforms } \mathbf{x} \text{ into } \mathbf{y}\right\}.$$

For each $\mathbf{x}$ of Hamming weight w, $\mathcal{E}_{1-\mathrm{S}}(\mathbf{x})$ contains exactly $1 + 2w_H(\mathbf{x})$ vectors. Since the code is single bitshift correcting, the sets $\mathcal{E}_{1-\mathrm{S}}(\mathbf{x})$ of vectors are non-overlapping.

□

Remark. The sphere-packing bounds have been developed further by Abdel-Ghaffar and Weber [13]. They consider spheres with respect to subsets of $\mathcal{A}_{\ldots}(\mathcal{M}(n,d,k,L,T))$. In the case of single shift error correcting codes, their bound is a substantial improvement over Theorem 3.8. However, it may be hard to calculate for large blocklengths.

Recursion Bounds

We use the shorthand notation $N_{L,T}(n) = N_{L,T;d,k,\Delta}(n)$

Theorem 3.9.

$$N_{L,T}(n) \leq \sum_{i=0}^{L} N_{k-d,T}(n-i-1-d) \qquad (4)$$

Proof. Suppose $\mathcal{C}$ is a maximum-size Δ-code. The result follows from partitioning $\mathcal{C}$ into subcodes starting with i "0"s, $l \leq i \leq L$, and observing that each of the subcodes are also Δ-codes.

□

For general L, T we can use the following result:

Theorem 3.10.

$$N_{L,T}(n) \leq \sum_{m=0}^{L+T} f(m) \cdot N_{k-d,k-d}(n-m-2-2d), \tag{5}$$

where $f(m)$ *is given by*

$$f(m) = min\{m+1, L+1, T+1, L+T-m+1\}.$$

Proof. By similar arguments as those used to show Theorem 3.9, it follows that

$$N_{L,T}(n) \leq \sum_{i=0}^{L} \sum_{j=0}^{T} N_{k-d,k-d}(n-i-j-2-2d). \tag{6}$$

But the RHS in (6) is equal to the RHS in (5).

□

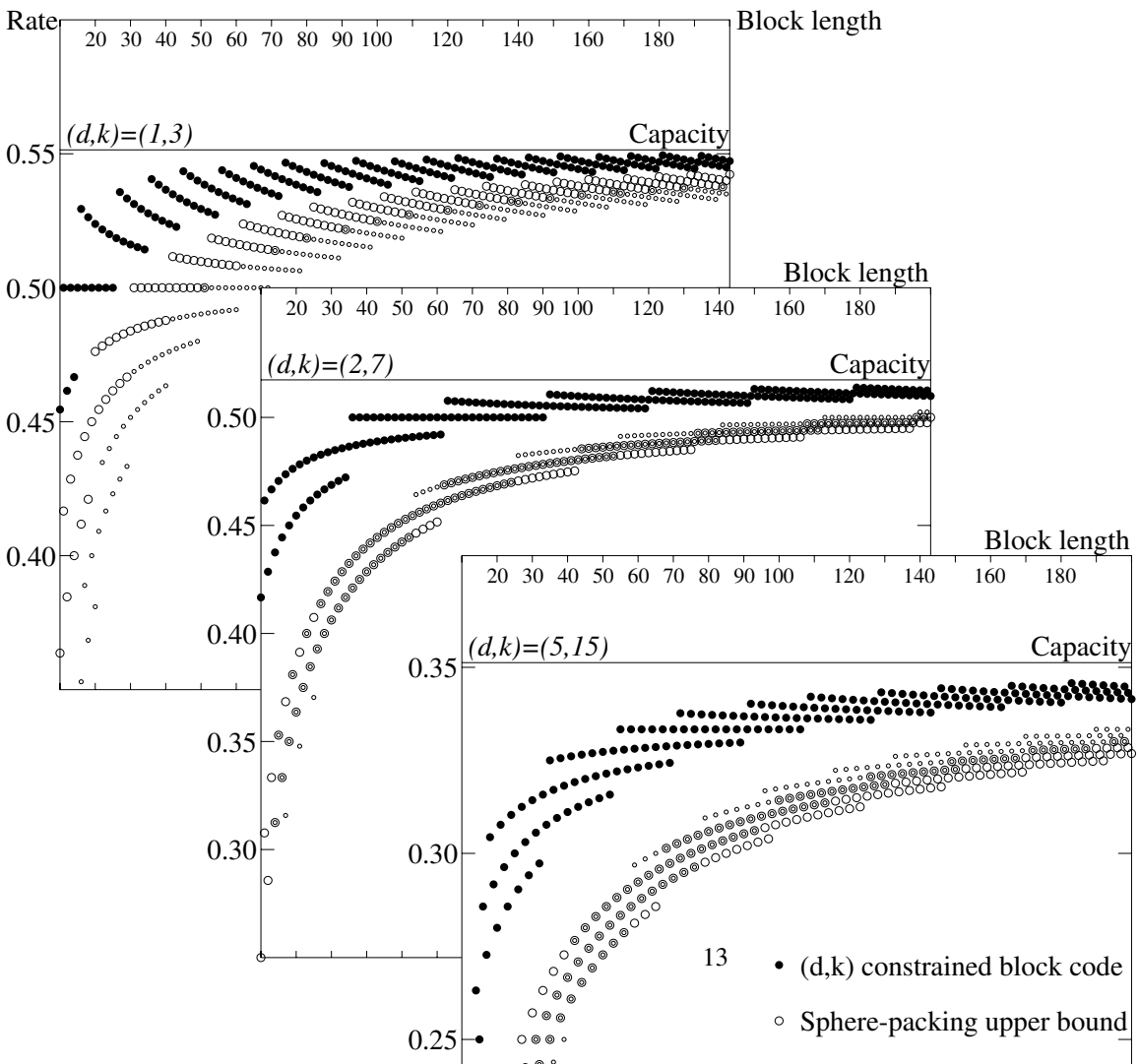
4. Examples, evaluation and conclusion

Figure I shows upper bounds on the maximum code rate obtained by single asymmetric error correcting block codes based on Construction 2. The (bounds on) the achievable code rate,

$$\frac{\lfloor \log_2 \left(N_{k-d,k-d;d,k,1-\text{AS correction}}(n) \right) \rfloor}{n+d},$$

is plotted against the block length n for three different (d, k) pairs. For reference, code rates obtained by Construction 2 (no-errors correcting) are also shown. Initial values for the recursion (4) can be found by a computer search as described in [9].

Figure I. Bounds on single asymmetric error correcting codes based on Construction 2



Similarly, Figure II shows upper bounds on the maximum code rate obtained by single shift error correcting block codes based on Construction 1. Here,

$$\frac{\left\lfloor \log_2 \left(N_{\lfloor \frac{k-d}{2} \rfloor, k-\lfloor \frac{k-d}{2} \rfloor - d; d, k, 1-\mathrm{S\ correction}}(n) \right) \right\rfloor}{n+d},$$

is plotted against the block length n for two different (d, k) pairs with $d \geq 2$. For reference, code rates obtained by Construction 1 (no-errors correcting) are also shown. In this case, initial values for the recursion (4) can be found by a combination of the methods of Abdel-Ghaffar and Weber [13], and a computer search.

Figure II. Bounds on single shift error correcting codes based on Construction 1

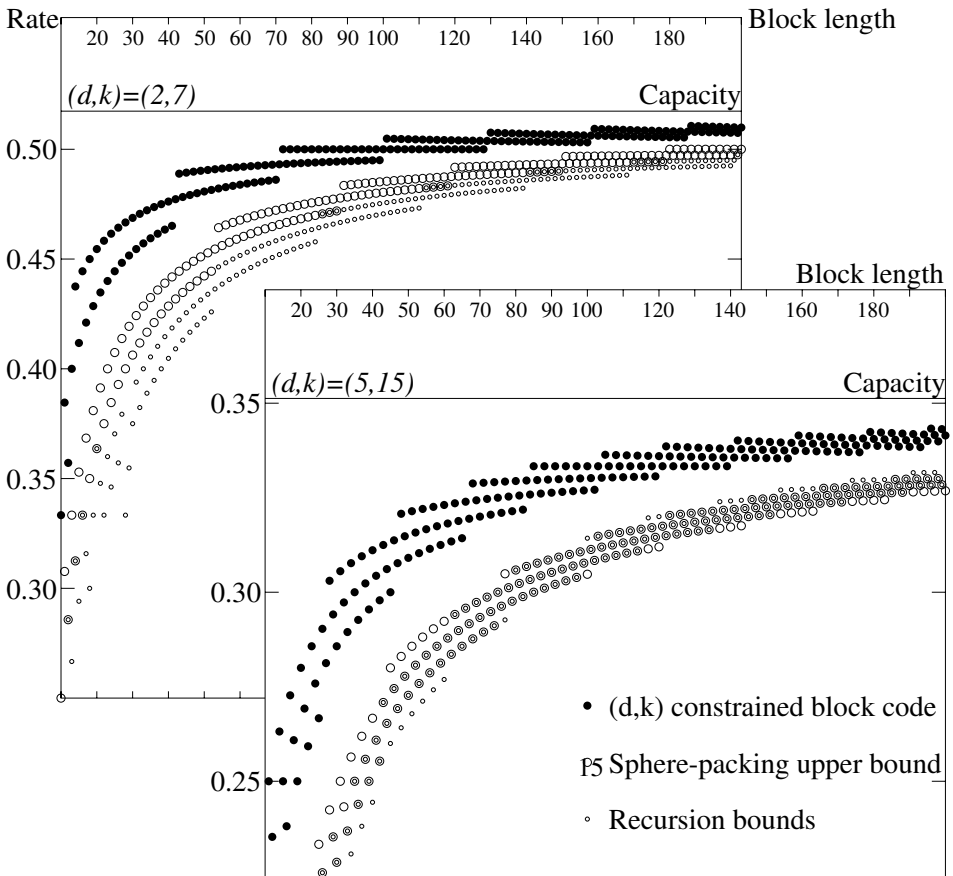


Table I shows the minimum block length necessary to achieve code rates of at least 80%, 90%, and 95% of the (d, k) capacity. This table suggests that even with very simple error correction schemes, such as single asymmetric or single shift error correction, one needs a much larger block length than in the no-error correction case, to preserve the code rate. This is somewhat disappointing, and weakens the hypothesis mentioned in Section 1: that some of the inherent redundancy of (d, k)-constrained codes can be exploited towards error-correction.

While the results are not very encouraging when considering *short* codes, it should be pointed out that for *long* block lengths, it has been demonstrated (see, for example, [14]) that combined coding can offer an improvement over the traditional concatenation of inner (d, k)-constrained codes and outer error correcting codes.

Acknowledgement

I would like to thank Patrick Perry for helpful suggestions.

Table I

(d,k)	$C(d,k)$	Con-struction	Error correction	Smallest block length necessary to achieve percentage of $C(d,k)$		
				95%	90%	80%
(1,3)	0.5515	2	None	16	1	
		2	1-AS	$\geq$100	$\geq$51	$\geq$26
(2,7)	0.5174	1	None	57	28	14
		1	1-S	$\geq$170	$\geq$86	$\geq$39
		2	None	36	14	5
		2	1-AS	$\geq$134	$\geq$62	$\geq$29
(5,15)	0.3513	1	None	105	51	23
		1	1-S		$\geq$112	$\geq$55
		2	None	72	35	16
		2	1-AS		$\geq$112	$\geq$48

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