1: Completeness

1.1: Completeness of Gentzen's system

Recall Exercise ... and the discussion just after the introduction of Gentzen's system for FOL. We assume that the new rules added in the FOL system are invertible in the same sense as are the propositional rules. More precisely, for any rule GiDi GD, if the conclusion is valid, then so are all the assumptions. We can allow ourselves to make an even stronger assumption, namely, that this implication holds when validity is replaced by truth in an arbitrary structure. I.e., for any structure M, if $M \models \forall(\bigwedge \Gamma \to \bigvee \Delta)$ then likewise $M \models \forall(\bigwedge \Gamma_i \to \bigvee \Delta_i)$ for each assumption. Verification of this fact is left as an exercise.

We proceed as in Exercise..., but with the additional complications of possibly non-terminating proofs. First, we have to ensure that even if a branch of a proof does not terminate, all formulae in the sequent are processed. Let us therefore view a sequent as a pair of sequences $\Gamma = G_1, ..., G_a$ and $\Delta = D_1, ..., D_c$. Such a sequent is processed by applying bottom-up the appropriate rule first to G_1, then to G_2, to G_3, etc. until G_a, and followed by D_1 through D_c. If no rule is applicable to a formula, it is skipped and one continues with the next formula. If no branching occurred, one starts anew from the first formula in the sequent and iterates through them all. New formulae arising from rule applications are always placed at the start of the sequence from which they arise (on the left or on the right of $\vdash_{\mathcal{G}}$). These restrictions are particularly important for the quantifier rules which introduce new formulae, i.e.:

$$6.\ \vdash\exists\ \frac{\Gamma \vdash_{\mathcal{G}} A^x_t \ldots \exists x A \ldots}{\Gamma \vdash_{\mathcal{G}} \ldots \exists x A \ldots}\ A^x_t\ legal \qquad 6'.\ \forall\vdash\ \frac{A^x_t \ldots \forall x A \ldots \vdash_{\mathcal{G}} \Delta}{\ldots \forall x A \ldots \vdash_{\mathcal{G}} \Delta}\ A^x_t\ legal$$

These two rules might start a non-terminating, repetitive process introducing new substitution instances of A without ever considering the remaining formulae. Processing formulae from left to right, and placing new formulae to the left of the actually processed ones, makes sure that all formulae in a sequent will be processed, before starting the processing of the newly introduced formulae.

We must also ensure that all possible substitution instances are attempted in search for axioms. We assume in addition a countable number of terms enumerated $t_1, t_2, t_3, ...$ Each of the above rules is treated separately. Its first application, to the given formula ($\exists x A$, respectively, $\forall x A$) substitutes first t_1, then t_2, etc. Hence, on any infinite branch, all instances $A^x_{t_i}$ will be present for all t_i.

It is clear that if there is a proof of $\Gamma \vdash_{\mathcal{G}} \Delta$, then it can be obtained in this way. The only difference is that an efficient proof may terminate quickly by choosing appropriate formulae to process or terms to substitute, while this exhaustive processing only makes sure that no formula nor term remains ignored. Since formulae have finite complexity, one can also easily show by induction that one will always, eventually, process the whole sequence on the left, and then on the right of $\vdash_{\mathcal{G}}$, even though new formulae can appear at the start of both.

Similar fairness principle is observed in the treatment of various branches of the proof tree. One processes them in a breadth-first manner, i.e., if at a given "height" h the tree has some current leafs, unprocessed sequents

$$\Gamma_1 \vdash_{\mathcal{G}} \Delta_1 \qquad \Gamma_2 \vdash_{\mathcal{G}} \Delta_2 \quad ... \quad \Gamma_k \vdash_{\mathcal{G}} \Delta_k$$

then, irrespectively of their mutual connections further down in the proof, one starts with the sequent $\Gamma_1 \vdash_{\mathcal{G}} \Delta_1$ – farthest to the left – and, having processed all its formulae, continues with $\Gamma_2 \vdash_{\mathcal{G}} \Delta_2$, then $\Gamma_3 \vdash_{\mathcal{G}} \Delta_3$, etc. In one such "big step", over each sequent $\Gamma_i \vdash_{\mathcal{G}} \Delta_i$ we may obtain a whole tree with many new sequents on its top. These, together from all initial sequents at the height h, constitute the new level $h + 1$ and the starting point for the next iteration of the same sequential processing.

Assume now validity of a given sequent, $\models \forall(\bigwedge \Gamma \to \bigvee \Delta)$, and that it has no proof, in particular, that the above strategy does not lead to a proof. There are two cases.

(1) If every branch in the obtained proof tree is finite, then all its leafs contain irreducible sequents, some of which are non-axiomatic. Select one such leaf, say, with $\Phi \vdash_{\mathcal{G}} \Psi$. Irreducibility means that no rule can be applied to this sequent, i.e., all its formulae are atomic. That it is non-axiomatic means that $\Phi \cap \Psi = \varnothing$. Construct a counter-model M by taking all terms occurring in the atoms of Φ, Ψ as the intepretation domain $\underline{M}$. In particular, if there are open atomic formulae, their variables are treated as elements on line with ground terms. Interpret the predicates over these elements by making all atoms in Φ true and all atoms in Ψ false. This is possible since $\Phi \cap \Psi = \varnothing$. (E.g., a leaf $P(x) \vdash_{\mathcal{G}} P(t)$ gives the structure with $\underline{M} = \{x, t\}$ where $x \in [\![P]\!]^M$ and $t \notin [\![P]\!]^M$.)

Invertibility of the rules implies now that this is also a counter-model to the initial sequent $\Gamma \vdash_{\mathcal{G}} \Delta$ – contradicting its assumed validity. Hence, all leafs of a finite proof tree for a valid sequent, constructed according to the strategy above, must be axiomatic, yielding a proof.

(2) In the other case, the resulting tree has an infinite branch. If it has a non-axiomatic sequent at some of its leafs (finite branches), we can construct a counter-models as we did in (1). Otherwise, we select an arbitrary infinite branch B and construct a counter-model M from all the terms occurring in the atomic formulae on B. (Again, variables occurring in such formulae are taken as elements on line with ground terms.) The predicates are defined by

$\bar{t} \in [\![P]\!]^M$ iff there is a node on B where $P(\bar{t})$ occurs on the left of $\vdash_{\mathcal{G}}$ (*)
Since atomic formulae remain unchanged once they appear on B, and no node is axiomatic, this never leads to a conflict.

The claim now is that M is a counter-model to every sequent on the whole B. We show, by induction on the complexity of the formulae, that all those occurring on B on the left of $\vdash_{\mathcal{G}}$ are true and all those on the right false. The claim is obvious for the atomic formulae by the definition (*), and is easily verified for the propositional combinations of formulae by the invertibility of the propositional rules. Consider now any quantified formula occurring on the left. If it is universal, it has been processed by the rule:

$$6'.\ \forall\vdash\ \frac{A^x_t \ldots \forall x A \ldots \vdash_{\mathcal{G}} \Delta}{\ldots \forall x A \ldots \vdash_{\mathcal{G}} \Delta}\quad A^x_t\ legal.$$

Then $[\![A^x_t]\!]^M = \mathbf{1}$ by IH and, moreover, since all such instances are tried on every infinite branch, so $[\![A^x_{t_i}]\!]^M = \mathbf{1}$ for all terms t_i. But this means that $M \models \forall x A$. If the formula is existential, it is processed by the rule:

$$7'.\ \exists\vdash\ \frac{\Gamma, A^x_{x'} \vdash_{\mathcal{G}} \Delta}{\Gamma, \exists x A \vdash_{\mathcal{G}} \Delta}\quad x'\ fresh.$$

Then x' occurs in the atomic subformula(e) of A and is an element of $\underline{M}$. By IH, $[\![A^x_{x'}]\!]^M = \mathbf{1}$ and hence also $[\![\exists x A]\!]^M = \mathbf{1}$.

A universally quantified formula occurring on the righ of $\vdash_{\mathcal{G}}$ is processed by the rule

$$7.\ \vdash\forall\ \frac{\Gamma \vdash_{\mathcal{G}} A^x_{x'}, \Delta}{\Gamma, \vdash_{\mathcal{G}} \forall x A, \Delta}\quad x'\ fresh$$

Then $x' \in \underline{M}$ and, by IH, $[\![A^x_{x'}]\!]^M = \mathbf{0}$. Hence also $[\![\forall x A]\!]^M = \mathbf{0}$. An existential formula on the right is processd by the rule

$$6.\ \vdash\exists\ \frac{\Gamma \vdash_{\mathcal{G}} A^x_t \ldots \exists x A \ldots}{\Gamma \vdash_{\mathcal{G}} \ldots \exists x A \ldots}\quad A^x_t\ legal$$

and, by the exhaustive fairness strategy, all such instances $A^x_{t_i}$ are included on the right in the proof process. By IH, $[\![A^x_{t_i}]\!]^M = \mathbf{0}$ for all t_i, which means that $[\![\exists x A]\!]^M = \mathbf{0}$.

Since the sequent $\Gamma \vdash_{\mathcal{G}} \Delta$ is on B, the above claim implies that $M \not\models \forall(\bigwedge \Gamma \to \bigvee \Delta)$, i.e., that the sequent is not valid, contrary to the assumption.

(1) and (2) together establish the implication from unprovability of a sequent to the existence of a counter-model for it. Formulated contrapositively, if a sequent is valid (has no counter-model), it is provable.