# Strong formulations for the pooling problem

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# Outline



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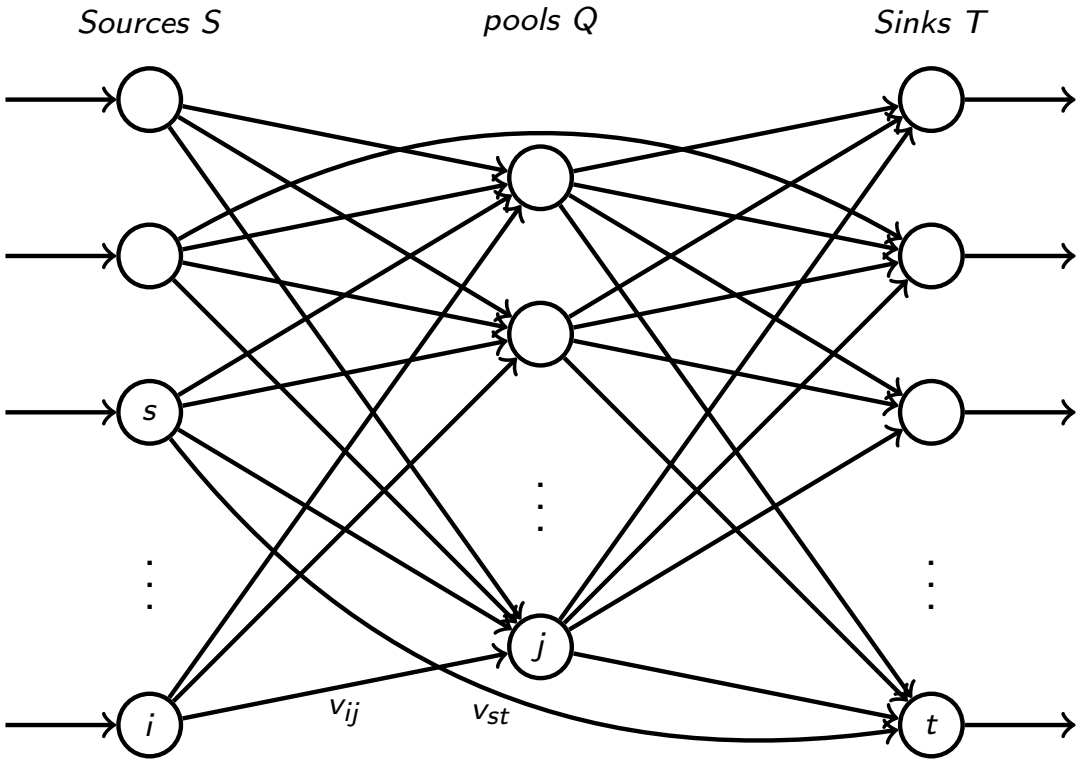

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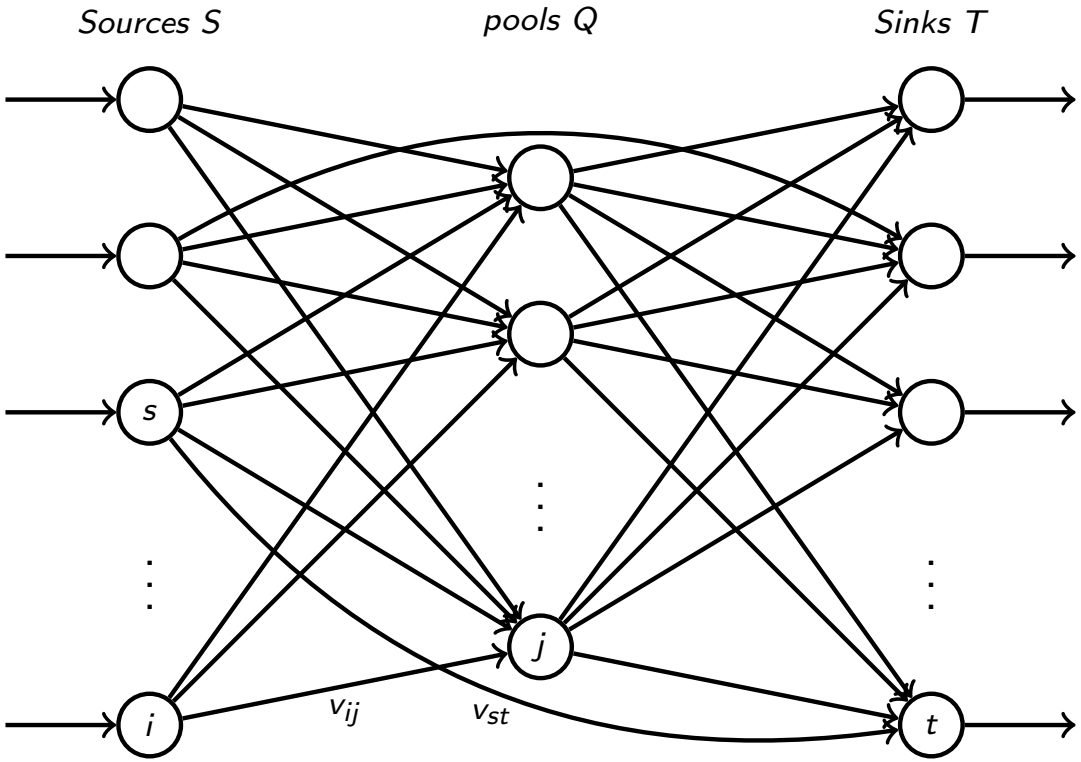


$v_{ij}$ is flow along the arc $(i, j) \in A$

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## $\mathcal{F}(G, S, T, u)$ is the polytope

$$\sum_{j:(s,j)\in A} v_{sj} \leq u_s,\ s \in S$$

$$\sum_{j:(j,i)\in A} v_{ji} \leq u_i,\ i \in N \setminus S$$

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$$\text{MCFP} = \min\{c^T v :\ v \in \mathcal{F}(G, S, T, u), 0 \leq v_{ij} \leq \max(u_i, u_j)\}$$

# The pooling problem

For any $v \in \mathcal{F}(G, S, T, u)$ associate a quality matrix $w \in \mathbb{R}^{N \times K}$:

$$w_i^k = \begin{cases} q_i^k & \text{if } i \in S \\ \dfrac{\sum_{j:(j,i)\in A} w_j^k v_{ji}}{\sum_{j:(j,i)\in A} v_{ji}} & \text{if } i \in N \setminus S \end{cases}$$

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The definition resembles the P-formulation

# Maximum 3-dimensional matching

**Definition**

Given 3 disjoint finite sets $I_1$, $I_2$ & $I_3$, and a set $E \subseteq I_1 \times I_2 \times I_3$, find $M \subseteq E$ of maximum cardinality s.t $\forall\ i \in I_1 \cup I_2 \cup I_3$ there is at most one $(i_1, i_2, i_3) \in M$ s.t $i \in \{i_1, i_2, i_3\}$

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Reducing max 3D matching to the pooling problem

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- $A = \{(i_1, i_2) : \exists\, (i_1, i_2, i_3) \in E\} \cup \{(i_2, (i_1, i_3)) : (i_1, i_2, i_3) \in E\}$

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- $u_i = 1,\ \forall\ i \in N$ and $c_{ij} = -1,\ \forall\ (i,j) \in A$
- $\forall\ t = (i_1, i_3)$, let $q_t^k = \begin{cases} 0 & k = k_{i_1} \\ 1 & k \neq k_{i_1} \end{cases}$, $q_{i_1}^k = \begin{cases} 0 & k = k_{i_1} \\ 1 & k \neq k_{i_1} \end{cases}$

# Consequence

Through out the proof we assume that $|K|$ is changing with the problem size

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## Open question

Is the pooling problem $\mathcal{NP}$-hard for fixed $|K|$?

# PQ-formulation

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$$\sum_{s\in S} y_i^s v_{it} = v_{it}, \ i \in Q, \ t \in T$$

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- PQ-formulation dominates both the P- and the Q-formulation

# TP-formulation

- analogous to the proportion $y_i^s$, define the terminal proportion

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  1. they do not have equal strength
  2. none of them dominates the other

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# Branching strategy

A procedure for choosing the branching pair in the STP-formulation

1. for each $(s, i, t)$

$$\text{violation} = \min\{v^* y^* - \text{Vex}(vy), \text{Cav}(vy) - v^* y^*\}$$

2. find the maximum violation pair in $\{y_i^s v_{it}\}$
3. find the maximum violation pair in $\{y_i^t v_{si}\}$
4. choose the pair with minimum violation from (2) and (3)

# Comparing the relaxations

Red value $\Longrightarrow$ better lower bounds

| Problem | Objective function value | | | Global Solution |
|---|---|---|---|---|
| | PQ-relaxation | TP-relaxation | STP-relaxation | |
| Adhya1 | −840.27 | −856.25 | −840.27 | −549.80 |
| Adhya2 | −574.78 | −574.78 | −574.78 | −549.80 |
| Adhya3 | −574.78 | −574.78 | −574.78 | −561.05 |
| Adhya4 | −961.93 | −967.43 | −961.93 | −877.65 |
| Bental4 | −550.00 | −541.67 | −541.67 | −450.00 |
| Bental5 | −3500.00 | −3500.00 | −3500.00 | −3500.27 |
| Foulds2 | −1100.00 | −1100.00 | −1100.00 | −1100.00 |
| Foulds3 | −8.00 | −8.00 | −8.00 | −8.00 |
| Foulds4 | −8.00 | −8.00 | −8.00 | −8.00 |
| Foulds5 | −8.00 | −8.00 | −8.00 | −8.00 |
| Haverly1 | −500.00 | −500.00 | −500.00 | −400.00 |
| Haverly2 | −1000.00 | −1000.00 | −1000.00 | −600.00 |
| Haverly3 | −800.00 | −875.00 | −800.00 | −750.00 |
| Rt2 | −6034.87 | −5528.00 | −5528.25 | −4391.83 |

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  - apply the `RELAXATION_ONLY_EQUATIONS` option in BARON

# Computational experiments with BARON

| Problem | PQ-formulation | | TP-formulation | | STP-formulation | |
|---|---|---|---|---|---|---|
| | #nodes | time(sec) | #nodes | time(sec) | #nodes | time(sec) |
| Adhya1 | 21 | 0.20 | 39 | 0.61 | 7 | 0.37 |
| Adhya2 | 33 | 0.19 | 15 | 0.30 | 1 | 0.27 |
| Adhya3 | 31 | 0.38 | 37 | 0.61 | 19 | 0.82 |
| Adhya4 | 1 | 0.18 | 17 | 0.27 | 1 | 0.30 |
| Bental4 | 1 | 0.02 | 1 | 0.01 | 1 | 0.02 |
| Bental5 | 0 | 0.03 | 0 | 0.10 | 0 | 0.17 |
| Foulds2 | 0 | 0.02 | 0 | 0.01 | 0 | 0.02 |
| Foulds3 | 0 | 0.84 | 0 | 0.36 | 0 | 8.95 |
| Foulds4 | 0 | 1.18 | 0 | 0.50 | 0 | 1.57 |
| Foulds5 | 0 | 0.30 | 0 | 0.42 | 0 | 1.64 |
| Haverly1 | 1 | 0.02 | 1 | 0.02 | 1 | 0.02 |
| Haverly2 | 5 | 0.02 | 1 | 0.01 | 1 | 0.03 |
| Haverly3 | 1 | 0.02 | 1 | 0.01 | 1 | 0.02 |
| Rt2 | 13 | 0.14 | 1 | 0.10 | 1 | 0.19 |

0 nodes means that the problem solved during the preprocessing

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Randomly generated problems with

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| problem | problem sets size | | | | probability |
|---|---|---|---|---|---|
| | $\|S\|$ | $\|Q\|$ | $\|T\|$ | $\|K\|$ | |
| problem1 | 25 | 15 | 20 | 10 | 0.60 |
| problem2 | 40 | 25 | 35 | 15 | 0.52 |
| problem3 | 60 | 35 | 50 | 20 | 0.30 |

# Computational results–large problems

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| Instance | PQ-formulation | | | | STP-formulation | | | |
|---|---|---|---|---|---|---|---|---|
| | BARON | | Our Code | | BARON | | Our Code | |
| | # node | time(sec) | # node | time(sec) | # node | time(sec) | # node | time(sec) |
| problem1 | 0 | 739.20 | 298 | 273.47 | – | > 2 days | 151 | 830.28 |
| problem2 | – | > 2 days | 532 | 4273.84 | – | > 2 days | 303 | 41100.15 |
| problem3 | – | > 2 days | 714 | 41387.13 | – | > 2 days | 640 | 162526.54 |

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- a branching strategy suite with the STP-formulation

Thank you.

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$$\sum_{j:(j,i_2)\in A} w_j^{k_{i_1}} v_{ji_2} = 0 \quad (u_{i_2} = 1) \quad \Longrightarrow v_{ji_2} = \begin{cases} 0 & \text{if } j \neq i_1 \\ 1 & \text{if } j = i_1 \end{cases}$$

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then the path $(i_1, i_2, t)$ is the only one