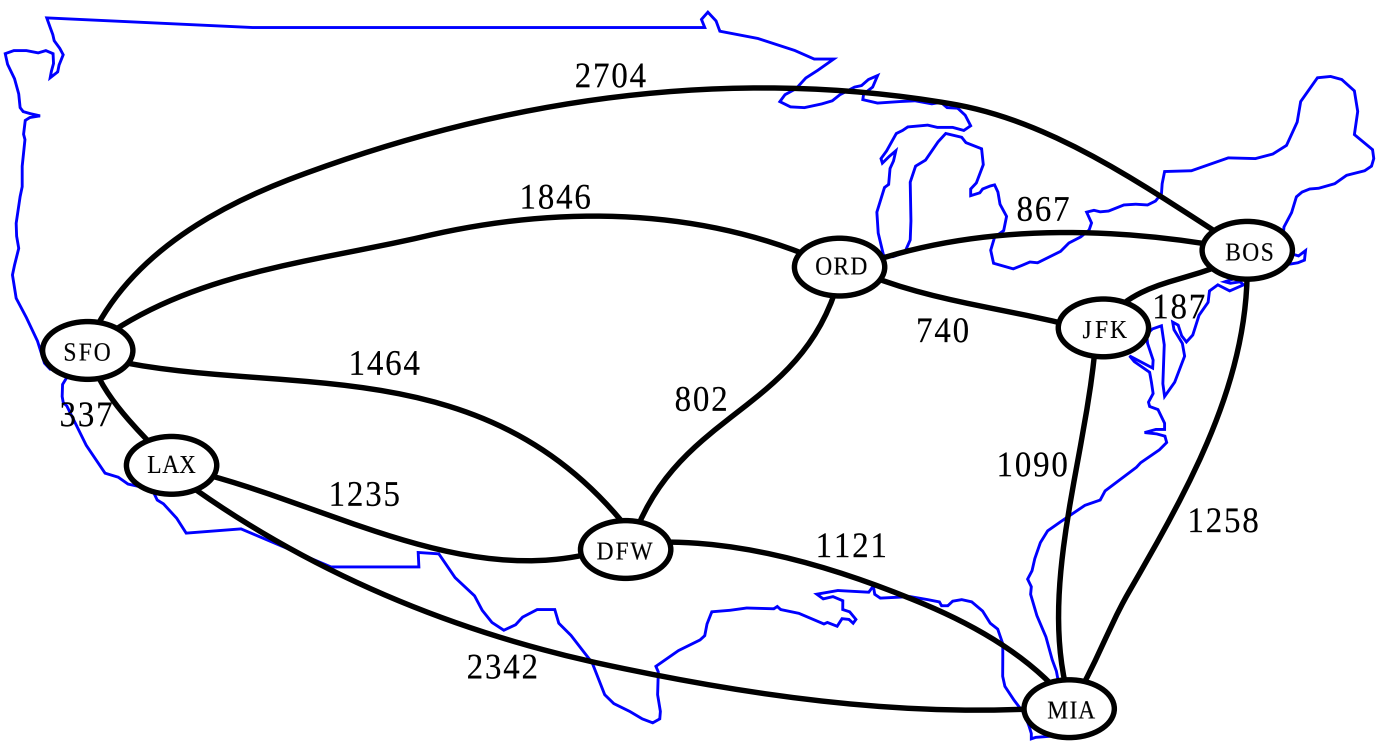
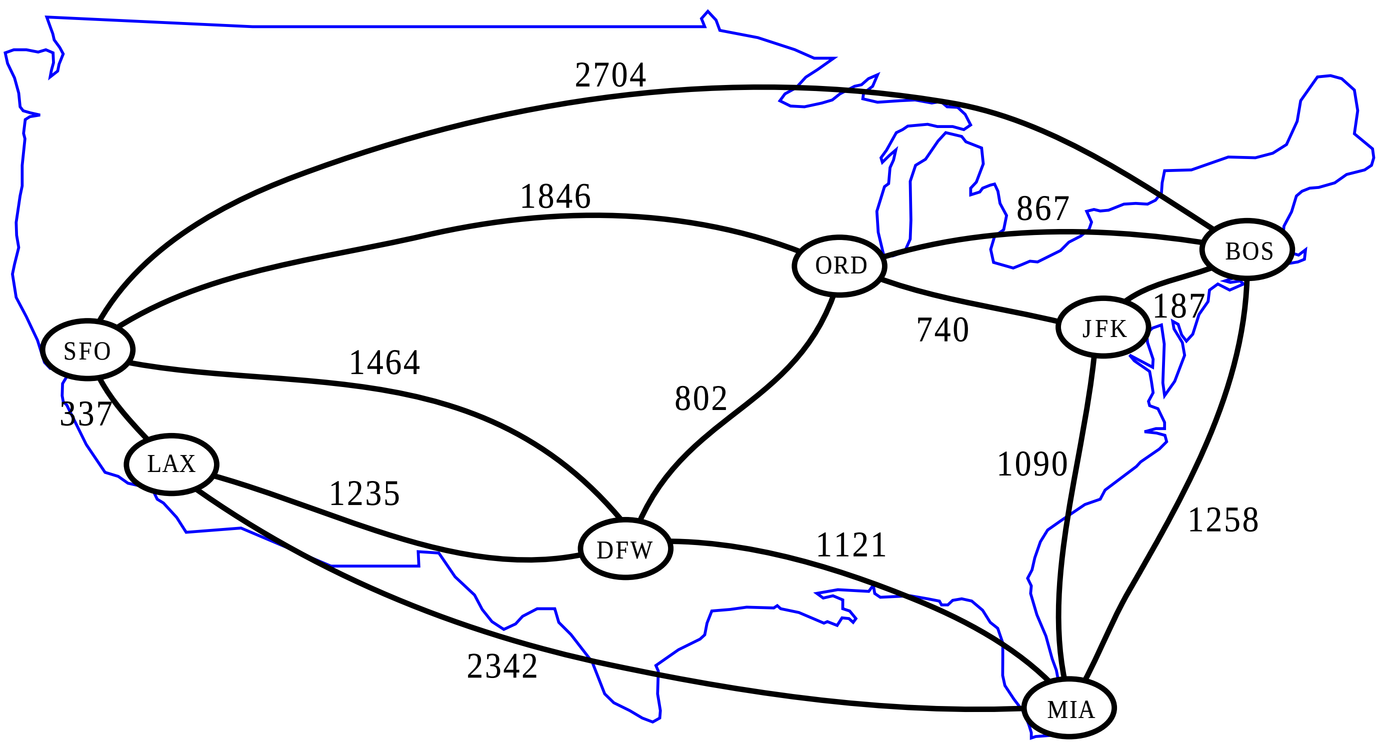
SHORTEST PATHS

- Weighted Digraphs
- Shortest paths



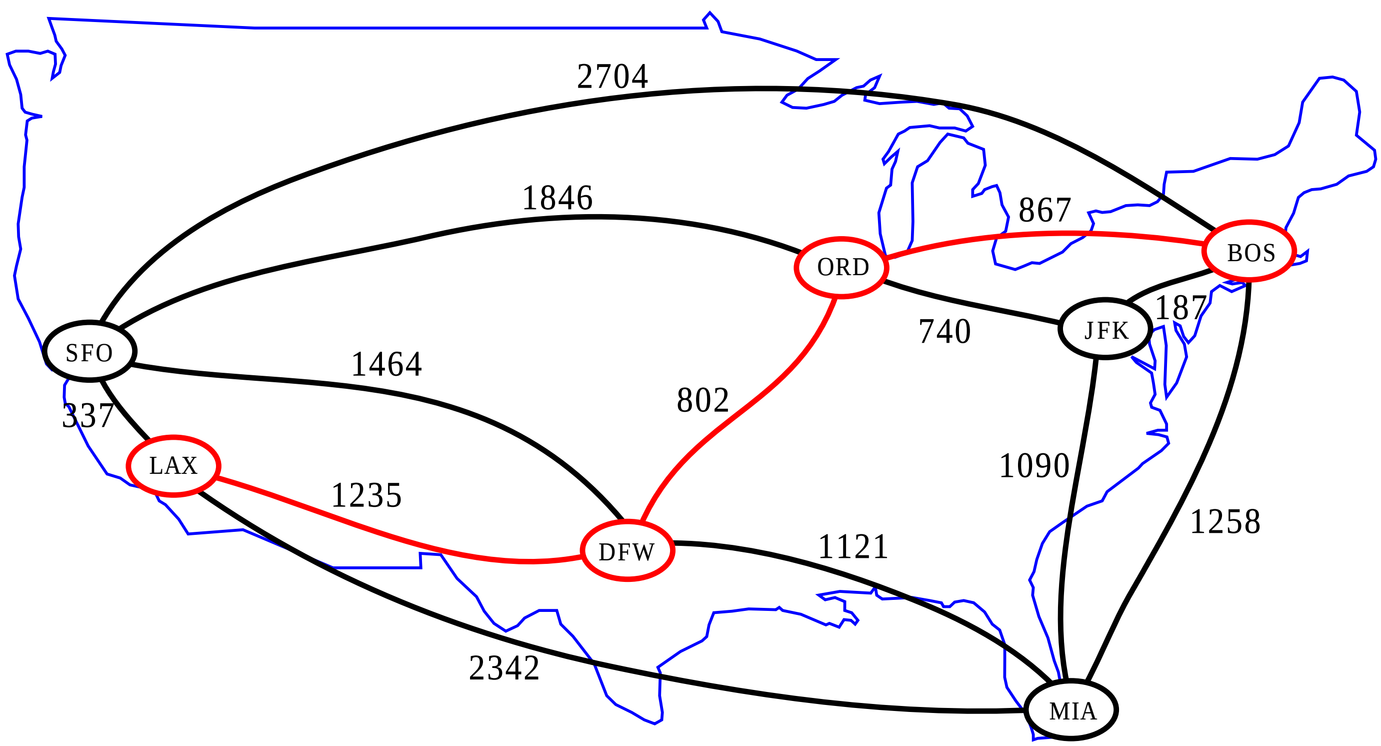
Weighted Graphs

- **weights** on the edges of a graph represent distances, costs, etc.
- An example of an undirected weighted graph:



Shortest Path

- BFS finds paths with the minimum number of edges from the start vertex
- Hencs, BFS finds shortest paths assuming that each edge has the same weight
- In many applications, e.g., transportation networks, the edges of a graph have different weights.
- How can we find paths of minimum total weight?
- Example - Boston to Los Angeles:



Dijkstra's Algorithm

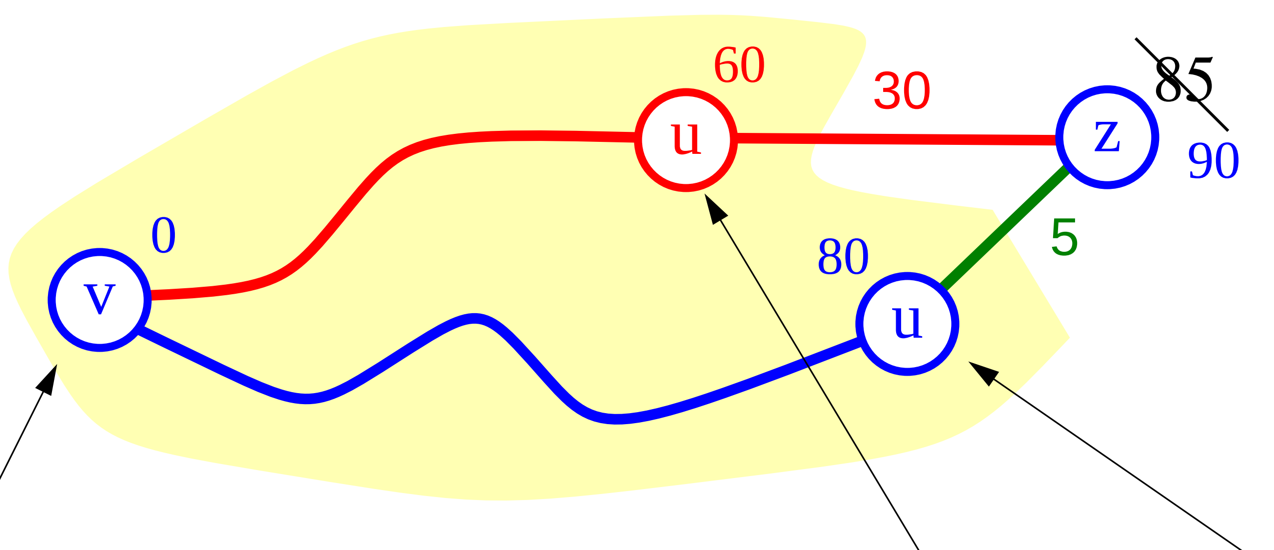
- Dijkstra's algorithm finds shortest paths from a start vertex *v* to all the other vertices in a graph with
  - **undirected edges**
  - **nonnegative edge weights**
- the algorithm computes for each vertex *u* the **distance** of *u* from the start vertex *v*, that is, the weight of a shortest path between *v* and *u*.
- the algorithm keeps track of the set of vertices for which the distance has been computed, called the **cloud** C
- Every vertex has a label D associated with it. For any vertex *u*, we can refer to its D label as D[*u*]. D[*u*] stores an approximation of the distance between *v* and *u*. The algorithm will update a D[*u*] value when it finds a shorter path from *v* to *u*.
- When a vertex *u* is added to the cloud, its label D[*u*] is equal to the actual (final) distance between the starting vertex *v* and vertex *u*.
- initially, we set
  - D[v] = 0 ...the distance from v to itself is 0...
  - D[u] = ∞ for u ≠ v ...these will change...

The Algorithm: Expanding the Cloud

- Repeat until all vertices have been put in the cloud:
  - let u be a vertex not in the cloud that has smallest label D[u]. (On the first iteration, naturally the starting vertex will be chosen.)
  - we add u to the cloud C
  - we update the labels of the adjacent vertices of u as follows

    for each vertex z adjacent to u **do**
      if z is not in the cloud C **then**
        if D[u] + weight(u,z) < D[z] **then**
          D[z] = D[u] + weight(u,z)

- the above step is called a **relaxation** of edge (u,z)



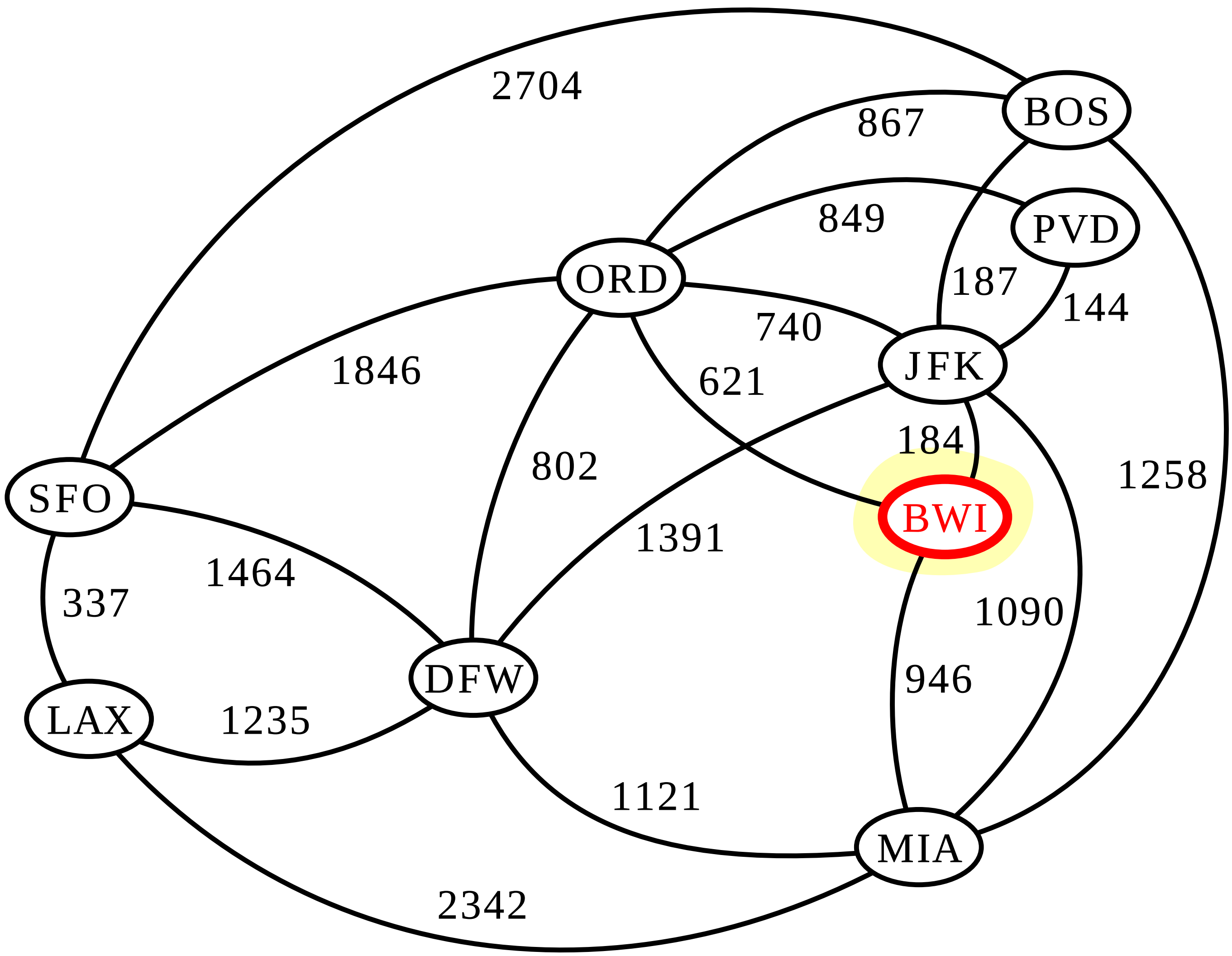
v was put in the cloud first. Then this u. Then this u.

Pseudocode

- we use a priority queue Q to store the vertices not in the cloud, where $D[v]$ the key of a vertex v in Q

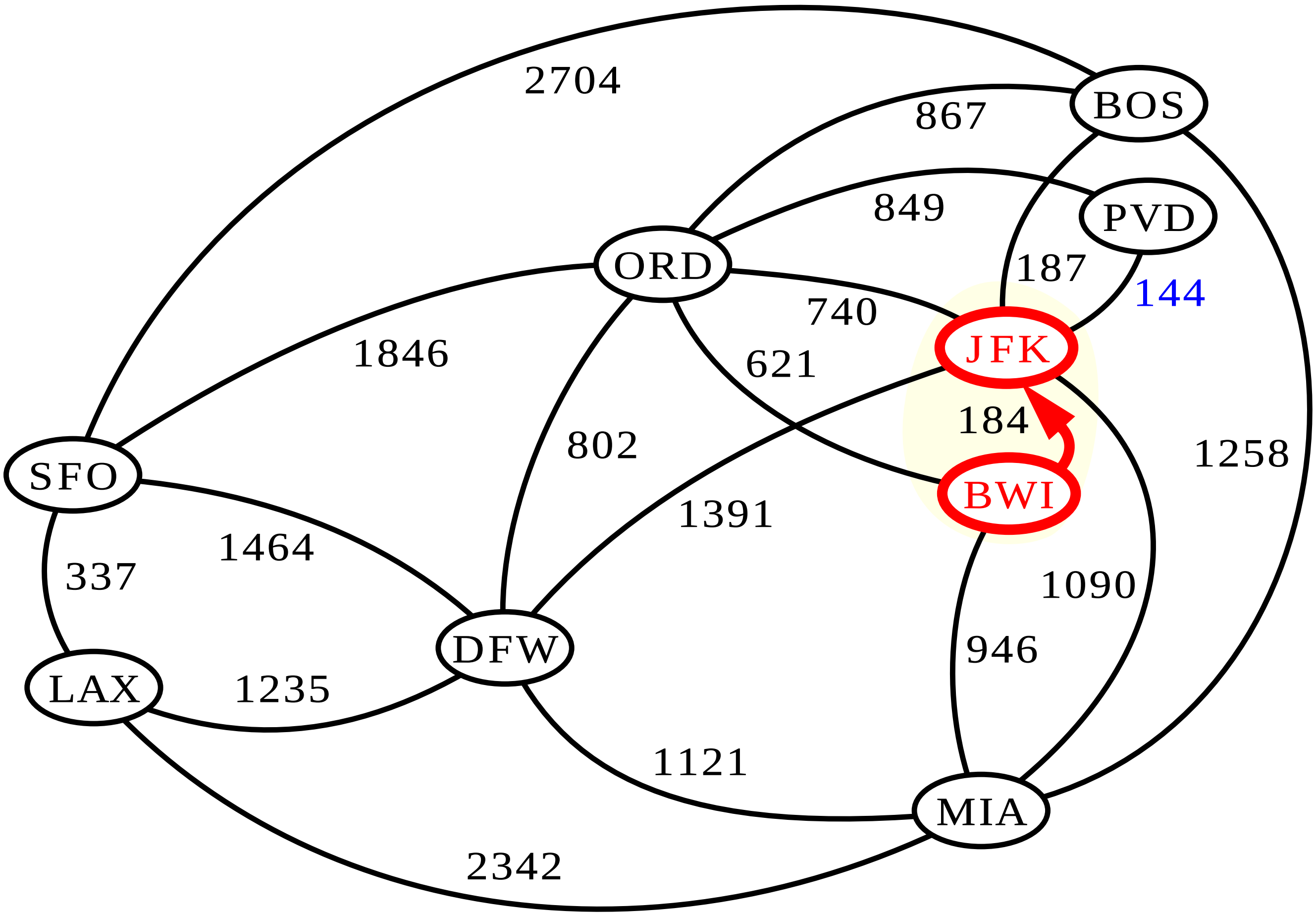
```
Algorithm ShortestPath(G, v):
  Input: A weighted graph G and a distinguished vertex
      v of G.
  Output: A label D[u], for each vertex that u of G,
      such that D[u] is the length of a shortest path
      from v to u in G.

  initialize D[v] ← 0 and D[u] ← +∞ for each
    vertex v ≠ u
  let Q be a priority queue that contains all of the
    vertices of G using the D lables as keys.
  while Q ≠ ∅ do
    {pull u into the cloud C}
    u ← Q.removeMinElement()
    for each vertex z adjacent to u such that z is in Q do
      {perform the relaxation operation on edge (u, z) }
      if D[u] + w((u, z)) < D[z] then
        D[z] ←D[u] + w((u, z))
        change the key value of z in Q to D[z]
  return the label D[u] of each vertex u.
```

Example: shortest paths starting from BWI



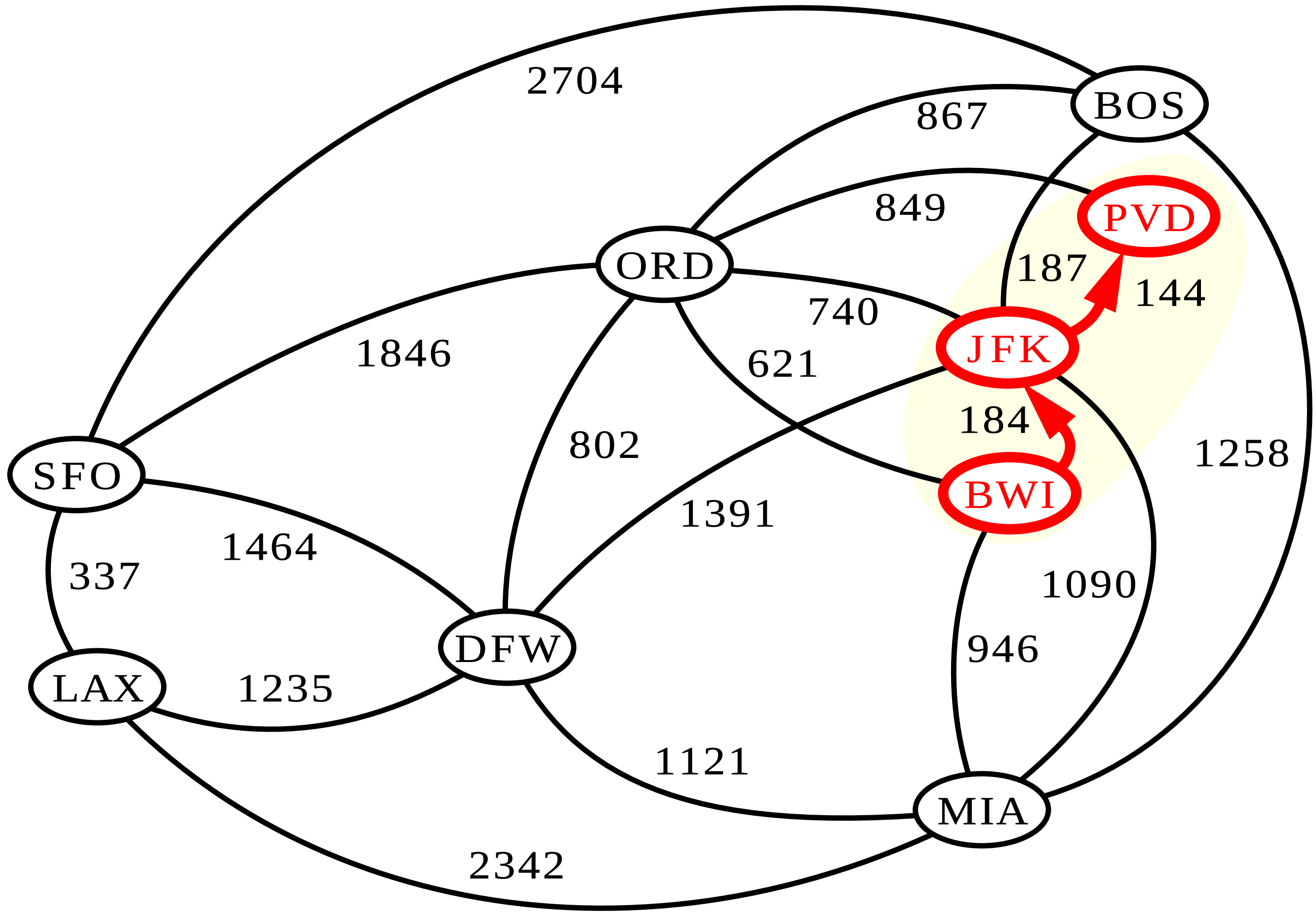
	parent	distance
BOS		∞
BWI		0
DFW		∞
JFK	BWI	184
LAX		∞
MIA	BWI	946
ORD	BWI	621
PVD		∞
SFO		∞

- JFK is the nearest...



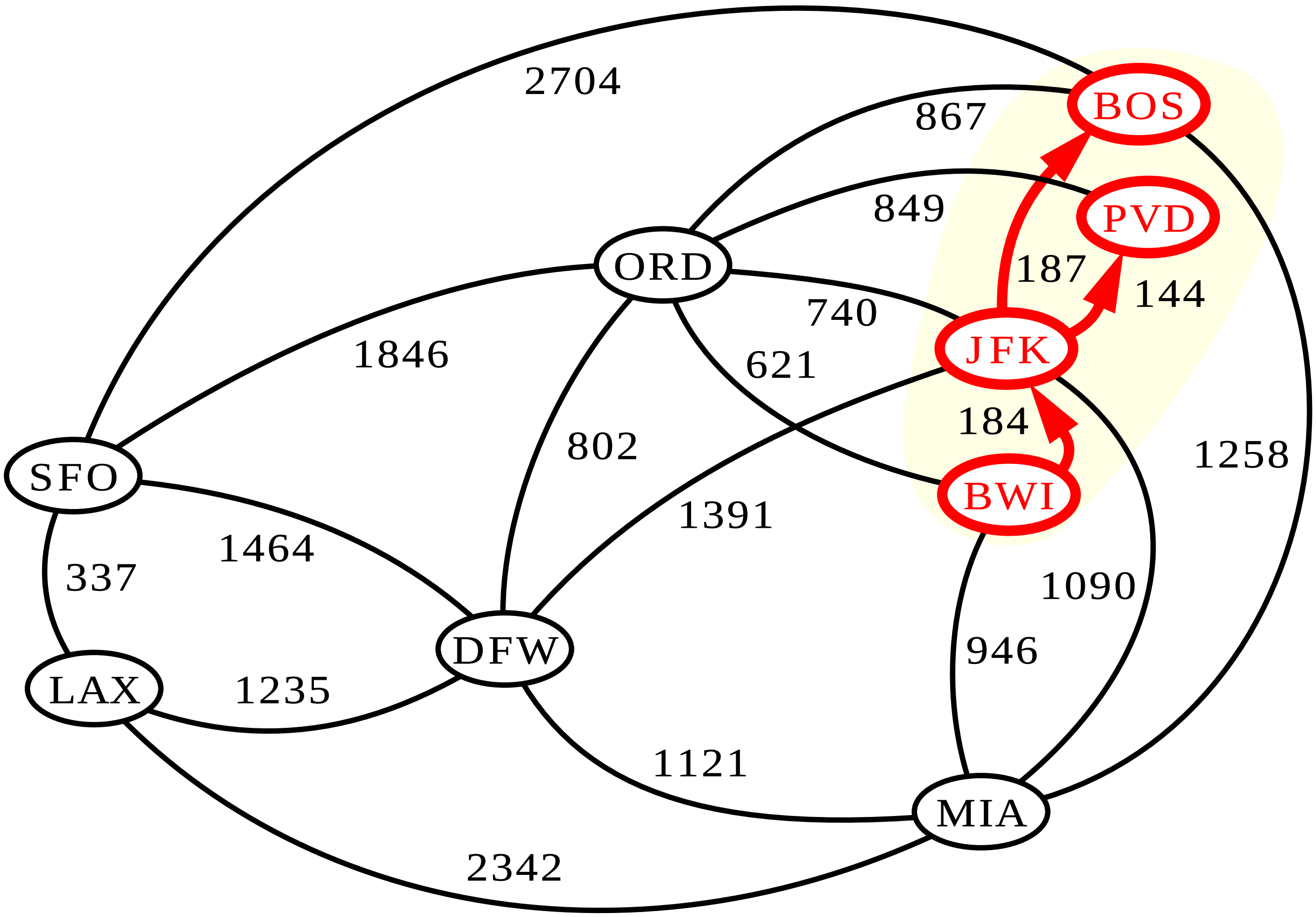
	parent	distance
BOS	JFK	371
BWI		0
DFW	JFK	1575
JFK	BWI	184
LAX		∞
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO		∞

- followed by sunny PVD.



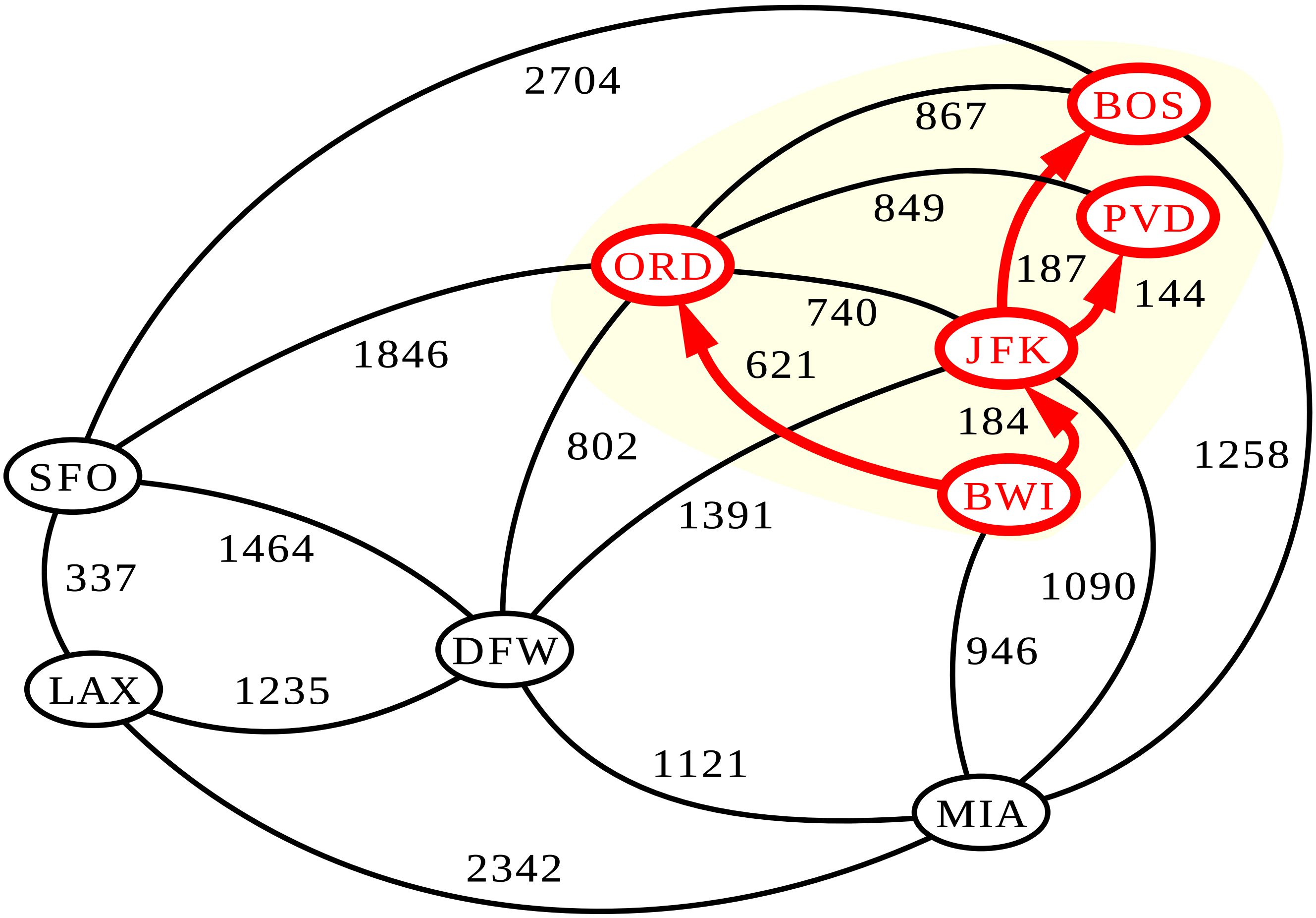
	parent	distance
BOS	JFK	371
BWI		**0**
DFW	JFK	1575
JFK	**BWI**	**184**
LAX		∞
MIA	BWI	946
ORD	BWI	621
PVD	**JFK**	**328**
SFO		∞

- BOS is just a little further.



	parent	distance
BOS	JFK	371
BWI		0
DFW	JFK	1575
JFK	BWI	184
LAX		∞
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	BOS	3075

- ORD: Chicago is my kind of town.

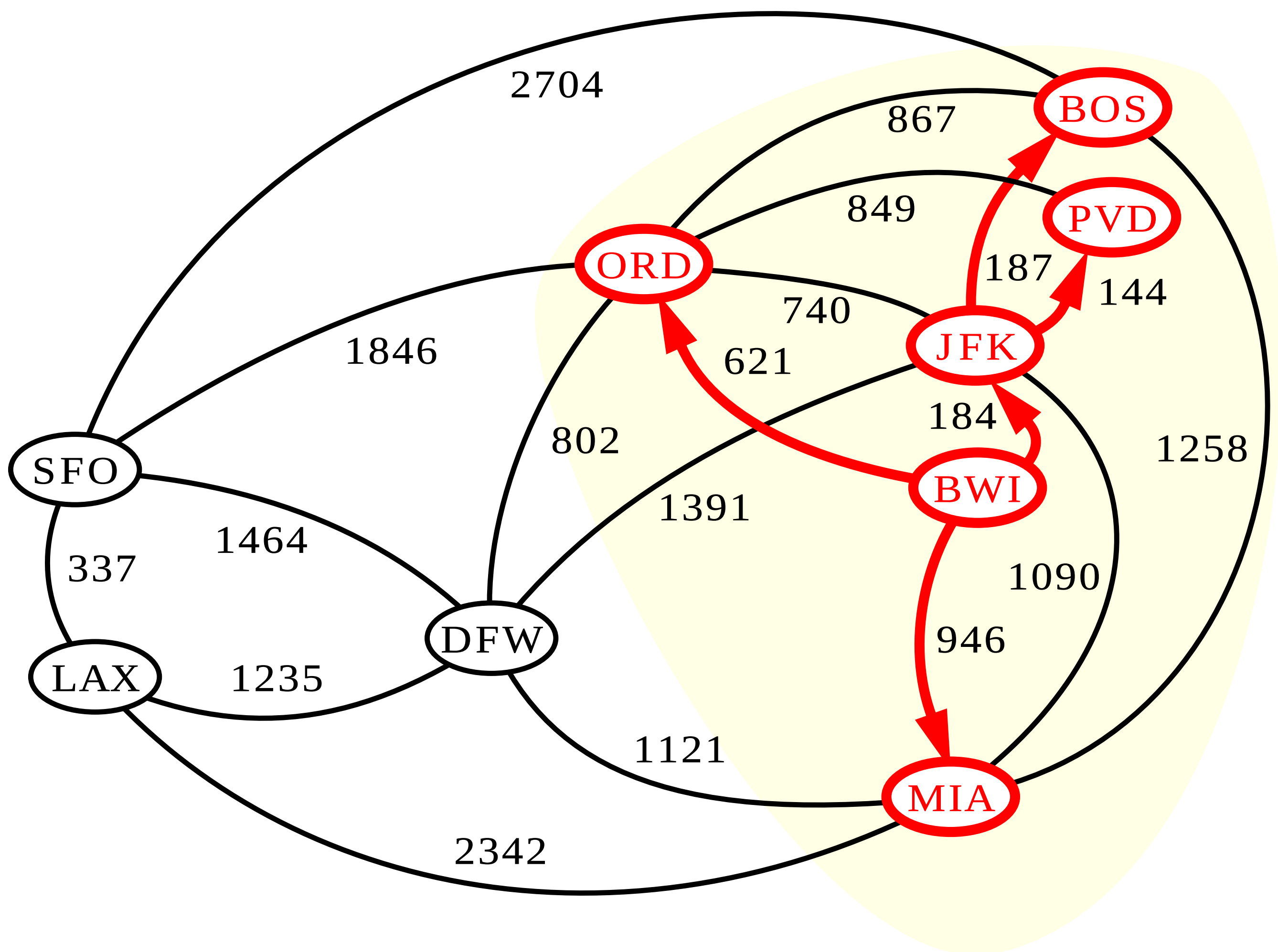


	parent	distance
BOS	JFK	371
BWI		0
DFW	ORD	1423
JFK	BWI	184
LAX		∞
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	ORD	2467

note that D for DWF was adjusted on this turn

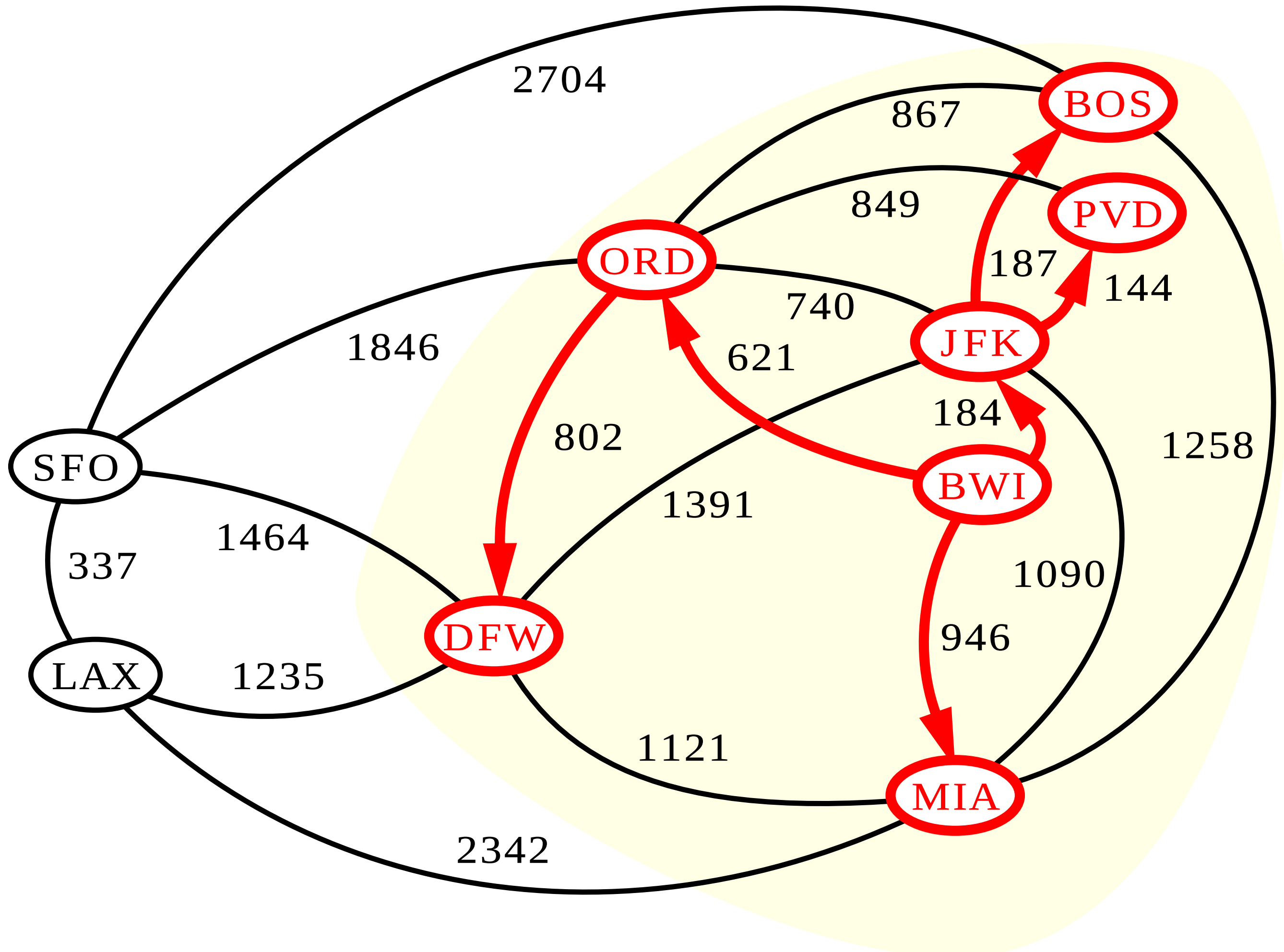
also for SFO

- MIA, just after Spring Break.



	parent	distance
BOS	JFK	371
BWI	[redacted]	0
DFW	JFK	1423
JFK	BWI	184
LAX	MIA	3288
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	BOS	2467

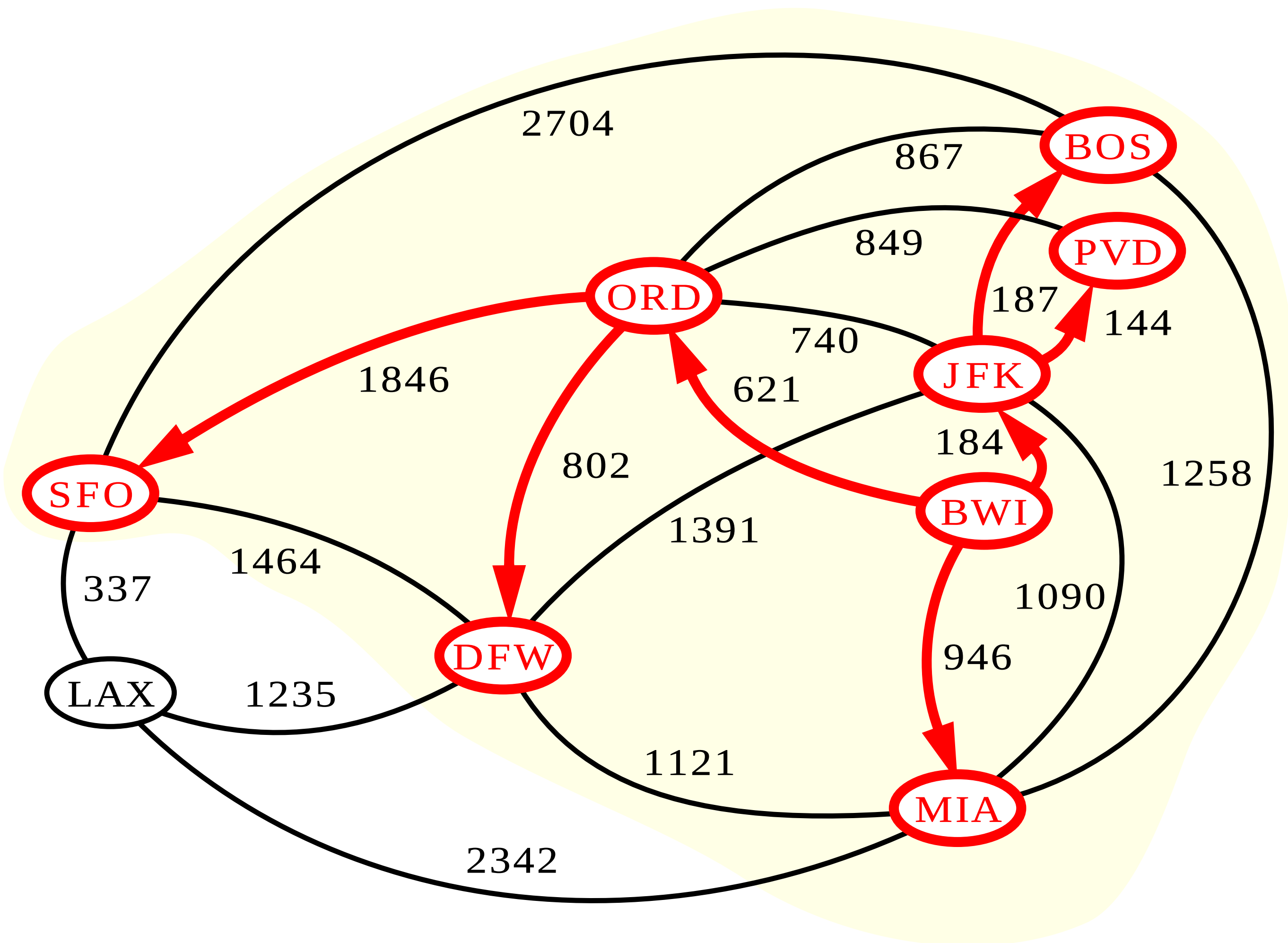
- DFW is huge like Texas.



	parent	distance
BOS	JFK	371
BWI		0
DFW	JFK	1423
JFK	BWI	184
LAX	DFW	2658
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	BOS	2467

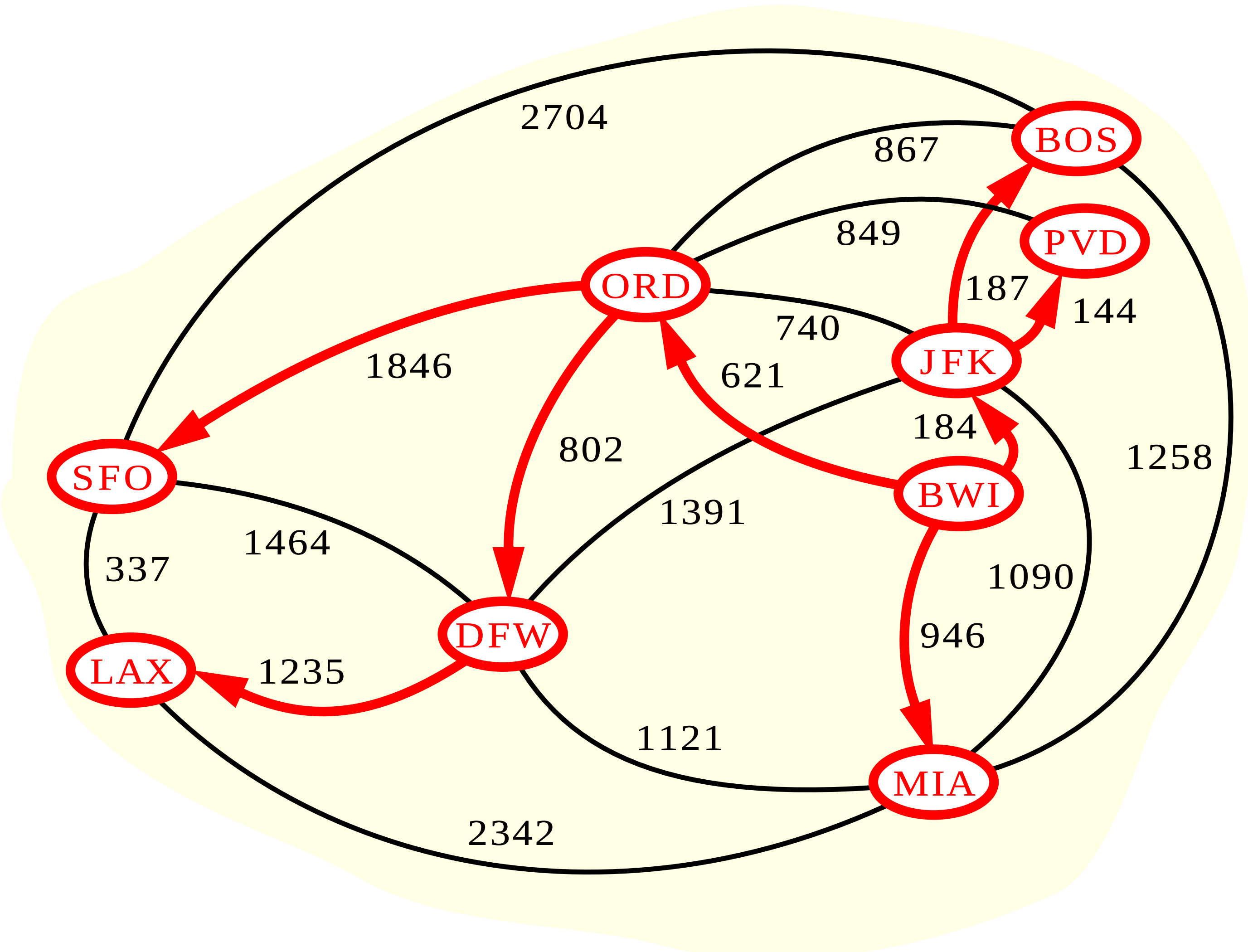
and D for LAX gets updated

- SFO: the 49’ers will take the prize next year.



	parent	distance
BOS	JFK	371
BWI		0
DFW	ORD	1423
JFK	BWI	184
LAX	MIA	2658
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	BOS	2467

- LAX is the last stop on the journey.



	parent	distance
BOS	JFK	371
BWI		0
DFW	ORD	1423
JFK	BWI	184
LAX	MIA	2658
MIA	BWI	946
ORD	BWI	621
PVD	JFK	328
SFO	BOS	2467

Running Time

- Let's assume that we represent G with an adjacency list. We can then step through all the vertices adjacent to u in time proportional to their number (i.e. **O**(j) where j in the number of vertices adjacent to u)

- The priority queue Q - we have a choice:
  - A **Heap**: Implementing Q with a heap allows for efficient extraction of vertices with the smallest D label(**O**($\log N$)). If Q is implented with locators, key updates can be performed in **O**($\log N$) time. The total run time is **O**($(n+m)\log n$) where n is the number of vertices in G and m in the number of edges. In terms of n, worst case time is **O**($n^2 \log n$)
  - An **Unsorted Sequence**: **O**(n) when we extract minimum elements, but fast key updates (**O**(1)). There are only n-1 extractions and m relaxations. The running time is **O**(n^2+m)

- In terms of **worst case** time, heap is good for small data sets and sequence for larger.

Running Time (cont)

- The ***average case*** is a slightly different story. Consider this:
  - If priority queue Q is implemented with a heap, the bottleneck step is updating the key of a vertex in Q. In the worst case, we would need to perform an update for every edge in the graph.
  - For most graphs, though, this would not happen. Using the **random neighbor-order** assumption, we can observe that for each vertex, its neighbor vertices will be pulled into the cloud in essentially random order. So here are only **O**(log$n$) updates to the key of a vertex.
  - Under this assumption, the run time of the heap implementation is **O**($n\log n+m$), which is always **O**(n^2). **The heap implementation is thus preferable for all but degenerate cases**.

Dijkstra's Algorithm, some things to think about...

- In our example, the **weight** is the geographical distance. However, the weight could just as easily represent the cost or time to fly the given route.
- We can easily **modify Dijkstra's algorithm for different needs**, for instance:
  - If we just want to know the shortest path from vertex v to a single vertex u, we can stop the algorithm as soon as u is pulled into the cloud.
  - Or, we could have the algorithm output a tree T rooted at v such that the path in T from v to a vertex u is a shortest path from v to u.
- **How to keep track of weights and distances?** Edges and vertices do not "know" their weights/ distances. Take advantage of the fact that D[u] is the key for vertex u in the priority queue, and thus D[u] can be retrieved if we know the locator of u in Q.
- Need some way of:
  - associating PQ locators with the vertices
  - storing and retrieving the edge weights
  - returning the final vertex distances